Multiattribute Utility Theory, Ordinal Regression and Robust Ordinal Regression

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13thMCDA/MCDM Summer School 2018
Chania, Greece - July 23th– August, 3rd 2018

Plan of the talk

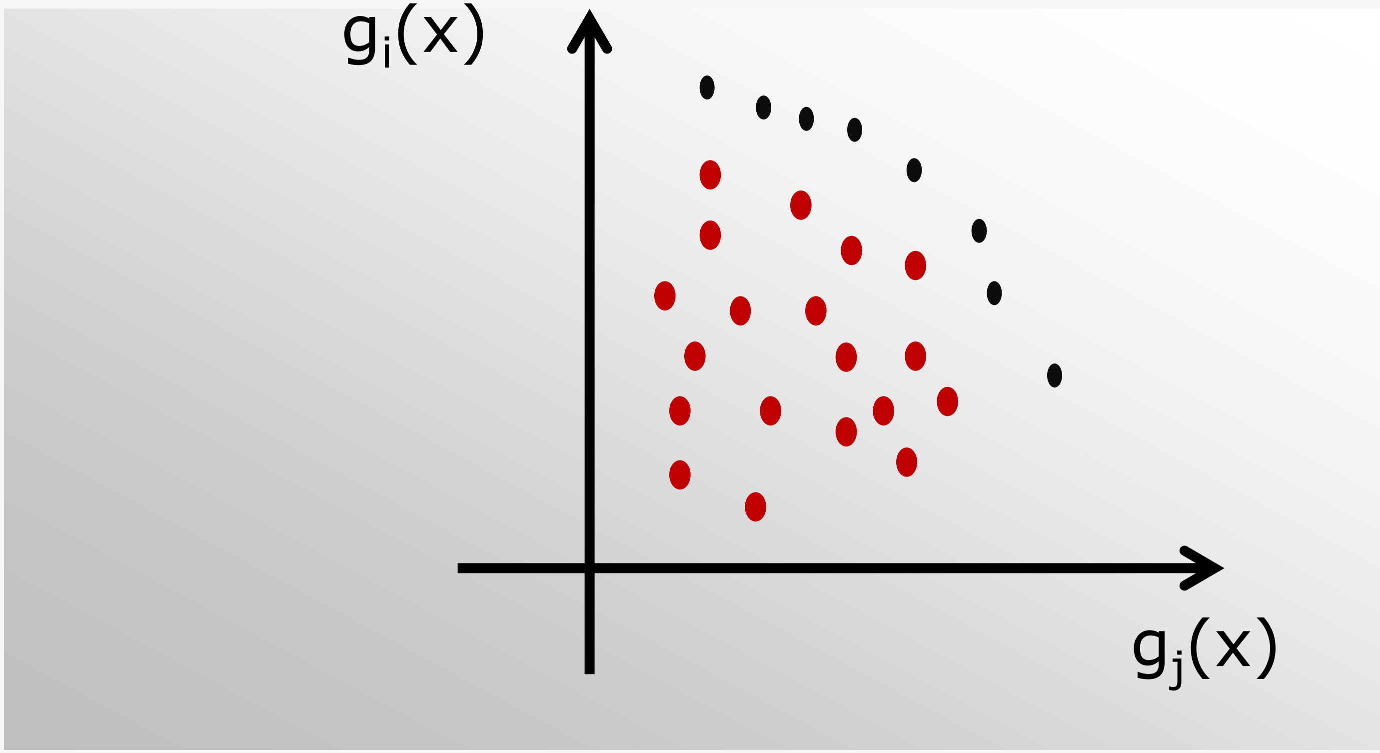
Plan

- Multiattribute Utility Theory
- Ordinal regression and inductive learning approaches
- The UTA method
- Robust ordinal regression
- The UTA^{GMS} method
- GRIP – *Generalized Regression with Intensities of Preference*
- Representative value functions
- UTADIS^{GMS}: Robust ordinal regression for sorting
- Robust ordinal regression for group decision
- Robust ordinal regression for outranking methods
- Robust ordinal regression for interacting criteria: NAROR and UTA^{GMS}-INT
- Stochastic Ordinal Regression and Subjective Stochastic Ordinal Regression
- Multiple Criteria Hierarchy Process
- Interactive optimization with Robust Ordinal Regression
- Conclusions

Ordinal regression and inductive learning approaches

Problem statement – multicriteria choice, ranking and sorting

- Consider a finite set *A* of actions (alternatives, solutions, objects) evaluated by *m* criteria from a consistent family $F=\{g_1,\ldots,g_m\}$; $I=\{1,\ldots,m\}$
- The only objective information is dominance relation in set *A*



Bernard Roy on the constructive approach of MCDA

"MCDA must be based on models that are, at least partially, co-constructed through interaction with the decision maker. The co-constructed model must be a tool for looking deeper into the subject, exploring, interpreting, debating and even arguing."(Roy 2010)

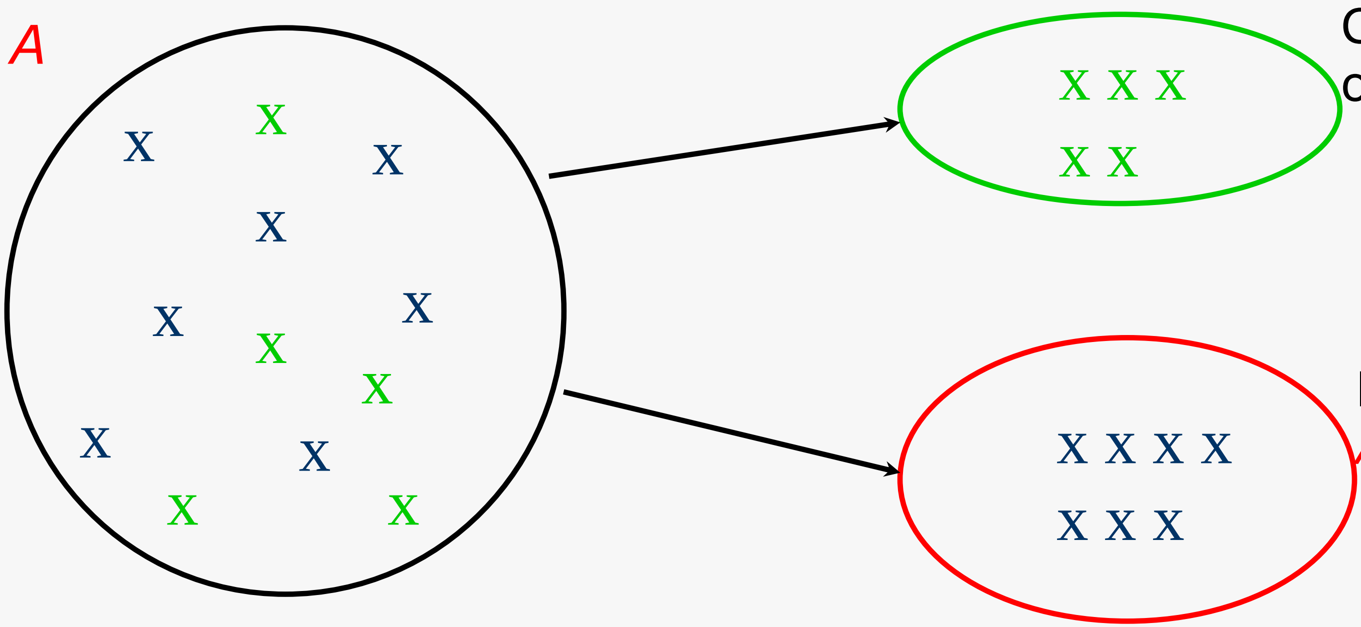
Bernard Roy on the recommendation in MCDA

"The content of the recommendation may be only the fruit of a conviction constructed in the course of a process necessitating multiple interactions, bringing into play a variety of actors involved in a complex managerial environment."

(Roy 1993)"

Taxonomy of Decision Problems

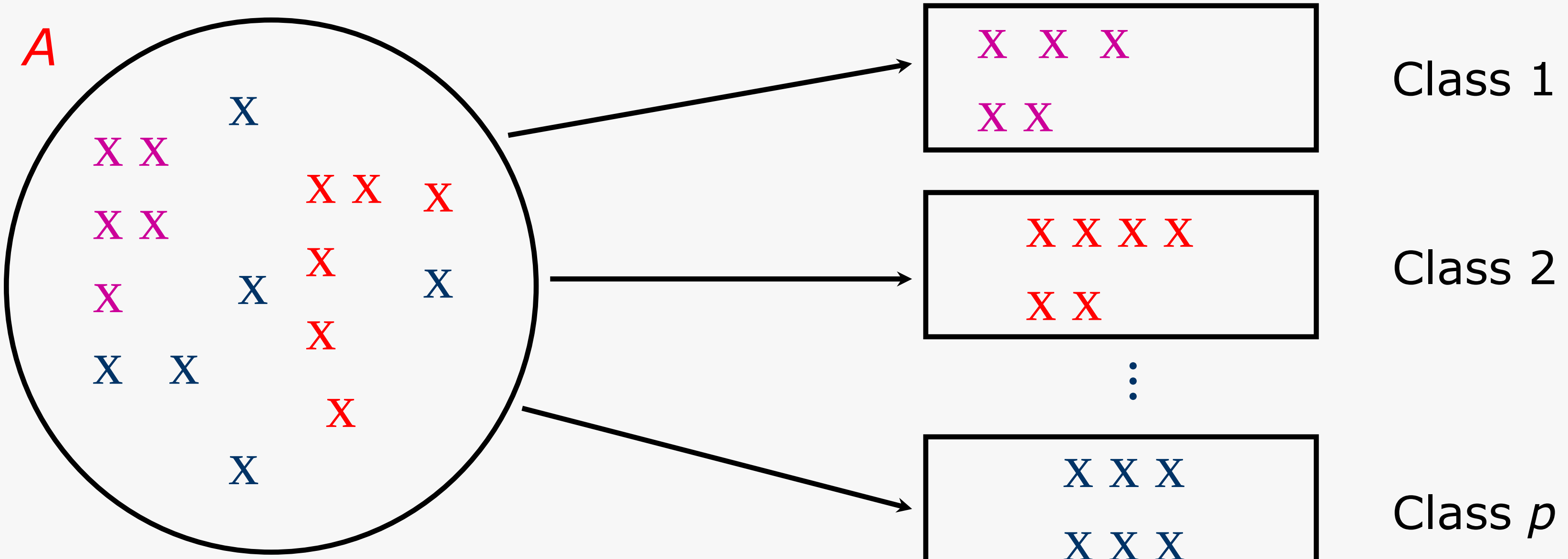
P_α : Choice problem (optimization)



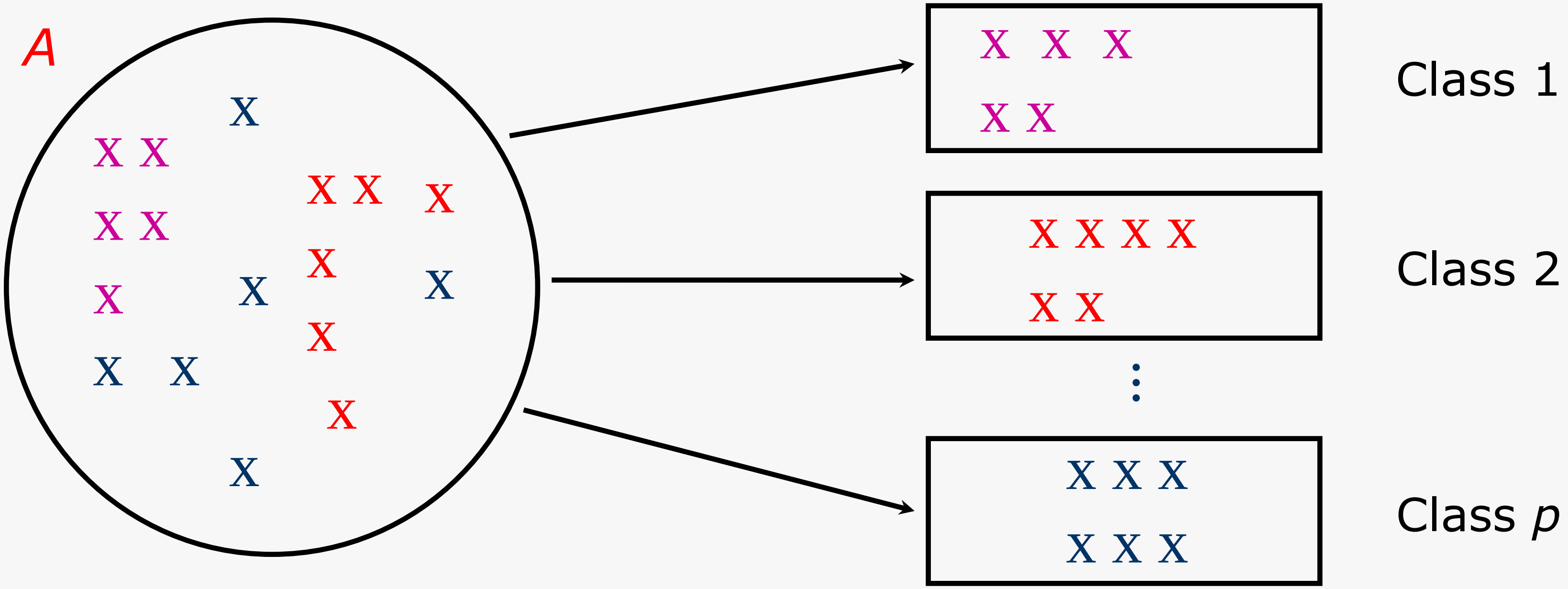
Chosen subset A' of best actions

Rejected subset $A \backslash A'$ of actions

$P_{\beta 1}$: Classification to preferentially non-ordered classes (classification in the strict sense)

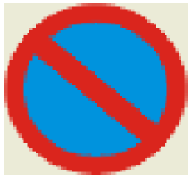


$P_{\beta 2}$: Classification to preferentially ordered classes (sorting)



Class 1 $\succ$ Class 2 $\succ$ … $\succ$ Class p

Classification in the strict sense – example of traffic signs

Traffic sign	Shape (S)	Primary Color (PC)	Class
a) 	triangle	yellow	W
b) 	circle	white	I
c) 	circle	blue	I
d) 	circle	blue	O

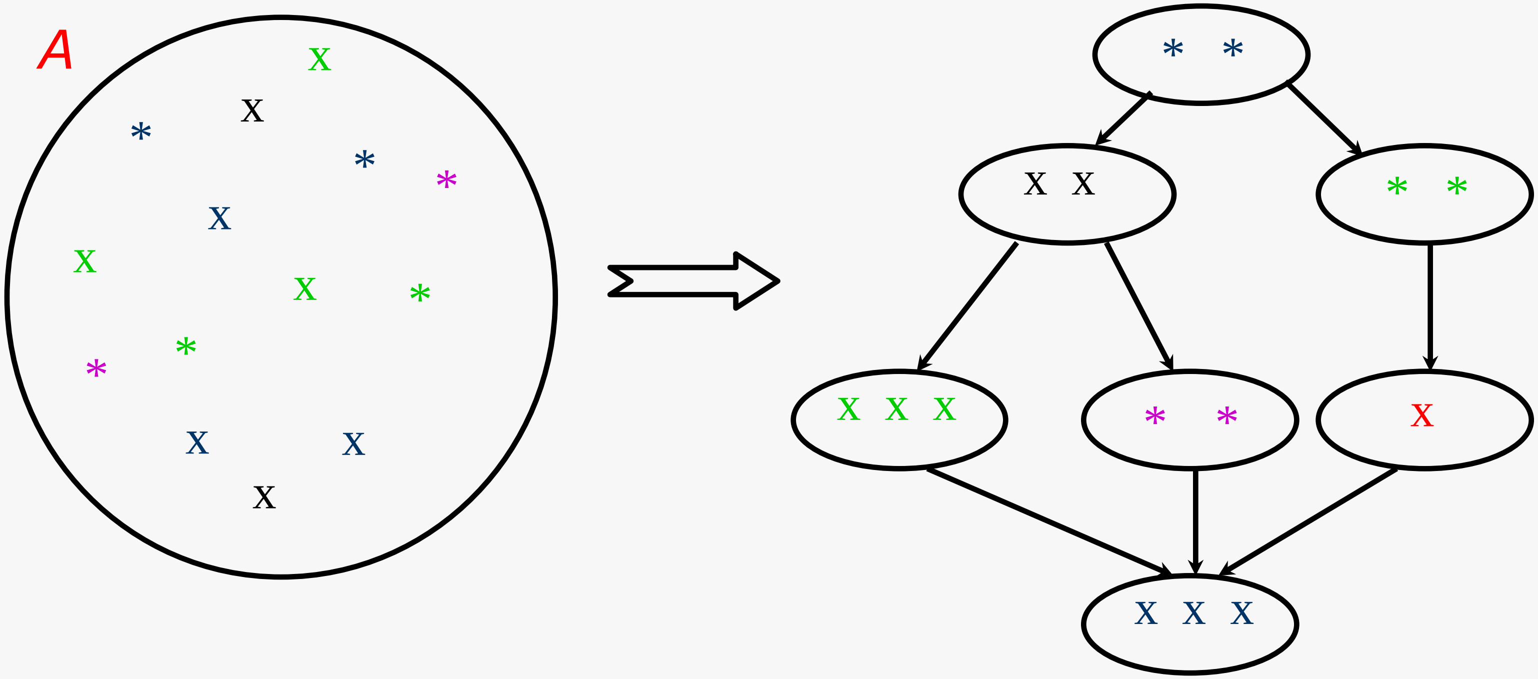
W: Warning; I: Interdiction; O: Obligation

Sorting – example of multiple criteria sorting of students

Student	Mathematics (**M**)	Physics (**Ph**)	Literature (**L**)	Overall class
S1	**good**	**medium**	**bad**	**bad**
S2	**medium**	**medium**	**bad**	**medium**
S3	**medium**	**medium**	**medium**	**medium**
S4	**medium**	**medium**	**medium**	**good**
S5	**good**	**medium**	**good**	**good**
S6	**good**	**good**	**good**	**good**
S7	**bad**	**bad**	**bad**	**bad**
S8	**bad**	**bad**	**medium**	**bad**

P_γ : Ordering problem (ranking)

Partial or complete ranking



Multiattribute Utility Theory (MAUT)

Setting

- $N=\{1,2,\ldots,n\}$ set of attributes
- X_i : set of possible values of the i-th attribute
- $X=\prod_{i=1}^{n} X_i = X_1 \times X_2 \times\ldots \times X_n=\{(x_1,\ldots, x_n):x_1\in X_1,\ldots, x_n\in X_n\}$: set of all **conceivable** alternatives
    - X includes the alternatives under study. . . and many others!
- $\succeq$: weak preference relation on X such that for all $x,y\in X$

$x \succeq y$

means

«x is at least as good as y»

- $x\succ y \Leftrightarrow x \succeq y$ and not $y \succeq x$ (which means «x is preferred to y»)
- $x\sim y \Leftrightarrow x \succeq y$ and $y \succeq x$ (which means «x and y are indifferent»)

Marginal preferences

- $J=\{i_1,\ldots,i_k\}\subseteq N$
- $X_J=\prod_{i\in J} X_i = X_{j1}\times X_{j2}\times\ldots\times X_{jk}=\{(x_{j1},\ldots,x_{jk}):x_{j1}\in X_{j1},\ldots,x_{jk}\in X_{i_k}\}$: set of all **conceivable** alternatives **with respect to attributes from** J
- $X_{-J}=\prod_{i\notin J} X_i$:set of all **conceivable** alternatives **with respect to attributes different from** J
- $\succeq_J$: weak **marginal preference** relation on X_J such that for all $x_J,y_J\in X_J$

$$x_J \succeq_J y_J \Leftrightarrow (x_J,z_{-J}) \succeq (y_J,z_{-J}) \text{ for all } z_{-J}\in X_{-J}$$

which means

«x_J is at least as good as y_J»

- In case J={i}, we write $\succeq_i$ instead of $\succeq_{\{i\}}$.

Additive value function model

- For all all x,y∈X

$$x \succeq y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)$$

with $u_i:X_i \rightarrow \mathbf{R}$.

- Sometimes a simplified model is considered: if $X \subseteq \mathbf{R}^n$ and for all attribute $i \in N$

$$x_i \geq y_i \Leftrightarrow x_i \succeq y_i$$

- For all x,y∈X

$$x \succeq y \Leftrightarrow \sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i y_i$$

with w_i non negative for all $i \in N$.

Independence

- $\succeq$ is independent for $J \subseteq N$ if for all $x_J, y_J \in X_J$

$$[(x_J, z_{-J}) \succeq (y_J, z_{-J}) \text{ for some } z_{-J} \in X_{-J}]$$

$$\Downarrow$$

$$[(x_J, z_{-J}) \succeq (y_J, z_{-J}) \text{ for all } z_{-J} \in X_{-J}]$$

- If $\succeq$ is independent for all $J \subseteq N$, with J non-empty, we say that $\succeq$ is **independent**.
- If $\succeq$ is independent for all $\{i\}$, $i \in N$, we say that $\succeq$ is **weakly independent**.
- If $\succeq$ is weakly independent, then **dominance** arguments apply, i.e. for all $x,y \in X$

$$[x_i \succeq_i y_i \text{ for all } i \in N] \Rightarrow x \succeq y$$

Independence: illustrative example

z_{-J}, z'_{-J} $x_J,\ y_J$

Students	Mathematics	Physics	Literature
S1	*Good*	*Medium*	*Bad*
S2	*Good*	*Bad*	*Medium*
S3	*Medium*	*Medium*	*Bad*
S4	*Medium*	*Bad*	*Medium*

If $S2 \succeq S1$, then $S4 \succeq S3$

If $(x_J\, ,\ z_{-J}) \succeq (y_J\, ,\ z_{-J})$, then $(x_J\, ,\ z'_{-J}) \succeq (y_J\, ,\ z'_{-J})$

Independence: illustrative example

Students	Mathematics	Physics	Literature
S1	~~*Good*~~	*Medium*	*Bad*
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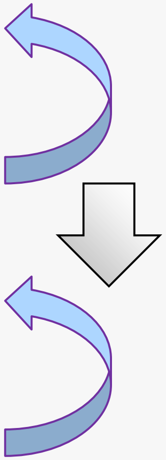
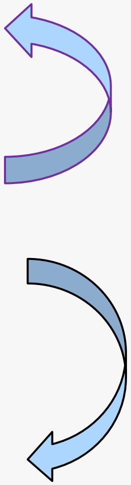
S3	~~*Medium*~~	*Medium*	*Bad*
S4	~~*Medium*~~	*Bad*	*Medium*



$$S2 \succ S1 \;\Rightarrow\; S4 \succ S3$$

Is independence a reasonable hypothesis?

Dinner	Main course	Wine
D1	*Meat*	*White*
D2	*Meat*	*Red*
D3	*Fish*	*White*
S4	*Fish*	*Red*



$D2 \succ D1$ and $D3 \succ D4$

Is independence a reasonable hypothesis?

Dinner	Main course	Wine
D1	*Meat*	*White*
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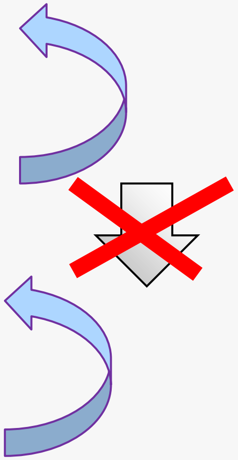
S4	*Fish*	*Red*



$$D2 \succ D1 \;\cancel{\Rightarrow}\; D4 \succ D3$$

Basic results for Multiattribute Utility Theory

If

- restricted solvability holds,
- each attribute is essential,

then the additive value function holds if and only if

$\succeq$ is an **independent** weak order satisfying the Thomsen and the Archimedean conditions.

In case there are more than two attributes, Thomsen condition can be forgotten.

How to assess a multiattribute value function?

Many methods:

- Direct rating
- Bisection techniques
- … (e.g. Peter C. Fishburn , Methods of Estimating Management Science, 13(7), 1967, 435-453, where **24** methods are presented)
-

How to assess tradeoff ?

- Consider the simplified model

$$x \succeq y \Leftrightarrow \sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i y_i$$

- For $x \in X$ and $i,j \in N$, consider k_{ij} such that

$(x_1,x_2,\dots, x_i+1,\dots, x_j,\dots, x_n) \sim (x_1,x_2,\dots,x_i,\dots,x_j+ k_{ij},\dots, x_n)$

- We get

$$w_1x_1+w_2x_2+\dots+w_i(x_i+1)+\dots+w_jx_j+\dots+w_nx_n$$

$$=$$

$$w_1x_1+w_2x_2+\dots+w_ix_i+\dots+w_j(x_j+ k_{ij})+\dots+w_nx_n$$

- From which…

How to assess tradeoff ?

- From which

$$w_i = w_j k_{ij}$$

and therefore

$$k_{ij} = w_i / w_j$$

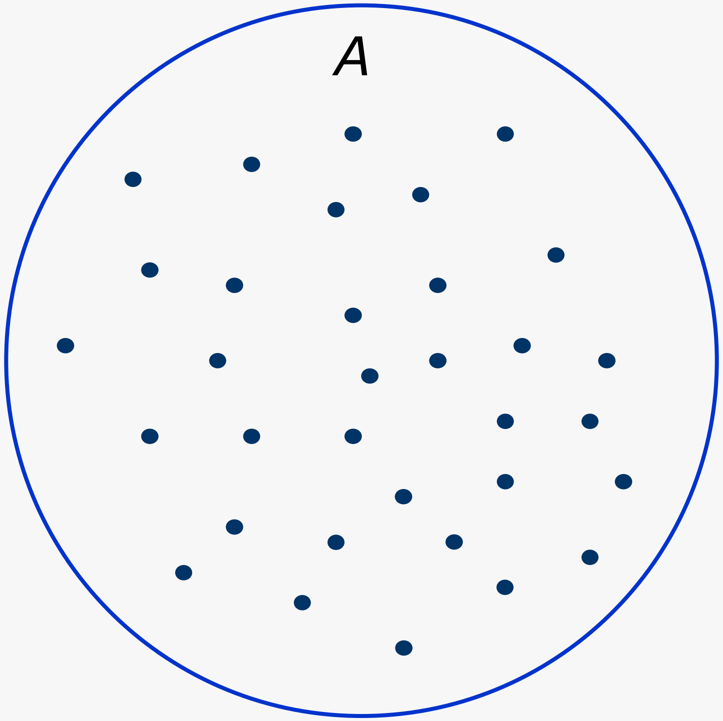
- This means that the weights in the MAUT model are related to the concept of tradeoff (I can renounce to k_{ij} on attribute j, in order to increase one unit on attribute i).
- Observe that coherence condition is that for all $i,j,l \in N$

$$k_{ij} = k_{il} \times k_{lj}$$

$$(w_i / w_j = w_i / w_l \times w_l / w_j)$$

Holistic preference information

- Psychologists confirm that Decision Makers (DMs) are more confident exercising their decisions than explaining them
- The most natural is a holistic pairwise comparison of some actions relatively well known to the DM, i.e. reference actions



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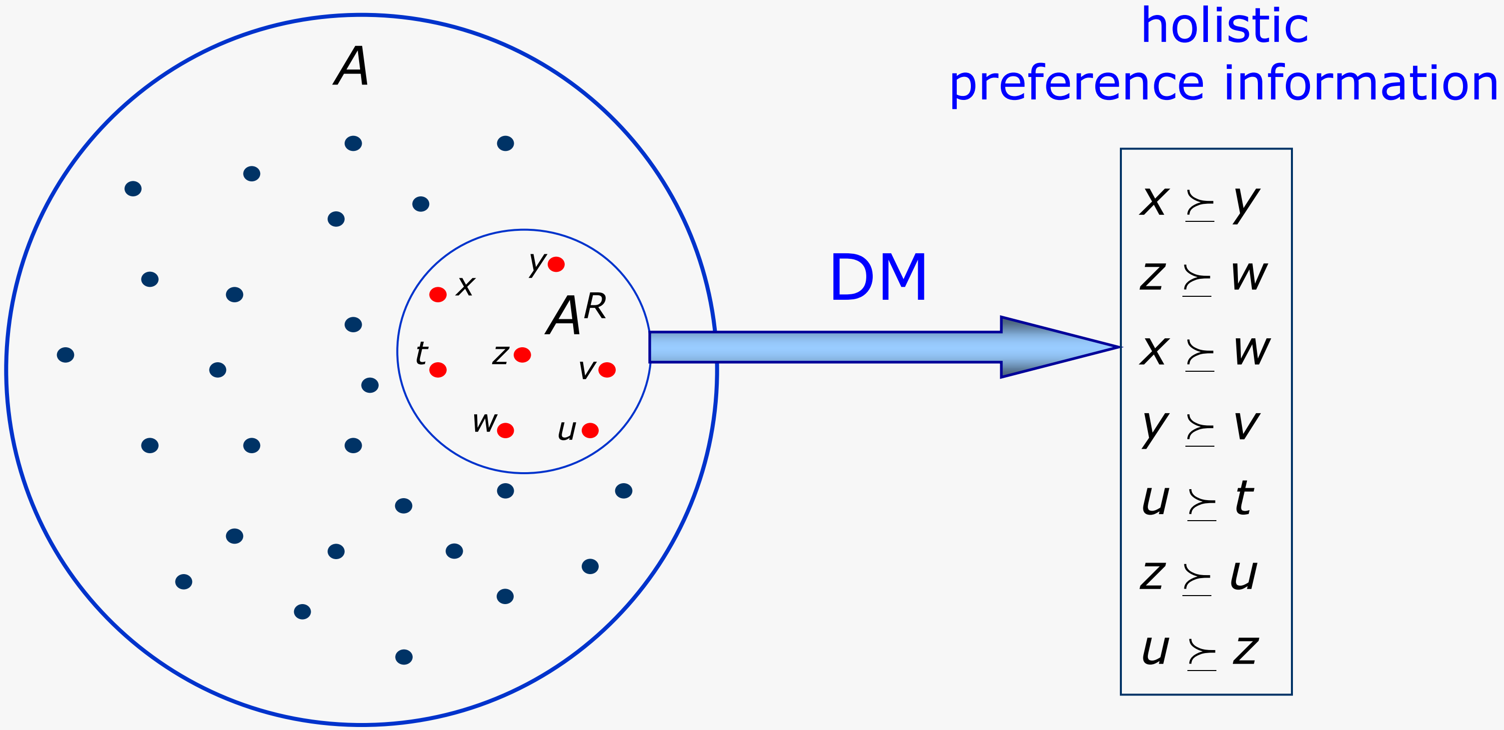


$x \succeq y$

$z \succeq w$

$x \succeq w$

$y \succeq v$

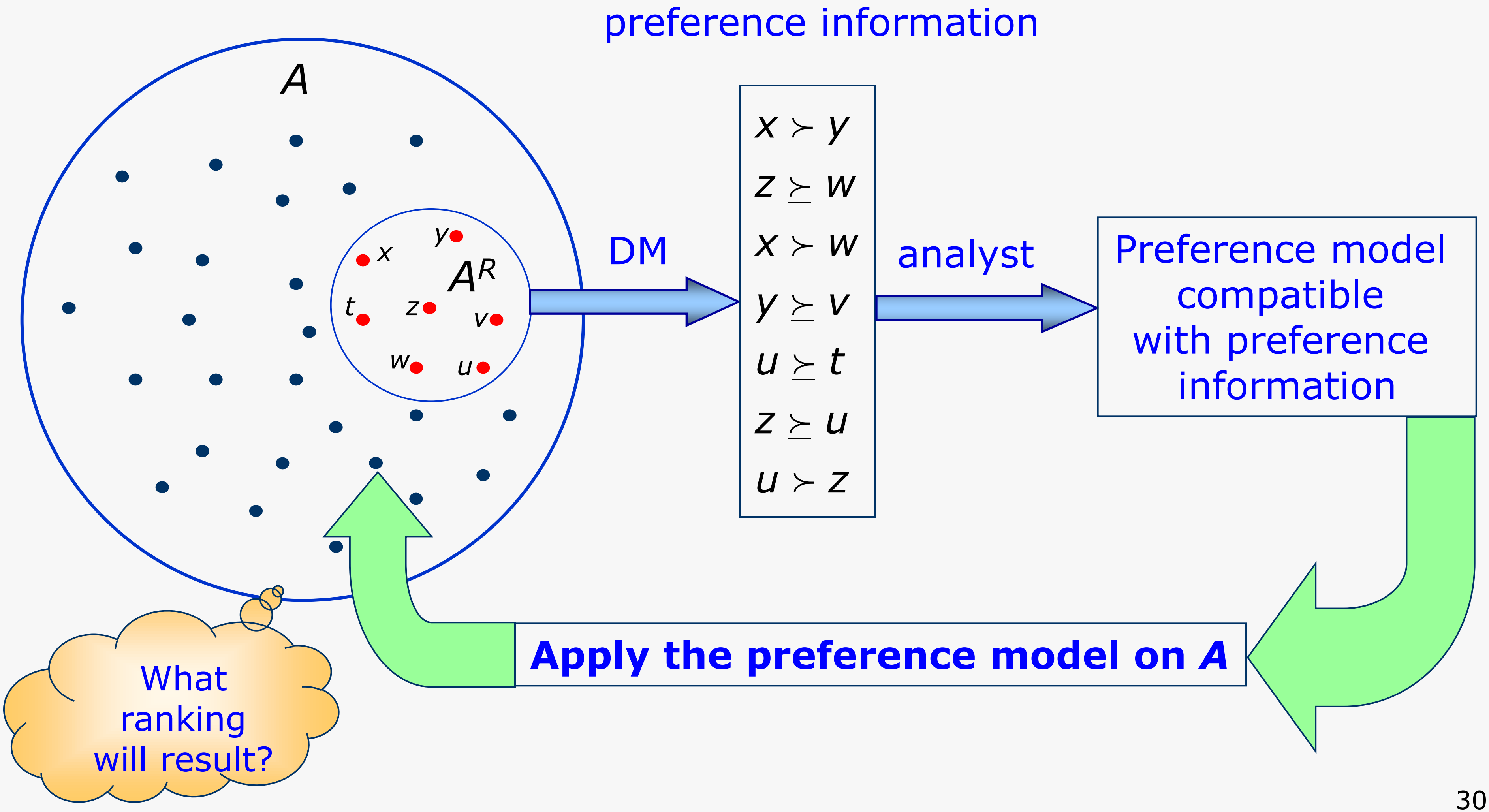
$u \succeq t$

$z \succeq u$

$u \succeq z$

Holistic preference information

- Question: what is the consequence of using on the whole set *A* this information transformed to a compatible preference model ?

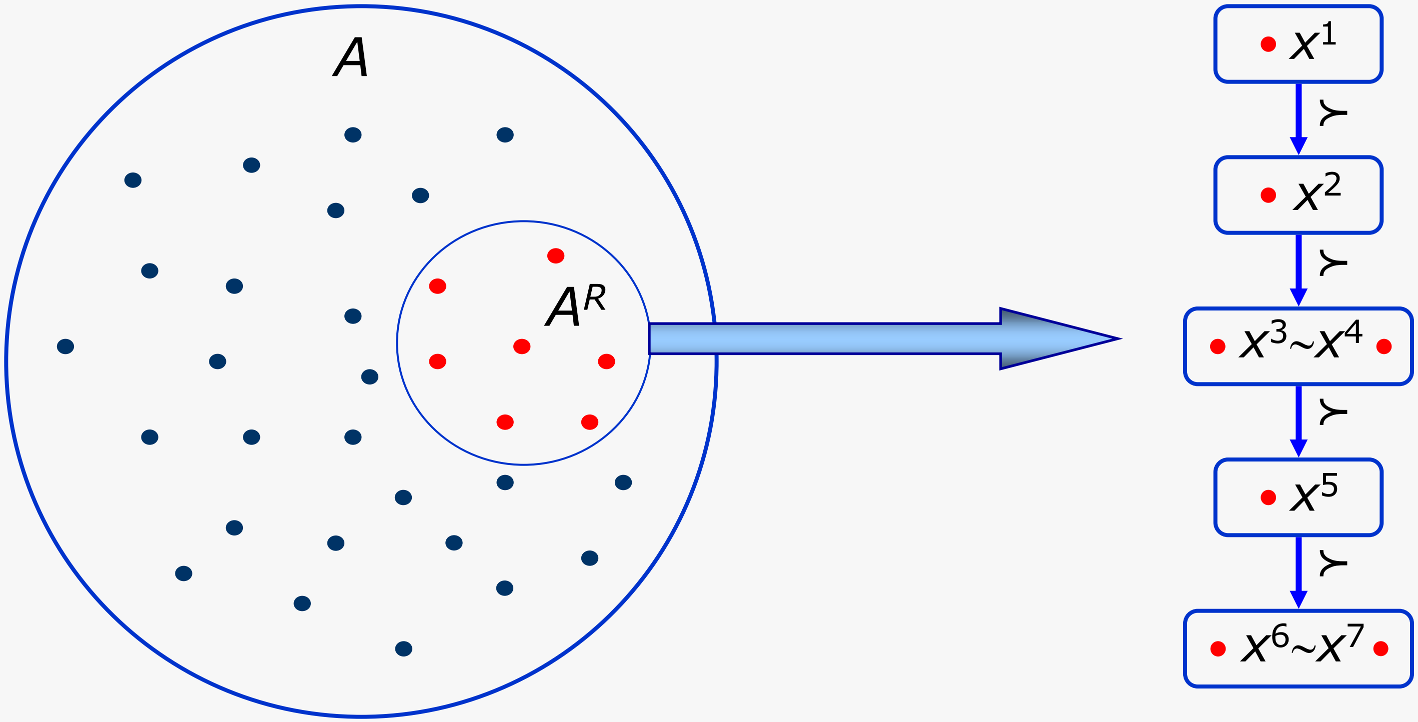


Principle of the ordinal regression

- The **preference information** is given in the form of partial preorder on a subset of reference actions $A^R \subseteq A$
- Additive value (or utility) function on A: for each $x \in A$

$$U(x) = \sum_{i=1}^{n} u_i[g_i(x)]$$

where u_i are non-decreasing marginal value functions



The UTA method

The UTA method (Jacquet-Lagreze & Siskos 1982)

- Marginal value of action $x_i \in A$ is approximated by linear interpolation

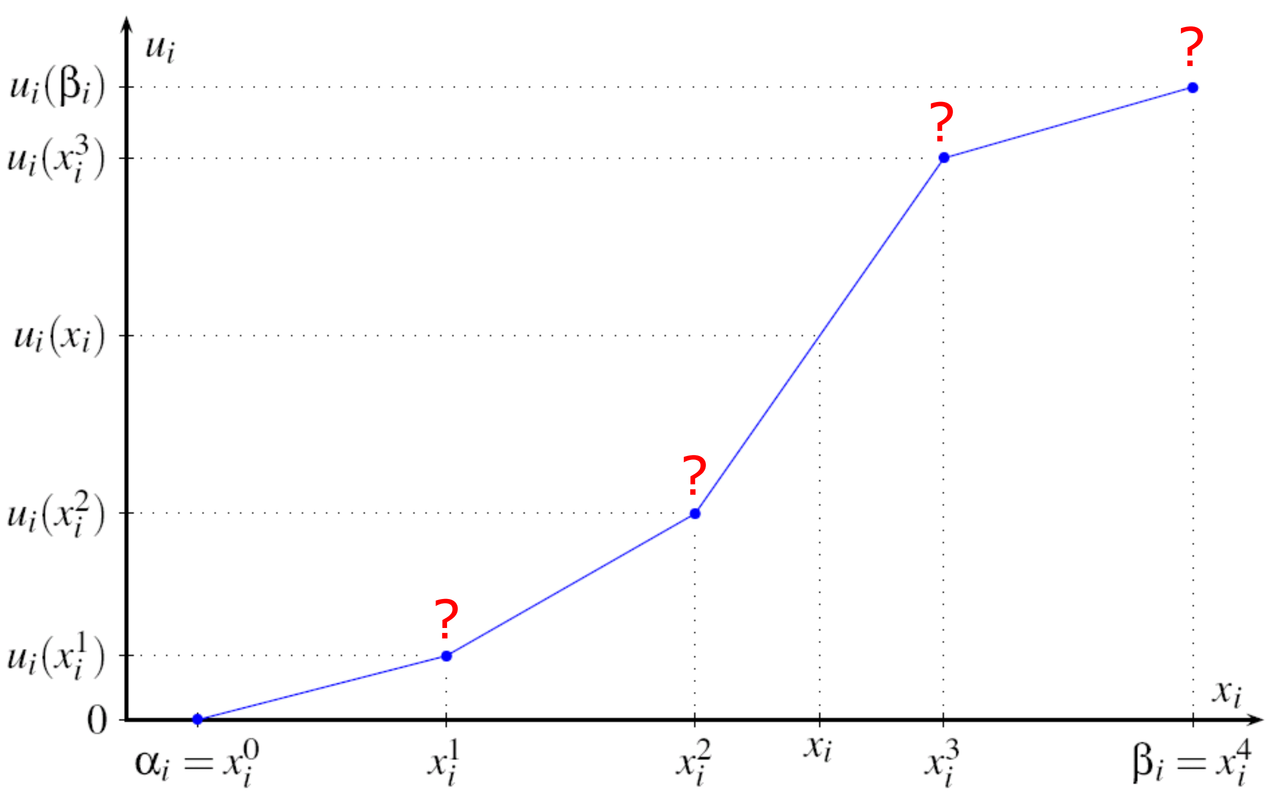


Figure 1: Piecewise linear marginal utility function

The principle of the ordinal regression – the UTA method

(Jacquet-Lagreze & Siskos 1982)

- The marginal value functions (breakpoint variables) are estimated by solving the LP problem

$$\text{Min} \quad \rightarrow \quad E^{UTA} = \sum_{a\in A^R}\left(\sigma^+(a)+\sigma^-(a)\right)$$

subject to

$$\left.\begin{array}{l}
\left.\begin{array}{l}
U(a)+\sigma^+(a)-\sigma^-(a)\ge U(b)+\sigma^+(b)-\sigma^-(b)+\varepsilon \Leftrightarrow \;\; a\succ b \\
U(a)+\sigma^+(a)-\sigma^-(a)= U(b)+\sigma^+(b)-\sigma^-(b) \Leftrightarrow \;\; a\sim b
\end{array}\right\}\;\forall a,b\in A^R \\
u_i\left(x_i^{j+1}\right)-u_i\left(x_i^{j}\right)\ge 0 \quad j=0,\ldots,\gamma_i-1\,;\;\; \forall i\in I \\
\sum_{i=1}^{n} u_i(\beta_i)=1 \\
u_i(\alpha_i)=0 \quad \forall i\in I \\
u_i\left(x_i^{j}\right)\ge 0,\;\; \sigma^+(a)\ge 0,\;\; \sigma^-(a)\ge 0,\;\; \forall a\in A^R,\;\; \forall i \;\text{ and }\; j
\end{array}\right\}\;(C)$$

where ε is a small positive constant, and σ^+ and σ^- are auxiliary variables (errors of approximation)

“The most representative value function” of UTA: UTAMP1 model

UTAMP1 model

(Siskos & Yannacopolous 1985)

- After verifying that the set of compatible value function is not empty, the “most representative” value function is estimated by solving the following LP problem,

$$\text{Max } \varepsilon$$

$$\xcancel{\text{Min} \quad \rightarrow \quad F^{UTA} = \sum_{x^k \in A^R} \left(\sigma^+\left(x^k\right) + \sigma^-\left(x^k\right) \right)}$$

subject to

$$\left.\begin{aligned}
&\left.\begin{aligned}
&U'\left(x^k\right) \geq U'\left(x^{k+1}\right) + \varepsilon \Leftrightarrow x^k \succ x^{k+1} \\
&U'\left(x^k\right) = U'\left(x^{k+1}\right) \Leftrightarrow x^k \sim x^{k+1}
\end{aligned}\right\} \; k=1,\dots,n\text{-}1 \\
&u_i\left(x_i^{j+1}\right) - u_i\left(x_i^j\right) \geq 0 \quad j = 0,\dots,\gamma_i \;;\; \forall i \in I \\
&\sum_{i=1}^{n} u_i\left(\beta_i\right) = 1 \\
&u_i\left(\alpha_i\right) = 0 \quad \forall i \in I \\
&u_i\left(x_i^j\right) \geq 0,\; \xcancel{\sigma^+\left(x^k\right) \geq 0,\; \sigma^-\left(x^k\right) \geq 0,}\; \forall x^k \in A^R,\; \forall i \text{ and } j
\end{aligned}\right\} \; (C)$$

$\sigma^+(x^k)=0$, $\sigma^-(x^k)=0$

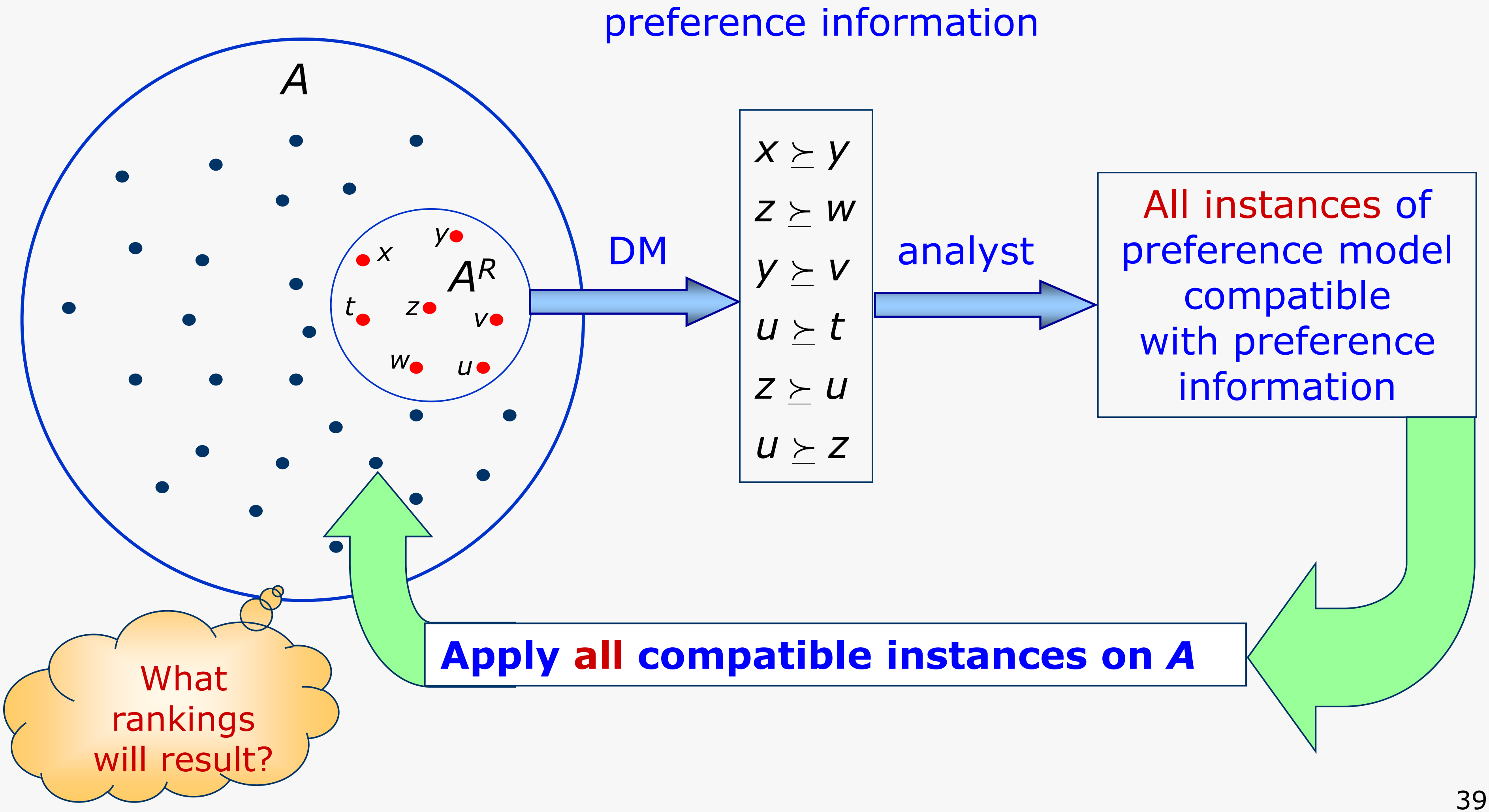
Intuition behind the Robust Ordinal Regression

Basic question

- Remark 1. If there is one value function representing the preferences of the DM, in general, there are infinitely many others.
- Remark 2. In general, each one of these infinitely many value functions, gives a different ranking of actions from A.
- **Why to consider only one of these infinitely many value functions?**

One should use all compatible preference models on set *A*

- Question: what is the consequence of using all compatible preference models on set *A* ?



preference information

DM

$x \succeq y$
$z \succeq w$
$y \succeq v$
$u \succeq t$
$z \succeq u$
$u \succeq z$

analyst

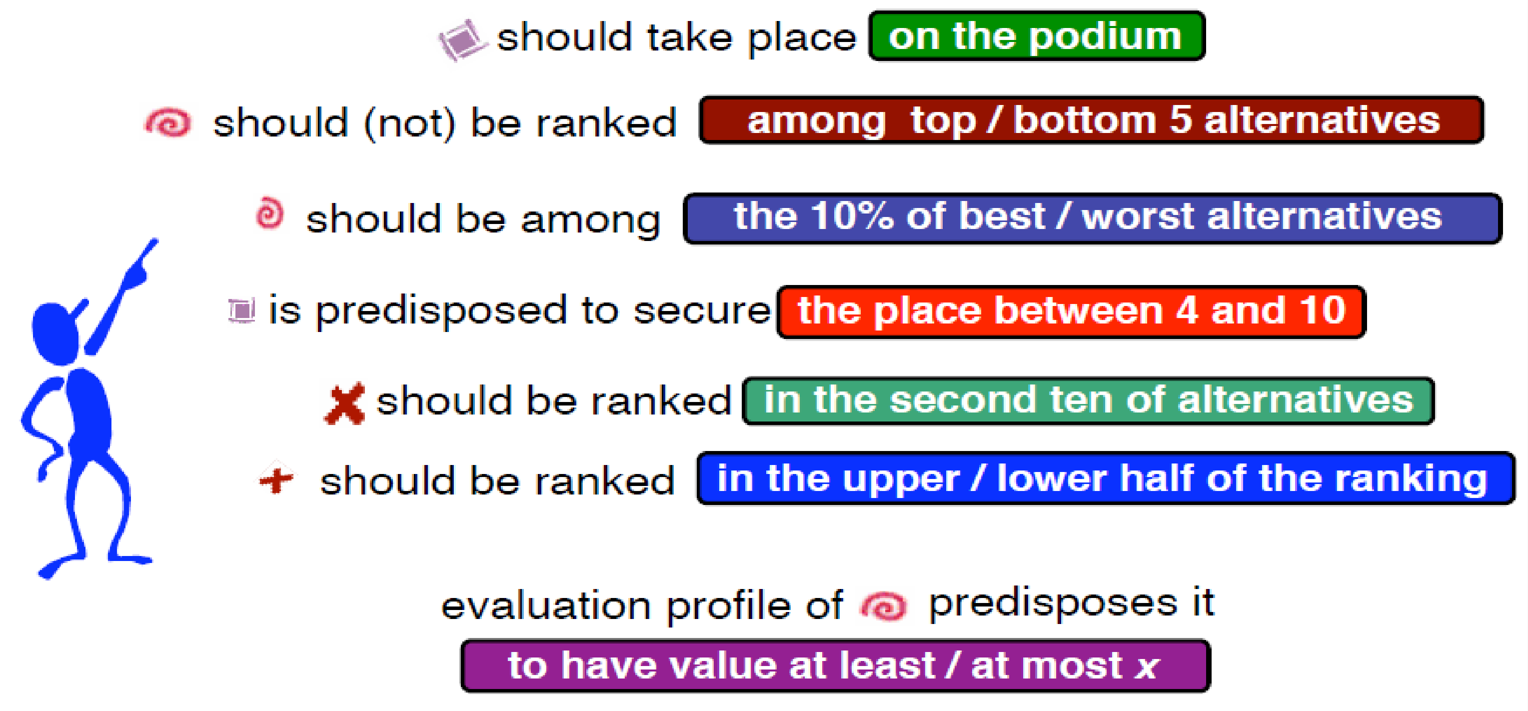
All instances of preference model compatible with preference information

Apply all compatible instances on *A*

What rankings will result?

Rank related preference information

- Types of indirect preference information in particular nodes of the tree:
  - Desired ranks of alternatives, e.g.,



◈ should take place **on the podium**

@ should (not) be ranked **among top / bottom 5 alternatives**

@ should be among **the 10% of best / worst alternatives**

▣ is predisposed to secure **the place between 4 and 10**

✗ should be ranked **in the second ten of alternatives**

✚ should be ranked **in the upper / lower half of the ranking**

evaluation profile of @ predisposes it **to have value at least / at most *x***

M. Kadziński, S. Greco, R. Słowiński: RUTA: a framework for assessing and selecting additive value functions on the basis of rank related requirements. *OMEGA*, 41 (2013) no.4, 735–751

The UTA^{GMS} method

The UTA^{GMS} method (Greco, Mousseau & Słowiński 2004, 2008)

- DM is supposed to provide the following preference information:
  - a partial preorder $\succeq$ on A^R, such that $\forall x,y \in A^R$

$$x \succeq y \;\Leftrightarrow\; \text{„}x \text{ is at least as good as } y\text{”}$$

The UTA^{GMS} method (Greco, Mousseau & Słowiński 2004, 2008)

- A value function U is called compatible if it satisfes the constraints corresponding to DM’s preference information:

a) $U(x) \geq U(y)$ iff $x \succeq y$

b) $U(x) > U(y)$ iff $x \succ y$

c) $U(x) = U(y)$ iff $x \sim y$

d) $u_i(x) \geq u_i(y)$ iff $x \succeq_i y$, $i \in I$

- Moreover, the following normalization constraints should also be taken into account:

e) $u_i(\alpha_i)=0$, $i \in I$

f) $\sum_{i \in I} u_i(\beta_i) = 1$

The UTA^{GMS} method (Greco, Mousseau & Słowiński 2004, 2008)

- If constraints *a) – f)* are consistent, then we get the two weak preference relations $\succeq^N$ and $\succeq^P$:
  - the necessary weak preference relation: for all $x,y \in A$,

    $x \succeq^N y \Leftrightarrow U(x) \geq U(y)$ for all compatible value functions

    (i.e. for all compatible value functions x is at least as good as y)
  - the possible weak preference relation: for all $x,y \in A$,

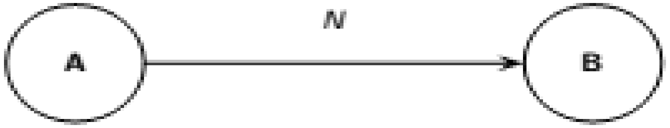
    $x \succeq^P y \Leftrightarrow U(x) \geq U(y)$ for at least one compatible value function

    (i.e. for at least one compatible value function

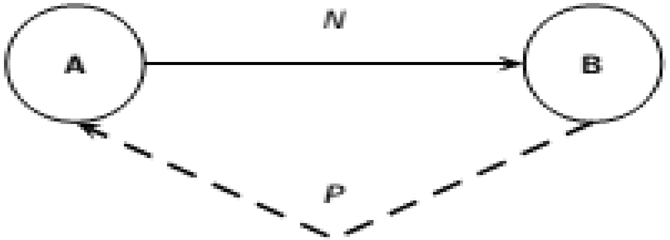
    x is at least as good as y)

Is it necessary the possible preference relation?

- If we do not consider the possible preference relation, we are not able to distinguish these following two cases.



$\mathbf{A} \succsim^N \mathbf{B}$ **and** $\mathbf{B} \not\succsim^P \mathbf{A}$



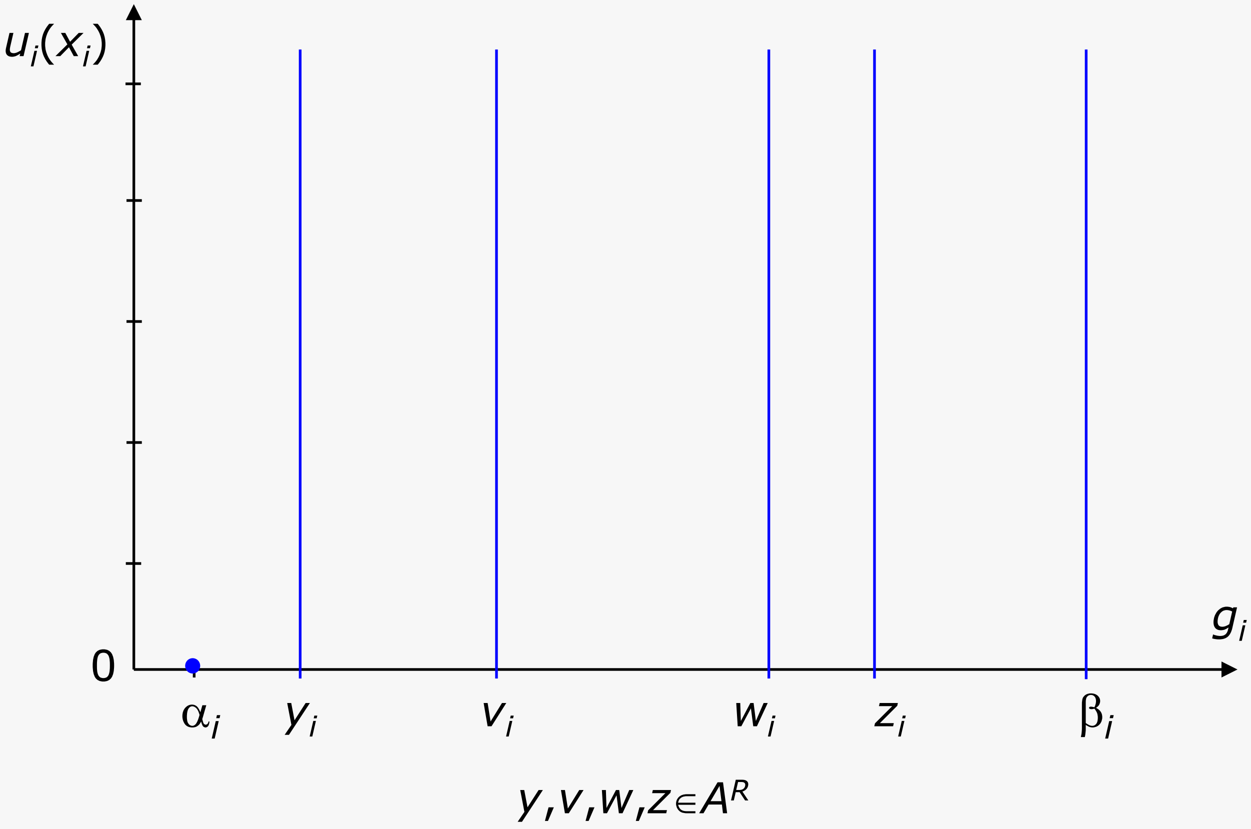
$\mathbf{A} \succsim^N \mathbf{B}$ **and** $\mathbf{B} \succsim^P \mathbf{A}$

The UTA^{GMS} method (Greco, Mousseau & Słowiński 2008)

- Basic properties: for all $x,y,z \in A$
  - $x \succsim^N y \;\Rightarrow\; x \succsim^P y$
  - $\succsim^N$ is a partial preorder (i.e. $\succsim^N$ is reflexive and transitive)
  - $x \succsim^N y$ and $y \succsim^P z \Rightarrow x \succsim^P z$
  - $x \succsim^P y$ and $y \succsim^N z \Rightarrow x \succsim^P z$
  - $x \succsim^N y$ or $y \succsim^P x$
  - $\succsim^P$ is strongly complete (i.e. for all $x,y \in A$, $x \succsim^P y$ or $y \succsim^P x$) and negatively transitive (i.e. for all $x,y,z \in A$, *not* $x \succsim^P y$ and *not* $y \succsim^P z \Rightarrow$ *not* $x \succsim^P z$), (in general, $\succsim^P$ is not transitive)
- Giarlotta and Greco (2013) proved that the first 5 properties characterize $\succsim^N$ and $\succsim^P$.

The UTA^{GMS} method (Greco, Mousseau & Słowiński 2004, 2008)

- The marginal value function $u_i(x_i)$



Characteristic points of marginal value functions are fixed on actual evaluations of actions from set *A*

GRIP – *Generalized Regression with Intensities of Preference*

GRIP – *Generalized Regression with Intensities of Preference*

(Figueira, Greco & Słowiński 2005, 2008)

- GRIP extends the UTA^{GMS} method by adopting all features of UTA^{GMS} and by taking into account additional preference information :
  - comprehensive comparisons of intensities of preference between some pairs of reference actions,
    e.g. „*x* is preferred to *y* at least as much as *w* is preferred to *z*”
  - partial comparisons of intensities of preference between some pairs of reference actions on particular criteria,
    e.g. „*x* is preferred to *y* at least as much as *w* is preferred to *z*, on criterion $g_i \in F$”

GRIP – *Generalized Regression with Intensities of Preference*

(Figueira, Greco & Słowiński 2005, 2008)

- DM is supposed to provide the following preference information :
  - a partial preorder $\succeq$ on A^R, such that $\forall x,y \in A^R$

$$x \succeq y \Leftrightarrow \text{„}x \text{ is at least as good as } y\text{”}$$

  - a partial preorder $\succeq^*$ on $A^R \times A^R$, such that $\forall x,y,w,z \in A^R$

$$(x,y) \succeq^* (w,z) \Leftrightarrow \text{„}x \text{ is preferred to } y \text{ at least as much as } w \text{ is preferred to } z\text{”}$$

  - a partial preorder $\succeq_i^*$ on $A^R \times A^R$, i=1,...,n, such that $\forall x,y,w,z \in A^R$

$$(x,y) \succeq_i^* (w,z) \Leftrightarrow \text{„}x \text{ is preferred to } y \text{ at least as much as } w \text{ is preferred to } z, \text{ on criterion } g_i \in F\text{”}.$$

GRIP – *Generalized Regression with Intensities of Preference*

(Figueira, Greco & Słowiński 2005, 2008)

- A utility function U is called compatible if it satisfes the constraints corresponding to DM's preference information:

a) $U(x) \geq U(y)$ iff $x \succeq y$

b) $U(x) > U(y)$ iff $x \succ y$

c) $U(x) = U(y)$ iff $x \sim y$

d) $U(x) - U(y) \geq U(w) - U(z)$ iff $(x,y) \succeq^* (w,z)$

e) $U(x) - U(y) > U(w) - U(z)$ iff $(x,y) \succ^* (w,z)$

f) $U(x) - U(y) = U(w) - U(z)$ iff $(x,y) \sim^* (w,z)$

g) $u_i(x) \geq u_i(y)$ iff $x \succeq_i y, \ i \in I$

h) $u_i(x) - u_i(y) \geq u_i(w) - u_i(z)$ iff $(x,y) \succeq_i^* (w,z), \ i \in I$

i) $u_i(x) - u_i(y) > u_i(w) - u_i(z)$ iff $(x,y) \succ_i^* (w,z), \ i \in I$

j) $u_i(x) - u_i(y) = u_i(w) - u_i(z)$ iff $(x,y) \sim_i^* (w,z), \ i \in I$

GRIP – *Generalized Regression with Intensities of Preference*

(Figueira, Greco & Słowiński 2005, 2008)

- Moreover, the following normalization constraints should also be taken into account:

k) $u_i(\alpha_i)=0, \quad i \in N$

l) $\sum_{i \in I} u_i(\beta_i) = 1$

GRIP – *Generalized Regression with Intensities of Preference*

(Figueira, Greco & Słowiński 2005, 2008)

- If constraints *a) – l)* are consistent, then we get two weak preference relations $\succeq^N$ and $\succeq^P$
  - a necessary weak preference relation: for all $x,y \in A$,

    $$x \succeq^N y \Leftrightarrow U(x) \geq U(y) \text{ for all compatible value functions}$$

  - a possible weak preference relation: for all $x,y \in A$,

    $$x \succeq^P y \Leftrightarrow U(x) \geq U(y) \text{ for at least one compatible value function}$$

GRIP – *Generalized Regression with Intensities of Preference*

(Figueira, Greco & Słowiński 2005, 2008)

- If constraints *a) – l)* are consistent, then we get also two overall binary relations comparing intensity of preference $\succeq^{*N}$ and $\succeq^{*P}$:
  - a necessary relation of preference intensity: for all $x,y,w,z \in A$,

$(x,y) \succeq^{*N} (w,z) \Leftrightarrow [U(x) - U(y)] \geq [U(w) - U(z)]$ for all compatible value functions

  - a possible relation of preference intensity: for all $x,y,w,z \in A$, $(x,y) \succeq^{*P} (w,z)$: $[U(x) - U(y)] \geq [U(w) - U(z)]$ for at least one compatible value function

GRIP – *Generalized Regression with Intensities of Preference*

(Figueira, Greco & Słowiński 2005,2008)

- If constraints *a) – l)* are consistent, then we get two binary relations comparing intensity of preference $\succeq_i^{*N}$ and $\succeq_i^{*P}$ for each criterion $g_i \in F$:
  - a necessary relation of preference intensity: for all $x,y,w,z \in A$,

$(x,y) \succeq_i^{*N} (w,z) \Leftrightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))]$

for all compatible value functions

  - a possible relation of preference intensity : for all $x,y,w,z \in A$,

$(x,y) \succeq_i^{*P} (w,z)$: $[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))]$

for at least one compatible value function

GRIP – fundamental properties of $\succsim^{N}$, $\succsim^{P}$, $\succsim^{*N}$, $\succsim^{*P}$, $\succsim_i^{*N}$, $\succsim_i^{*P}$

- Some properties:
  - $x \succsim^{N} y \;\Rightarrow\; x \succsim^{P} y$,
  - $(x,y) \succsim^{*N} (w,z) \;\Rightarrow\; (x,y) \succsim^{*P} (w,z)$,
  - $(x,y) \succsim_i^{*N} (w,z) \;\Rightarrow\; (x,y) \succsim_i^{*P} (w,z)$, $g_i \in F$
  - $\succsim^{N}$, $\succsim^{*N}$ and $\succsim_i^{*N}$, $i \in N$, are partial preorders
  - $\succsim^{P}$, $\succsim^{*P}$ and $\succsim_i^{*P}$ are strongly complete and negatively transitive, (in general, $\succsim^{P}$ $\succsim^{P}$, $\succsim^{*P}$ and $\succsim_i^{*P}$ are not transitive)

GRIP – the linear programming problem: the result is independent of ε

- Strict inequalities such as *b), e), i)* are rewritten as:

  b') $U(x) \geq U(y) + \varepsilon$

  e') $U(x) - U(y) \geq U(w) - U(z) + \varepsilon$

  i') $u_i(x) - u_i(y) \geq u_i(w) - u_i(z) + \varepsilon$

- $x \succsim^P y \Leftrightarrow$ the set of constraints is feasible and $\varepsilon^* > 0$, where $\varepsilon^* = \text{Max } \varepsilon$, subject to constraints *a)–l)*, with *b),e),i)* written as *b'),e'),i')* and $U(x) \geq U(y)$
- $x \succsim^N y \Leftrightarrow$ the set of constraints is infeasible or $\varepsilon^* \leq 0$, where $\varepsilon^* = \text{Max } \varepsilon$, subject to constraints *a)–l)*, with *b),e),i)* written as *b'),e'),i')* and $U(y) \geq U(x) + \varepsilon$

The “most representative” value function

The need for a representative value function

- Recommendations taking into account the whole set of admissible value functions answer to robustness concerns, since they are in general “more robust” than a single ranking obtained by an arbitrarily chosen compatible value function.
- However, in practice, for some decision-making situations, a score is needed to assign to the different actions and despite the interest of the two rankings provided, some users would like to see the “most representative” value function among all the compatible ones.
- This value function should allow assigning a score to each action.
- We propose a way to identify the “most representative” value function in GRIP, without loosing the advantage of taking into account all compatible value functions.

The idea of the „most representative” value function (Figueira, Greco, Slowinski 2008; see also Kadzinski, Greco, Slowinski 2010, 2011)

- The idea is to select among compatible value functions that value function which better highlights the necessary ranking, maximizing the difference of evaluations between actions for which there is a preference in the necessary ranking.
- As secondary objective, one can consider minimizing the difference of evaluations between actions for which there is not a preference in the necessary ranking.

Procedure to determine the "most representative" value function

- 1) Determine the necessary and the possible preferences in the considered set of actions.
- 2) For all pairs of actions (a,b), such that a is necessarily preferred to b ($a \succsim^N b$ but not $b \succsim^N a$), add the following constraints to the linear programming constraints of GRIP: $U(a) \geq U(b)+\varepsilon$.
- 3) Maximize ε
- 4) Add the constraint $\varepsilon = \varepsilon^*$, with ε^*=Max ε of point 3), to the linear programming constraints of point 2)
- 5) For all pairs of actions (a,b), such that neither a is necessarily preferred to b nor b is necessarily preferred to a (not $a \succsim^N b$ and not $b \succsim^N a$), add the following constraints to the linear programming constraints of GRIP: $U(a)-U(b) \leq \delta$ and $U(b)-U(a) \leq \delta$.
- 6) Minimize δ

Alternative procedure to determine the "most representative" value function

- 1) Determine the necessary and the possible preferences in the considered set of actions.
- 2) For all pairs of actions (a,b), such that a is necessarily preferred to b ($a \succsim^N b$ but not $b \succsim^N a$), add the following constraints to the linear programming constraints of GRIP: $U(a) \geq U(b) + \varepsilon$.
- 3) For all pairs of actions (a,b), such that neither a is necessarily preferred to b nor b is necessarily preferred to a (not $a \succsim^N b$ and not $b \succsim^N a$), add the following constraints to the linear programming constraints of GRIP: $U(a) - U(b) \leq \delta$ and $U(b) - U(a) \leq \delta$.
- 4') Maximize the following objective function: $M\varepsilon - \delta$, where M is a "big value".

First illustrative example:
ROR is easy!

Illustrative example

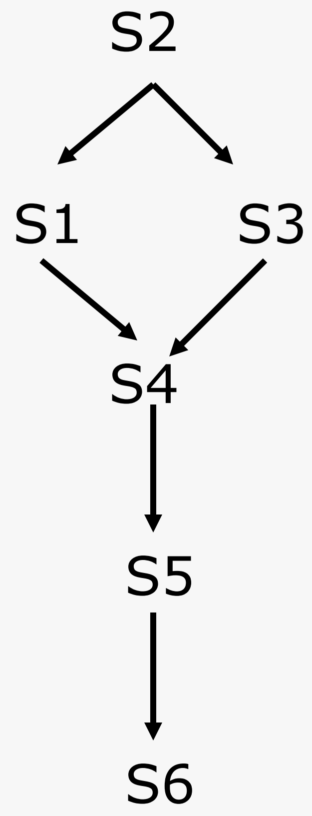
Students	Mathematics	Physics	Literature
S1	Medium	Medium	Good
S2	Good	Good	Medium
S3	Medium	Good	Medium
S4	Medium	Medium	Medium
S5	Good	Good	Bad
S6	Medium	Bad	Good

Information on preferences given by the DM

- **Preferences between students**
- S2 $\succ$ S1
- S4 $\succ$ S5
- S5 $\succ$ S6
- **Overall intensity of preferences**
- (S5,S6) $\succ^*$ (S2,S1)
- **Intensity of preference relative to single criteria**
- (Good,Medium) $\succ^*_{Mathematics}$ (Medium,Bad)

Necessary weak preference $\succeq^{*N}$ from GRIP

(Hasse Diagram)



“The most representative value function”

- $\varepsilon=0.1$, $\delta=0$

	Mathematics	Physics	Literature
Bad	0	0	0
Medium	0	0.4	0.3
Good	0.1	0.5	0.4

Evaluation of students by means of “the most representative value function”

Students	Mathematics	Physics	Literature	Value
S1	Medium (0)	Medium (0.4)	Good (0.4)	0.8
S2	Good (0.1)	Good (0.5)	Medium (0.3)	0.9
S3	Medium (0)	Good (0.5)	Medium (0.3)	0.8
S4	Medium (0)	Medium (0.4)	Medium (0.3)	0.7
S5	Good (0.1)	Good (0.5)	Bad (0)	0.6
S6	Medium (0)	Bad (0)	Good (0.4)	0.4

Value function given by UTAMP1

- $\varepsilon=0.167$

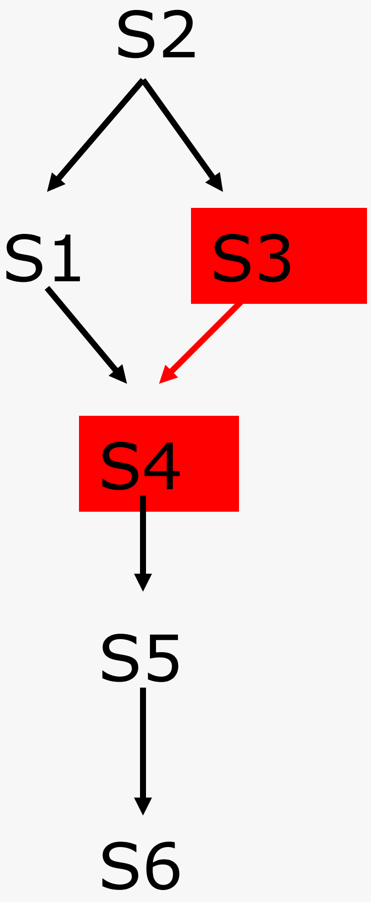
	Mathematics	Physics	Literature
Bad	0	0	0
Medium	0	0.5	0.33
Good	0.17	0.5	0.33

Evaluation of students by means of UTAMP1

Students	Mathematics	Physics	Literature	Value
S1	Medium (0)	Medium (0.5)	Good (0.33)	0.83
S2	Good (0.17)	Good (0.5)	Medium (0.33)	1
S3	Medium (0)	Good (0.5)	Medium (0.33)	0.83
S4	Medium (0)	Medium (0.5)	Medium (0.33)	0.83
S5	Good (0.17)	Good (0.5)	Bad (0)	0.67
S6	Medium (0)	Bad (0)	Good (0.33)	0.33

Comparison of GRIP, “the most representative value function” and UTAMP1

GRIP



Students	**“The most representative value function”**	**UTAMP1**
S1	0.8	0.83
S2	0.9	1
S3	0.8	0.83
S4	0.7	0.83
S5	0.6	0.67
S6	0.4	0.33

UTAMP1 does not represent the necessary weak preference of S3 over S4

A didactic example:
ROR is interactive!

UTA^{GMS} : an illustrative example

Ranking problem: 20 actions evaluated on 5 criteria

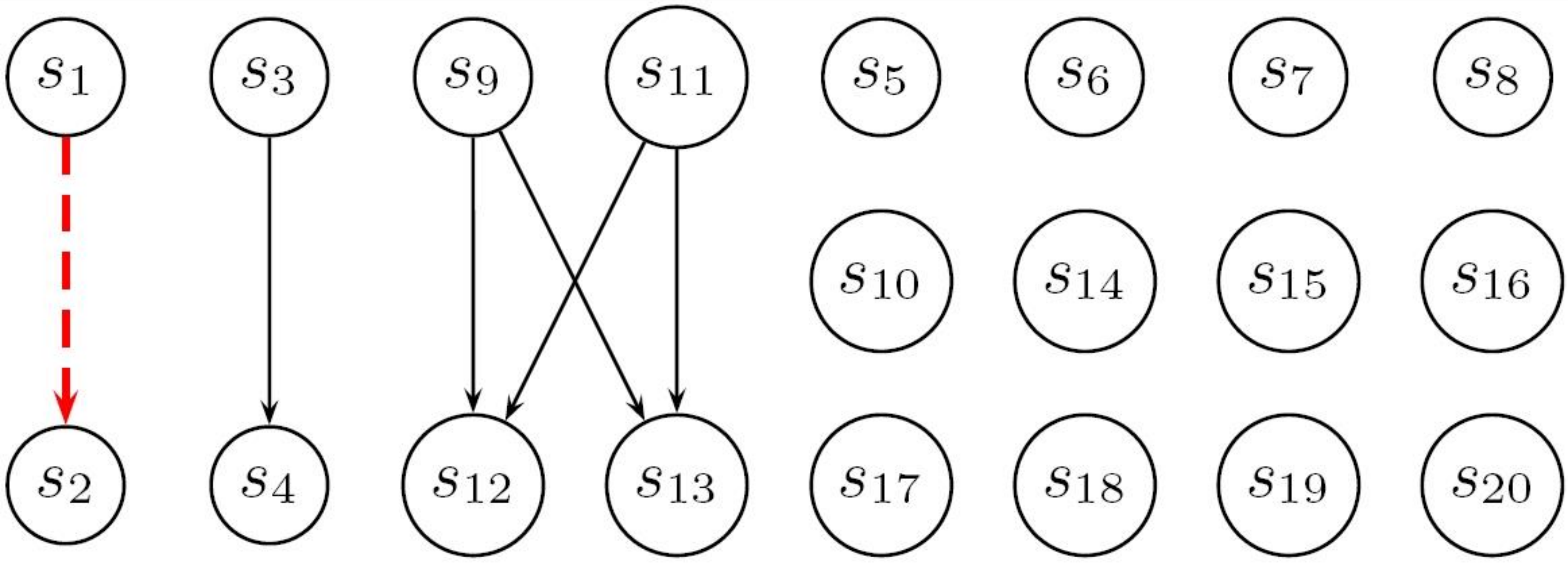
	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}	s_{17}	s_{18}	s_{19}	s_{20}
g_1	2	1	3	0	1	3	0	4	3	3	3	3	3	1	1	4	1	3	3	3
g_2	0	3	1	2	1	3	0	4	0	4	1	2	3	0	3	1	2	1	2	2
g_3	0	0	1	1	4	2	3	1	3	3	3	3	3	1	1	4	1	3	3	3
g_4	5	5	4	4	4	3	3	2	2	3	1	1	1	3	1	1	2	4	0	1
g_5	3	2	3	2	2	3	3	0	3	0	4	2	1	3	4	2	3	2	3	1

Evaluation matrix

Empty dominance relation !

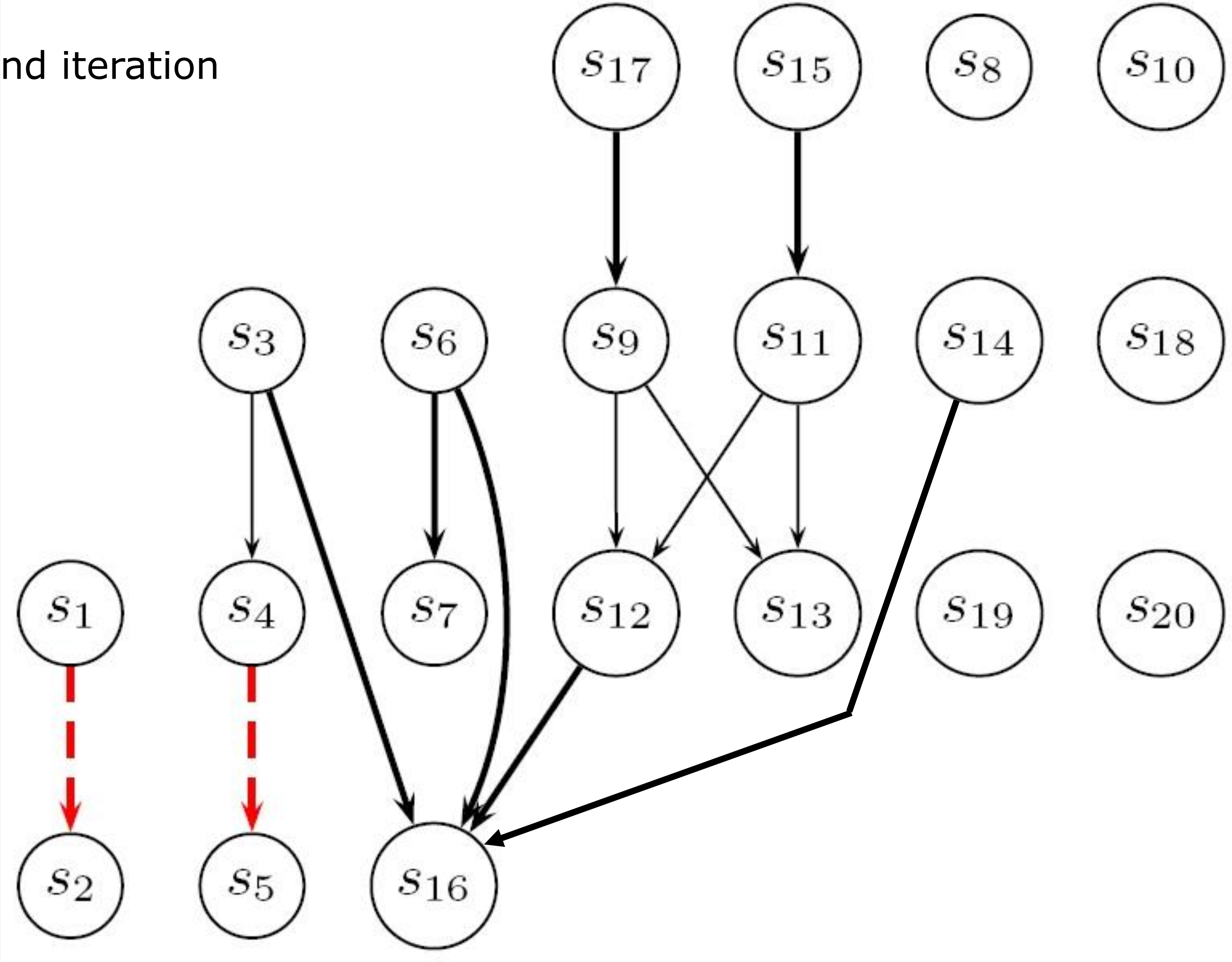
UTA^{GMS} : an illustrative example

First iteration



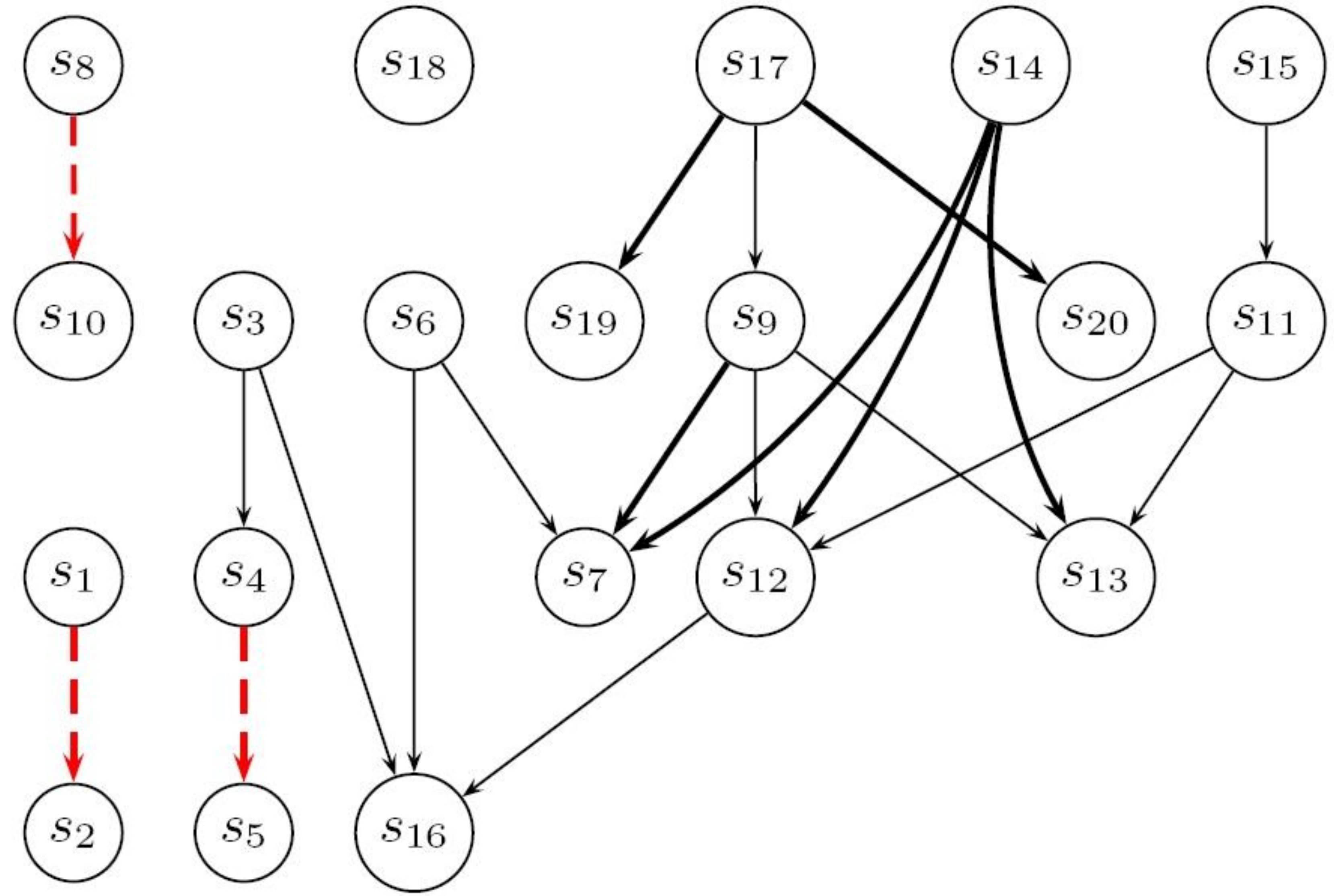
UTA^{GMS} : an illustrative example

Second iteration



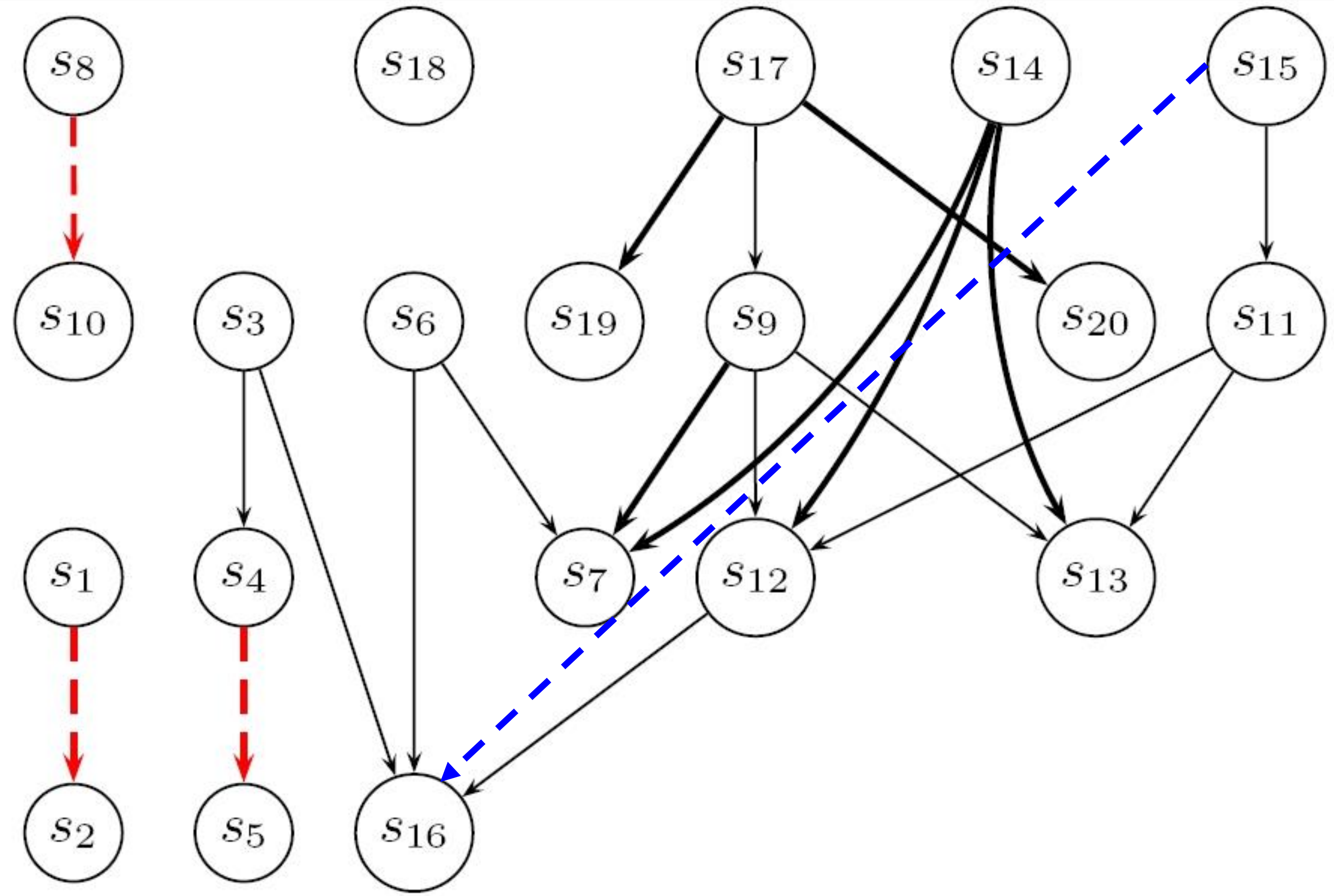
UTA^{GMS} : an illustrative example

Third iteration



GRIP : an illustrative example

Fourth iteration, after addition of intensity condition: $(s_8,s_{10}) \sim (s_1,s_2)$



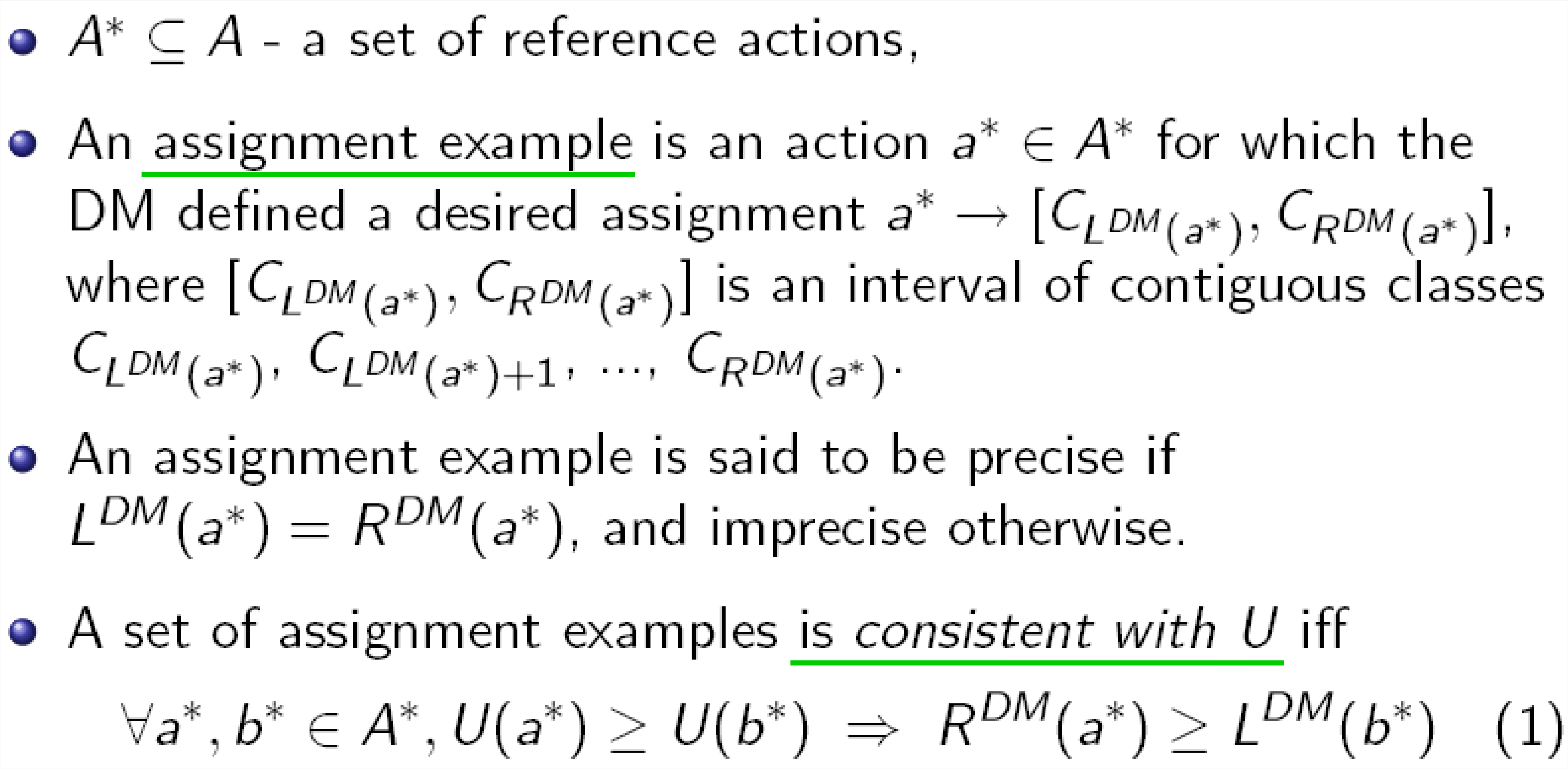
UTA-DIS^{GMS} for multicriteria sorting problems

- Actions from set *A* are to be assigned to pre-defined and preference-ordered classes
- Classes have a semantic definition
- Assignment to classes is grounded on absolute evaluation of actions on multiple criteria
- No relative comparison is required because sorting is „context-free”, which is not the case of choice and ranking

UTA-DIS^{GMS} – given data

- $A = \{a_1, a_2, \ldots, a_i, \ldots, a_m\}$ actions to be assigned to classes,
- $g_1, g_2, \ldots, g_n$, n criteria, $g_j : A \rightarrow \mathbb{R}$ for $j \in G = \{1, 2, \ldots, n\}$,
- $C_1, C_2, \ldots, C_p$, p ordered classes, $C_{h+1} \gg C_h$, $H = \{1, \ldots, p\}$,
- $X_j = \{x_j \in \mathbb{R} : g_j(a_i) = x_j, a_i \in A\}$ - the set of all different evaluations on g_j, $j \in G$,
- $x_j^0, x_j^1, \ldots, x_j^{m_j}$ - the ordered values of X_j, $(x_j^k < x_j^{k+1})$.

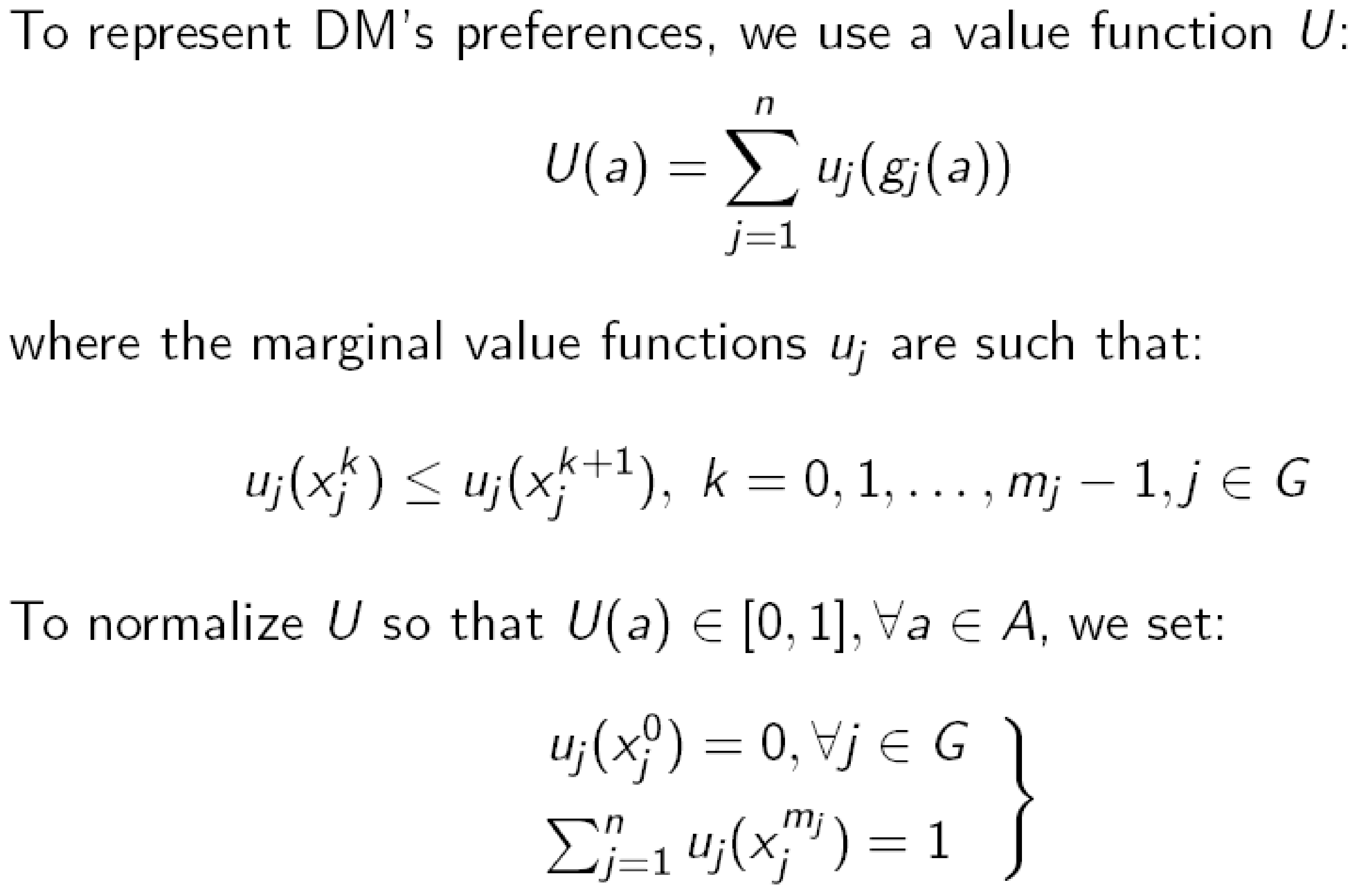
UTA-DIS^{GMS} – preference information



- $A^* \subseteq A$ - a set of reference actions,
- An assignment example is an action $a^* \in A^*$ for which the DM defined a desired assignment $a^* \to [C_{L^{DM}(a^*)}, C_{R^{DM}(a^*)}]$, where $[C_{L^{DM}(a^*)}, C_{R^{DM}(a^*)}]$ is an interval of contiguous classes $C_{L^{DM}(a^*)}, C_{L^{DM}(a^*)+1}, \ldots, C_{R^{DM}(a^*)}$.
- An assignment example is said to be precise if $L^{DM}(a^*) = R^{DM}(a^*)$, and imprecise otherwise.
- A set of assignment examples is *consistent with U* iff

$$\forall a^*, b^* \in A^*, U(a^*) \geq U(b^*) \;\Rightarrow\; R^{DM}(a^*) \geq L^{DM}(b^*) \quad (1)$$

UTA-DIS^{GMS} – preference model



To represent DM's preferences, we use a value function U:

$$U(a) = \sum_{j=1}^{n} u_j(g_j(a))$$

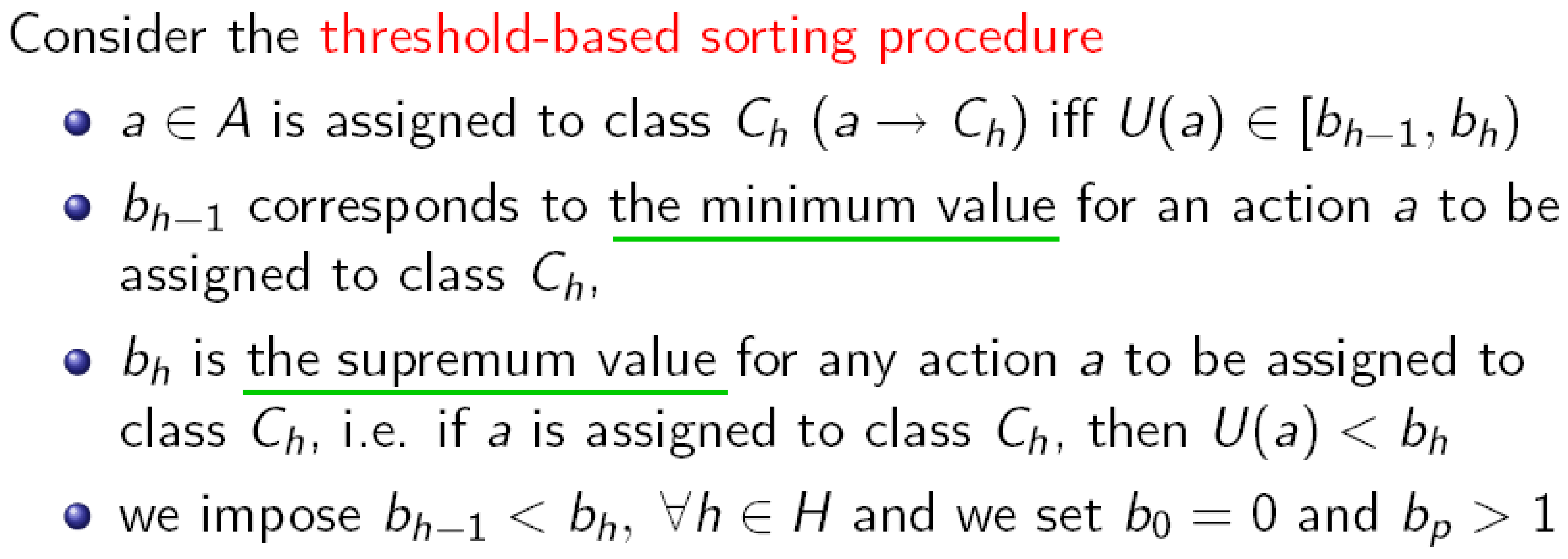
where the marginal value functions u_j are such that:

$$u_j(x_j^k) \leq u_j(x_j^{k+1}),\ k = 0, 1, \ldots, m_j - 1, j \in G$$

To normalize U so that $U(a) \in [0, 1], \forall a \in A$, we set:

$$\left.\begin{array}{l} u_j(x_j^0) = 0, \forall j \in G \\ \sum_{j=1}^{n} u_j(x_j^{m_j}) = 1 \end{array}\right\}$$

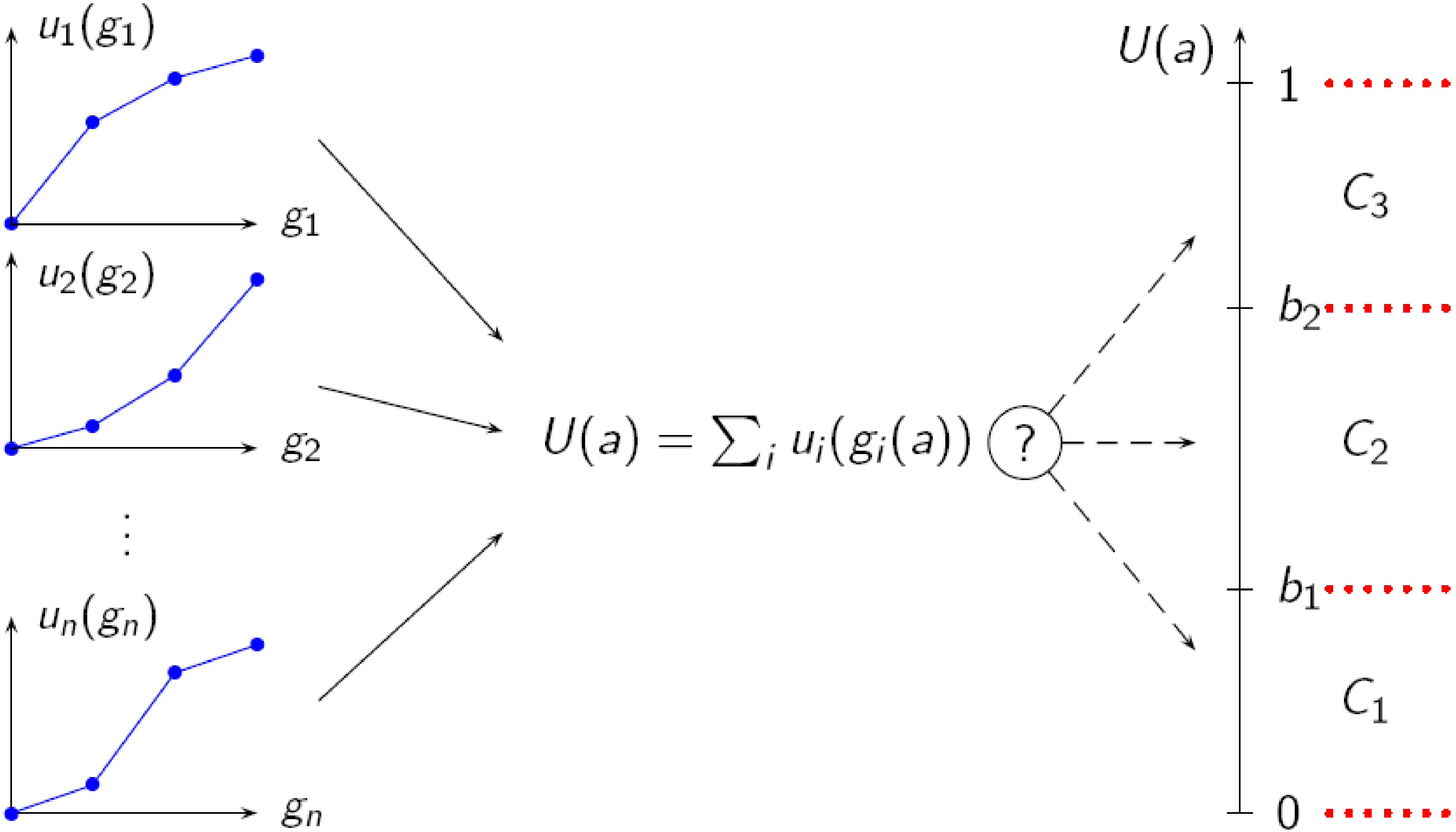
UTA-DIS^{GMS}



Consider the threshold-based sorting procedure

- $a \in A$ is assigned to class C_h $(a \rightarrow C_h)$ iff $U(a) \in [b_{h-1}, b_h)$
- b_{h-1} corresponds to the minimum value for an action a to be assigned to class C_h,
- b_h is the supremum value for any action a to be assigned to class C_h, i.e. if a is assigned to class C_h, then $U(a) < b_h$
- we impose $b_{h-1} < b_h$, $\forall h \in H$ and we set $b_0 = 0$ and $b_p > 1$

Threshold-based sorting



UTA-DIS^{GMS}

We consider the example-based sorting procedure

- The *example-based sorting procedure* is driven by a value function U and its associated assignment examples $A^* \subset A$. It assigns an action a to an interval of classes $[C_{L^U(a)}, C_{R^U(a)}]$:

$$
\begin{aligned}
L^U(a) &= Max\left\{L^{DM}(a^*) : U(a^*) \leq U(a),\ a^* \in A^*\right\} \quad (2) \\
R^U(a) &= Min\left\{R^{DM}(a^*) : U(a^*) \geq U(a),\ a^* \in A^*\right\} \quad (3)
\end{aligned}
$$

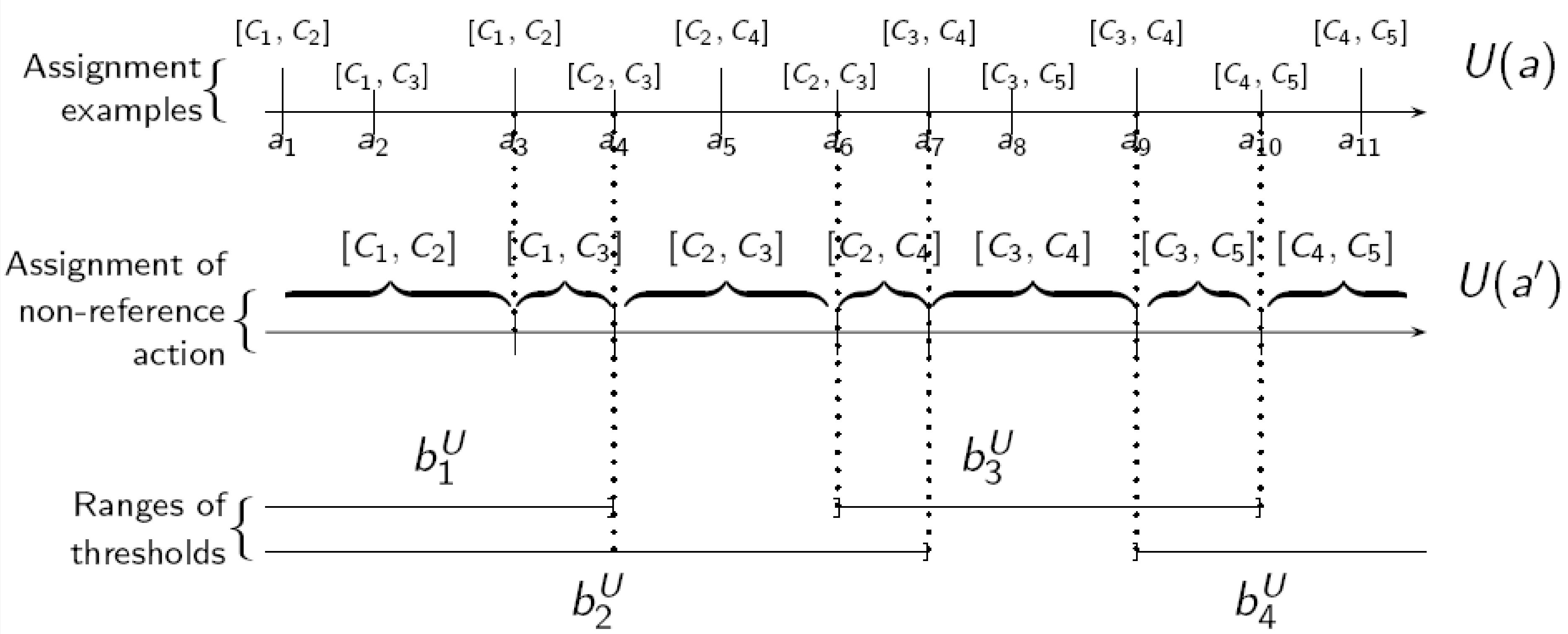
- Procedure considered in [Köksalan, Ulu 2003] with linear value functions.

UTA-DIS^{GMS}

Proposition

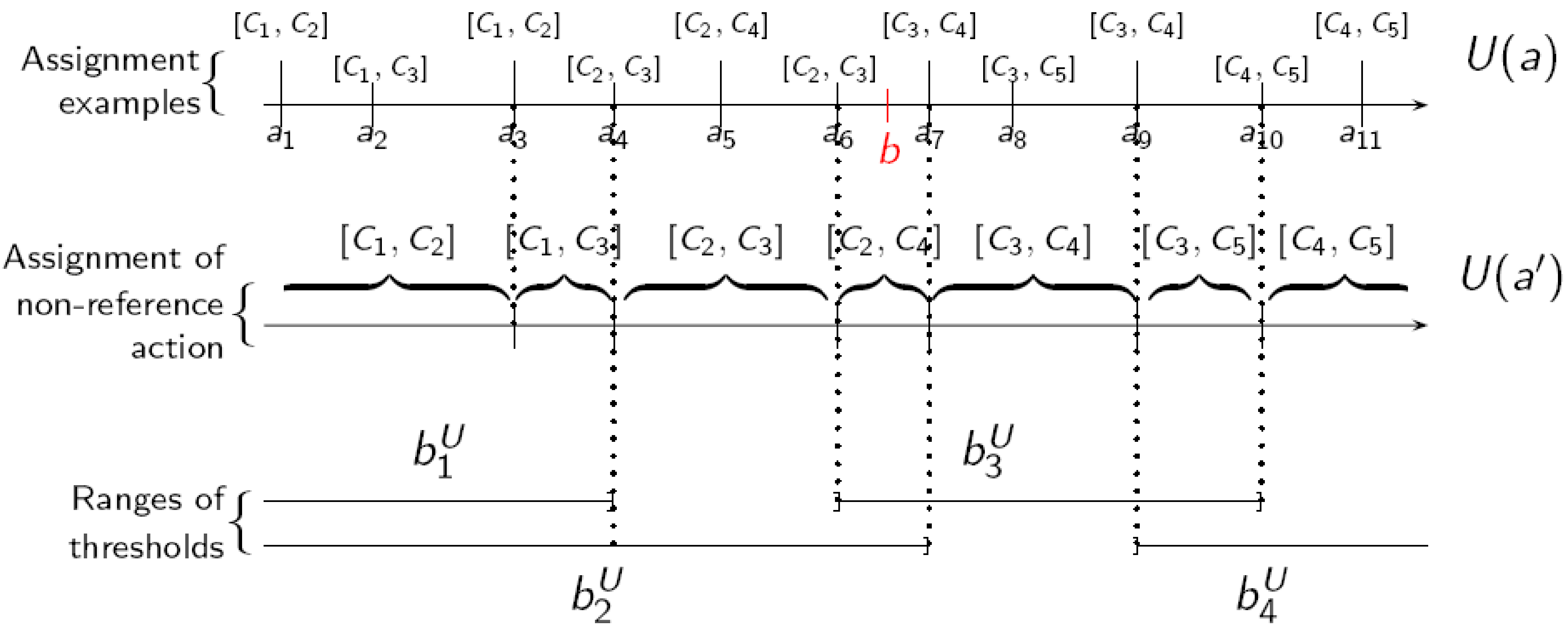
- Consider the case where $L^{DM}(a^*) \leq R^{DM}(a^*)$, $\forall a^* \in A^*$,
- Assuming the use of a $\boxed{\text{single value function } U}$ in the example-based sorting procedure,
- if we choose the b_h^U, $h = 1, ..., p-1$ in the interval $\left]Max_{a^*:R^{DM}(a^*)\leq h}\left\{U(a^*)\right\}, Min_{a^*:L^{DM}(a^*)>h}\left\{U(a^*)\right\}\right[$, with $b_h^U < b_{h+1}^U$,
- we obtain a threshold-based sorting procedure that restores the assignment examples and assigns each non-reference action $a \in A \setminus A^*$ to a single class in the interval $[C_{L^U(a)}, C_{R^U(a)}]$ stemming from the example-based sorting procedure.

UTA-DIS^{GMS}



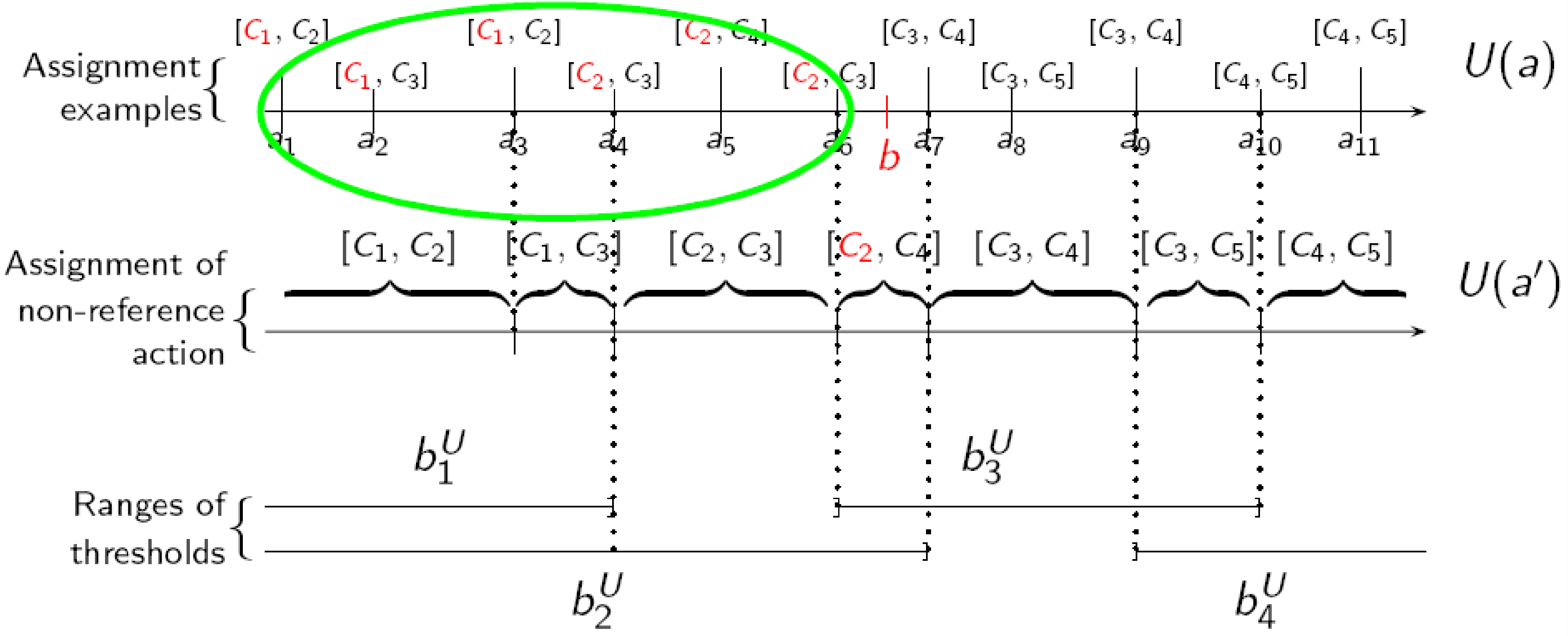
Assignment examples
$[C_1, C_2]$
$[C_1, C_3]$
$[C_1, C_2]$
$[C_2, C_3]$
$[C_2, C_4]$
$[C_2, C_3]$
$[C_3, C_4]$
$[C_3, C_5]$
$[C_3, C_4]$
$[C_4, C_5]$
$[C_4, C_5]$
$U(a)$
a_1
a_2
a_3
a_4
a_5
a_6
a_7
a_8
a_9
a_{10}
a_{11}
Assignment of non-reference action
$[C_1, C_2]$
$[C_1, C_3]$
$[C_2, C_3]$
$[C_2, C_4]$
$[C_3, C_4]$
$[C_3, C_5]$
$[C_4, C_5]$
$U(a')$
b_1^U
b_3^U
Ranges of thresholds
b_2^U
b_4^U

UTA-DIS^{GMS}



Consider $b \in A$, with $U(b) \in]U(a_6), U(a_7)[$

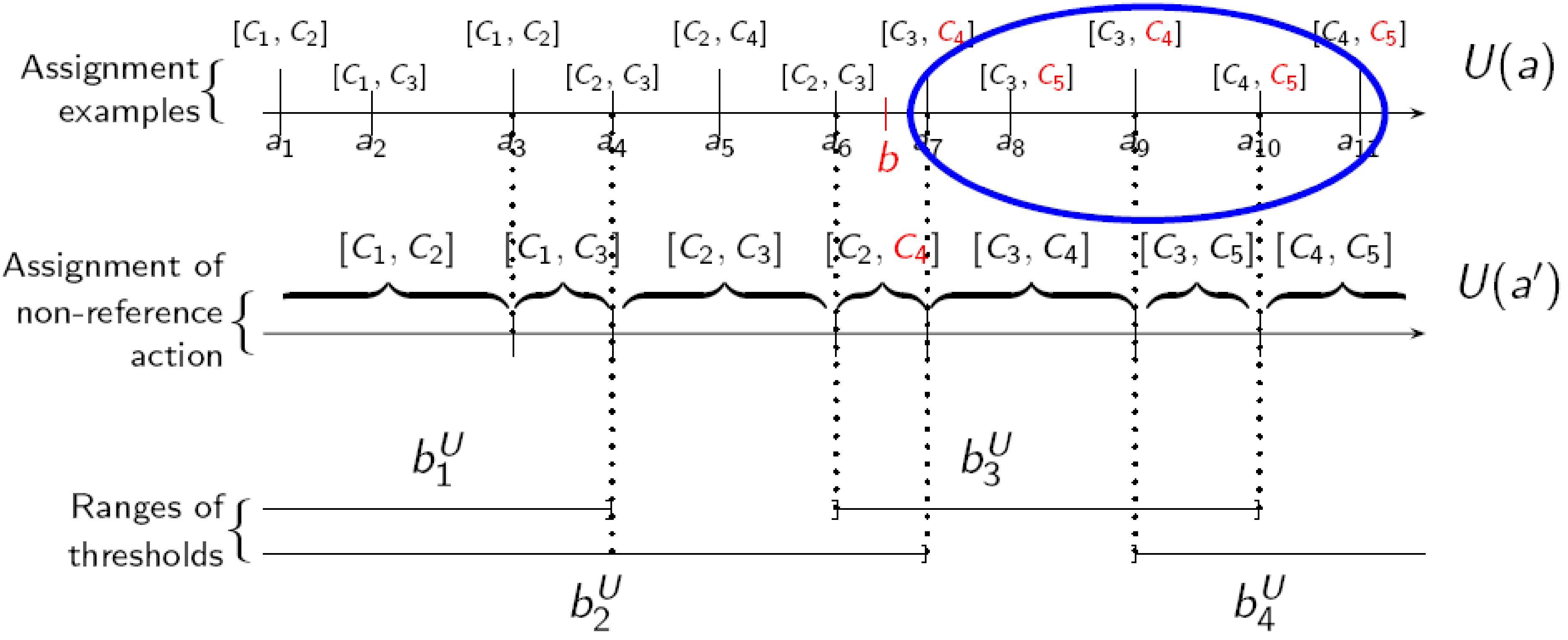
UTA-DIS^{GMS}



Consider $b \in A$, with $U(b) \in]U(a_6), U(a_7)[$

$L^U(b) = C_2$

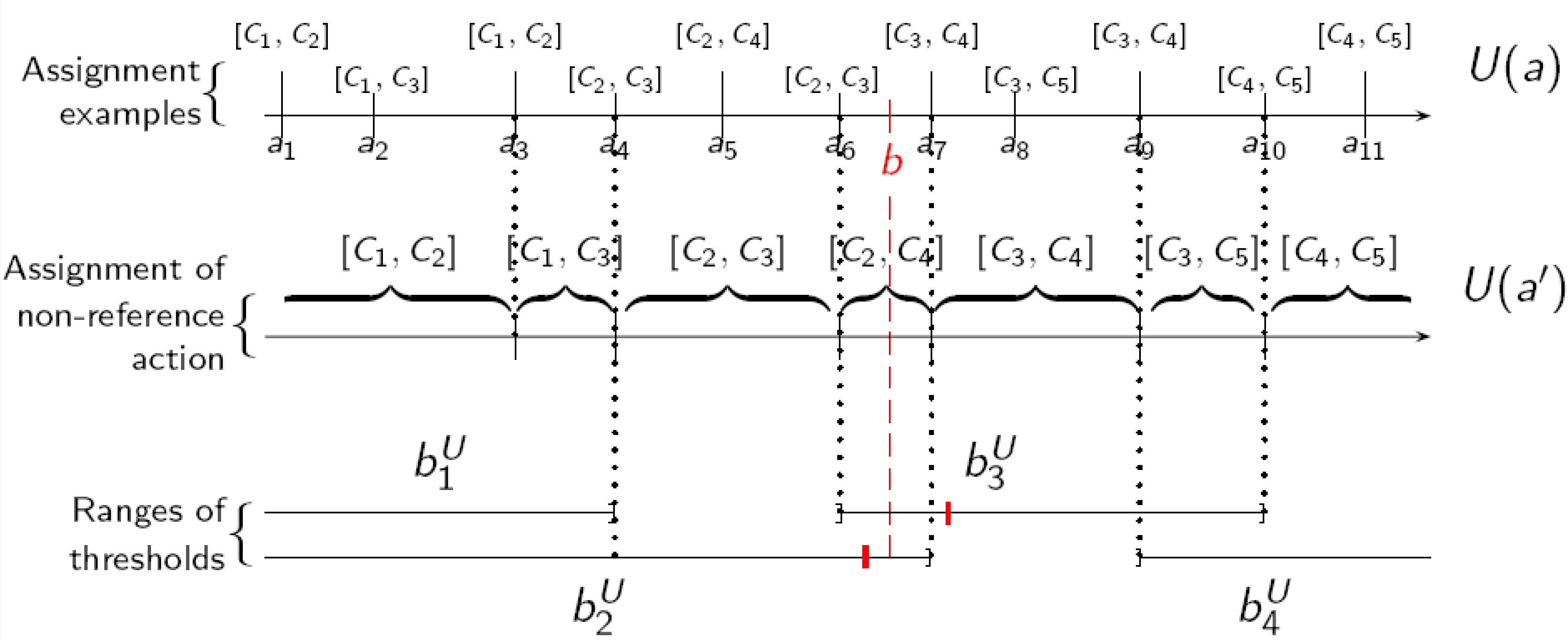
UTA-DIS^{GMS}



Consider $b \in A$, with $U(b) \in]U(a_6), U(a_7)[$

$L^U(b) = C_2$, $R^U(b) = C_4$

UTA-DIS^{GMS}

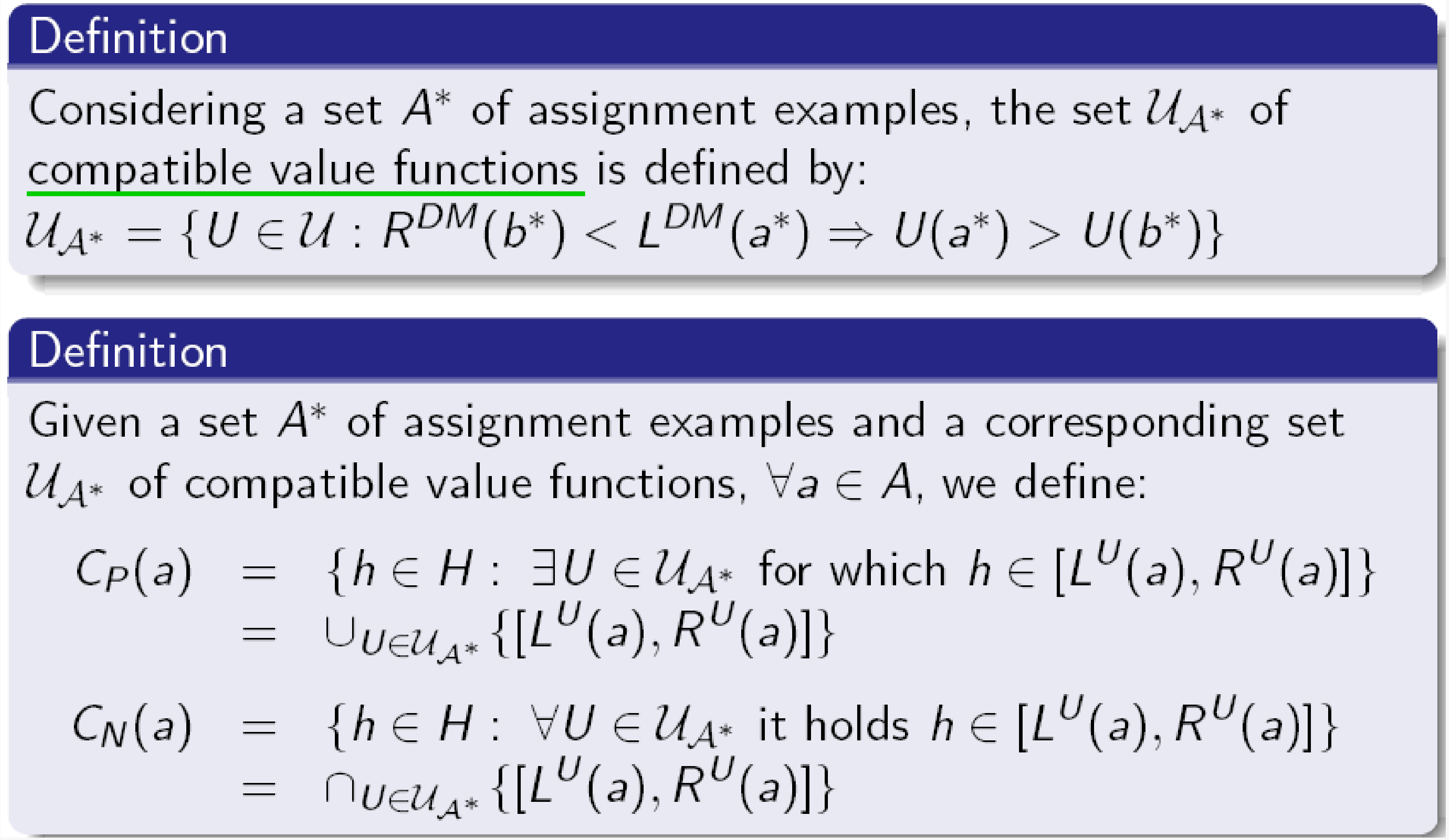


Consider $b \in A$, with $U(b) \in]U(a_6), U(a_7)[$

$L^U(b) = C_2$, $R^U(b) = C_4$

UTA-DIS^{GMS} – posible and necessary assignments

Application of a set of compatible value functions $\mathcal{U}$



Definition

Considering a set A^* of assignment examples, the set $\mathcal{U}_{A^*}$ of compatible value functions is defined by:

$$\mathcal{U}_{A^*} = \{U \in \mathcal{U} : R^{DM}(b^*) < L^{DM}(a^*) \Rightarrow U(a^*) > U(b^*)\}$$

Definition

Given a set A^* of assignment examples and a corresponding set $\mathcal{U}_{A^*}$ of compatible value functions, $\forall a \in A$, we define:

$$\begin{aligned} C_P(a) &= \{h \in H : \exists U \in \mathcal{U}_{A^*} \text{ for which } h \in [L^U(a), R^U(a)]\} \\ &= \cup_{U \in \mathcal{U}_{A^*}} \{[L^U(a), R^U(a)]\} \\ C_N(a) &= \{h \in H : \forall U \in \mathcal{U}_{A^*} \text{ it holds } h \in [L^U(a), R^U(a)]\} \\ &= \cap_{U \in \mathcal{U}_{A^*}} \{[L^U(a), R^U(a)]\} \end{aligned}$$

UTA-DIS^{GMS} – posible and necessary assignments

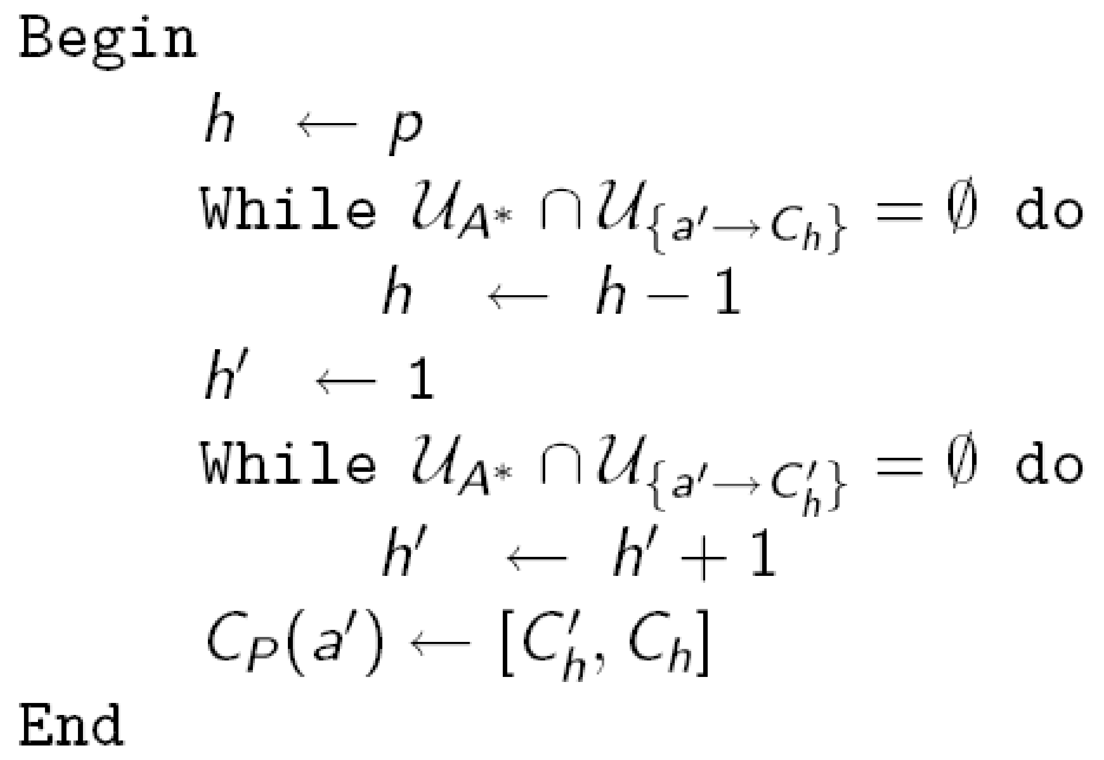
- Consider all increasing value functions (not piece-wise only),
- Consider $a^* \to [C_{L^{DM}(a^*)}, C_{R^{DM}(a^*)}]$, $a^{*\prime} \to [C_{L^{DM}(a^{*\prime})}, C_{R^{DM}(a^{*\prime})}]$, such that the two intervals of classes have an empty intersection. It holds:

$$U(a^*) < U(a^{*\prime}) \text{ iff } R^{DM}(a^*) < L^{DM}(a^{*\prime}) \tag{4}$$

- Assignment of $a^*, a^{*\prime} \in A^*$, to intervals of classes having an non-empty intersection, does not induce constraints,
- if $|A^*| > 2$, one should consider constraints (4) for all pairs assigned to series of classes with an empty intersection,
- these constraints (with monotonicity and normality) define (if consistent) a non-empty set of value functions $\mathcal{U}_{A^*}$ compatible with the preference information,

UTA-DIS^{GMS} – posible assignments

- Computing possible assignments



```
Begin
    $h \leftarrow p$
    While $\mathcal{U}_{A^*} \cap \mathcal{U}_{\{a' \rightarrow C_h\}} = \emptyset$ do
        $h \leftarrow h - 1$
    $h' \leftarrow 1$
    While $\mathcal{U}_{A^*} \cap \mathcal{U}_{\{a' \rightarrow C'_h\}} = \emptyset$ do
        $h' \leftarrow h' + 1$
    $C_P(a') \leftarrow [C'_h, C_h]$
End
```

UTA-DIS^{GMS} – posible assignments

$$u_j(x_j^k) \leq u_j(x_j^{k+1}),\ k = 0, 1, \ldots, m_j - 1, j \in G \tag{5}$$

$$u_j(x_j^0) = 0, \forall j \in G \tag{6}$$

$$\sum_{j=1}^n u_j(x_j^{m_j}) = 1 \tag{7}$$

$$\sum_{j=1}^n u_j(a) \geq b_{C_{min}(a)-1},\ \forall a \in A^* \tag{8}$$

$$\sum_{j=1}^n u_j(a) \leq b_{C_{max}(a)} - \varepsilon,\ \forall a \in A^* \tag{9}$$

$$\sum_{j=1}^n u_j(a') \geq b_{h-1} \tag{10}$$

$$\sum_{j=1}^n u_j(a') \leq b_h - \varepsilon \tag{11}$$

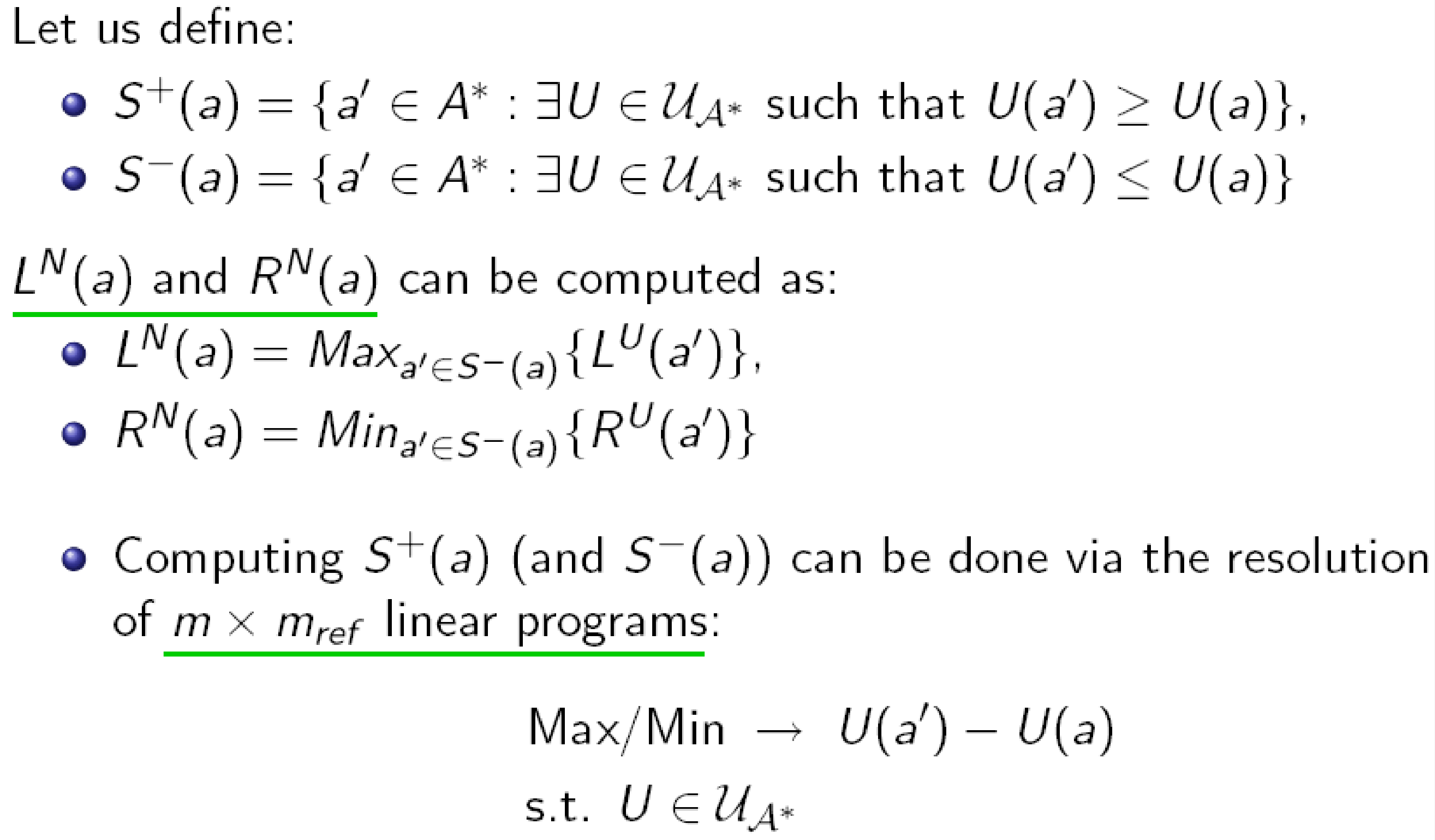
- Contraints (5)-(9) define $\mathcal{U}_{A^*}$,
- Contraints (10),(11) define $\mathcal{U}_{a' \to C_h}$,
- The set of constraints (5)-(11) is feasible iff the following linear program has an optimal value $\varepsilon^* > 0$:

$$\text{Max} \ \rightarrow \ \varepsilon$$

$$\text{s.t. constraints } (5) - (11)$$

UTA-DIS^{GMS} – necessary assignments

- Computing necessary assignments



Let us define:

- $S^{+}(a) = \{a' \in A^{*} : \exists U \in \mathcal{U}_{A^{*}} \text{ such that } U(a') \geq U(a)\}$,
- $S^{-}(a) = \{a' \in A^{*} : \exists U \in \mathcal{U}_{A^{*}} \text{ such that } U(a') \leq U(a)\}$

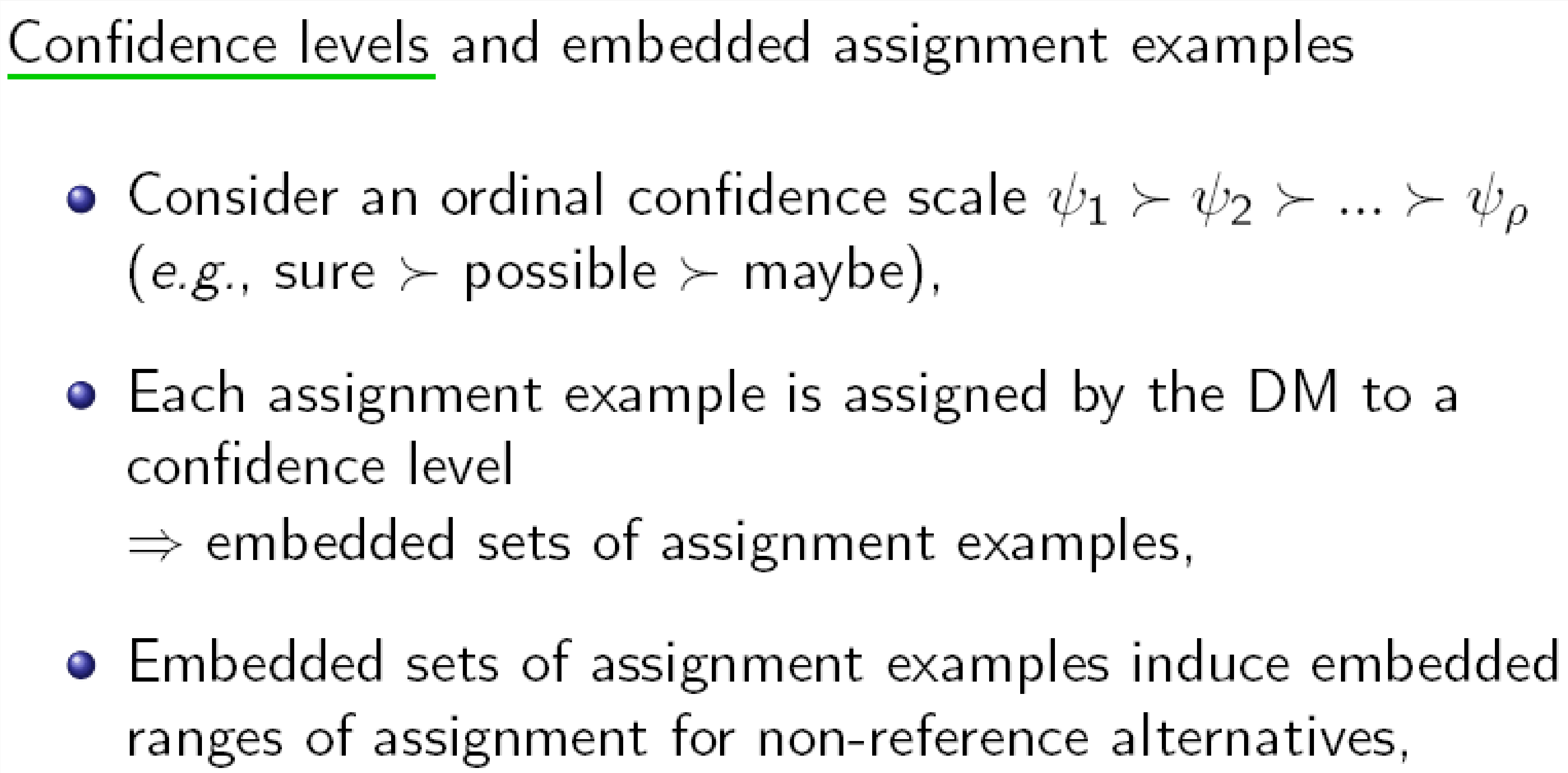
$L^{N}(a)$ and $R^{N}(a)$ can be computed as:

- $L^{N}(a) = Max_{a' \in S^{-}(a)}\{L^{U}(a')\}$,
- $R^{N}(a) = Min_{a' \in S^{-}(a)}\{R^{U}(a')\}$

- Computing $S^{+}(a)$ (and $S^{-}(a)$) can be done via the resolution of $m \times m_{ref}$ linear programs:

$$\begin{aligned} &\text{Max/Min} \ \rightarrow \ U(a') - U(a) \\ &\text{s.t.} \ U \in \mathcal{U}_{A^{*}} \end{aligned}$$

UTA-DIS^{GMS} – incremental definition of set A^* with decrasing confidence



Confidence levels and embedded assignment examples

- Consider an ordinal confidence scale $\psi_1 \succ \psi_2 \succ \ldots \succ \psi_\rho$ (*e.g.*, sure $\succ$ possible $\succ$ maybe),
- Each assignment example is assigned by the DM to a confidence level
  $\Rightarrow$ embedded sets of assignment examples,
- Embedded sets of assignment examples induce embedded ranges of assignment for non-reference alternatives,

Group multicriteria ranking problem statement

- Several DMs cooperate in a decision problem to construct a collective ranking
- DMs share the same „description” of the decision problem (set of actions, evaluation criteria, evaluation matrix)
- Each DM provides his/her own preference information
- The collective ranking should account for the preference expressed by each DM

Ordinal regression for group ranking: UTA-GROUP^{GMS}

- Set of DMs: $D=\{d_1,\ldots,d_p\}$

- Preference information provided by DM d_h, $h=1,\ldots,p$:
  $B^R(d_h)$ a partial preorder on a set of reference actions

Ordinal regression for group ranking: UTA-GROUP^{GMS}

- We consider the set of value functions for each $d_h \in D^* \subset D$ stemming from UTA^{GMS}
- For each $D^* \subset D$, 4 situations are interresting for $(x,y) \in A$:
  - $x \succsim^{N,N}(D^*)\ y$: $x \succsim^{N} y$ for all $d_h \in D^*$,
  - $x \succsim^{N,P}(D^*)\ y$: $x \succsim^{N} y$ for at least one $d_h \in D^*$,
  - $x \succsim^{P,N}(D^*)\ y$: $x \succsim^{P} y$ for all $d_h \in D^*$,
  - $x \succsim^{P,P}(D^*)\ y$: $x \succsim^{P} y$ for at least one $d_h \in D^*$

Ordinal regression for group ranking: UTA-GROUP$^{\text{GMS}}$

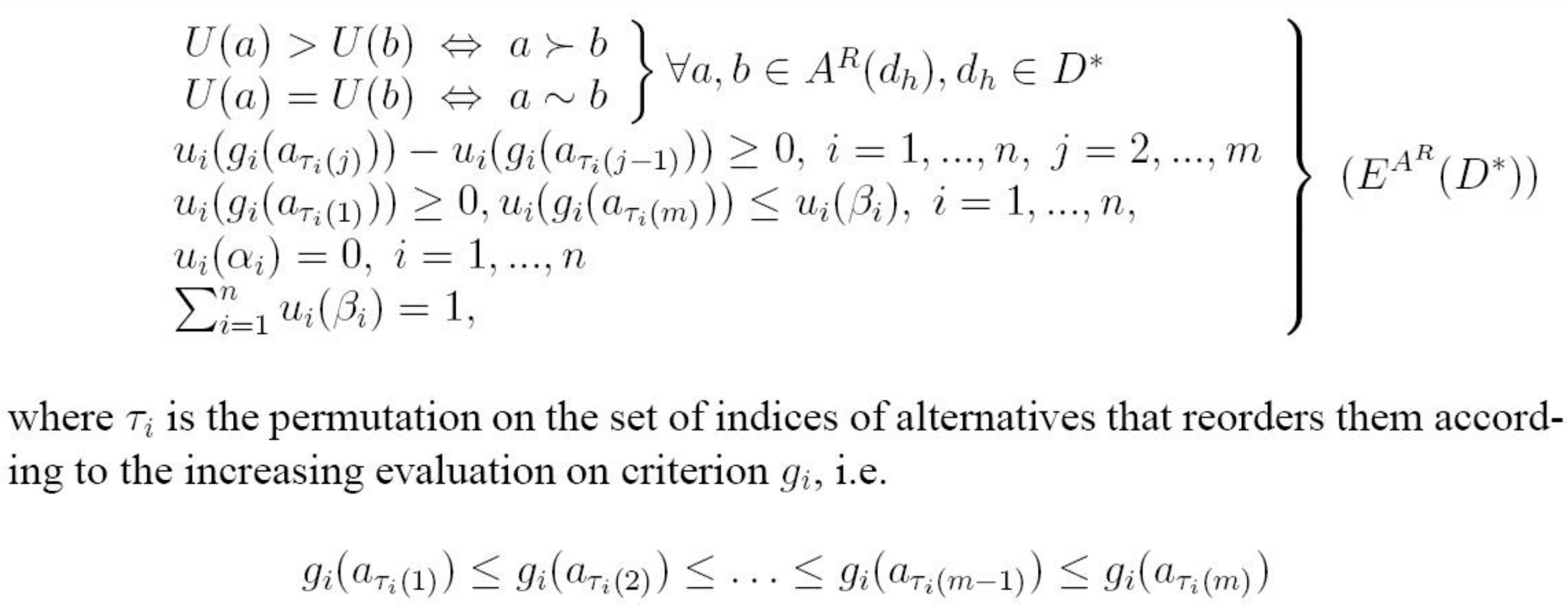
- Properties
  - $\succsim^{N,N}(D^*)$ is a partial preorder
  - $\succsim^{N,P}(D^*)$ is not necessarily transitive
  - $\succsim^{P,P}(D^*)$ is strongly complete
  - $x \succsim^{N,N}(D^*)\ y \Rightarrow x \succsim^{N,P}(D^*)\ y$
  - $x \succsim^{N,P}(D^*)\ y \Rightarrow x \succsim^{P,P}(D^*)\ y$

  When $D^* \subset D^{**}$, it holds

  - $x \succsim^{N,N}(D^{**})\ y \ \Rightarrow \ x \succsim^{N,N}(D^*)\ y$
  - $x \succsim^{N,P}(D^{**})\ y \ \Rightarrow \ x \succsim^{N,P}(D^*)\ y$
  - $x \succsim^{P,N}(D^{**})\ y \ \Rightarrow \ x \succsim^{P,N}(D^*)\ y$
  - $x \succsim^{P,P}(D^{**})\ y \ \Rightarrow \ x \succsim^{P,P}(D^*)\ y$

Ordinal regression for group ranking: UTA-GROUP^{GMS}

- Given a set of DMs $D^* \subseteq D$, a value function U is compatible if it satisfies the following set of constraints:



$$\left.\begin{array}{l}
\left.\begin{array}{lcl} U(a) > U(b) & \Leftrightarrow & a \succ b \\ U(a) = U(b) & \Leftrightarrow & a \sim b \end{array}\right\} \forall a,b \in A^R(d_h), d_h \in D^* \\
u_i(g_i(a_{\tau_i(j)})) - u_i(g_i(a_{\tau_i(j-1)})) \geq 0, \ i=1,...,n, \ j=2,...,m \\
u_i(g_i(a_{\tau_i(1)})) \geq 0, u_i(g_i(a_{\tau_i(m)})) \leq u_i(\beta_i), \ i=1,...,n, \\
u_i(\alpha_i) = 0, \ i=1,...,n \\
\sum_{i=1}^{n} u_i(\beta_i) = 1,
\end{array}\right\} (E^{A^R}(D^*))$$

where τ_i is the permutation on the set of indices of alternatives that reorders them according to the increasing evaluation on criterion g_i, i.e.

$$g_i(a_{\tau_i(1)}) \leq g_i(a_{\tau_i(2)}) \leq \ldots \leq g_i(a_{\tau_i(m-1)}) \leq g_i(a_{\tau_i(m)})$$

- Suppose that set $\mathcal{U}_{D^*}$ of compatible value functions is not empty (DMs statements are not contradictory)...

Ordinal regression for group ranking: UTA-GROUP^{GMS}

- One obtains two rankings such that for any pair of actions $(x,y) \in A$:
    - $x \succsim^{N}(D^*)\ y$: x is ranked at least as good as y iff $U^{D^*}(x) \geq U^{D^*}(y)$ for all value functions compatible with the preference information (necessary weak preference relation $\succsim^{N}$ being a partial preorder)
    - $x \succsim^{P}(D^*)\ y$: x is ranked at least as good as y iff $U^{D^*}(x) \geq U^{D^*}(y)$ for at least one value function compatible with the preference information (possible weak preference relation $\succsim^{P}$ being a strongly complete and negatively transitive binary relation)

- However, the set $\mathcal{U}_{D^*}$ of compatible value function can be empty...

Ordinal regression for group ranking: UTA-GROUP^{GMS}

Suppose $\mathcal{U}_{D*}=\varnothing$

- $\mathcal{U}_{D*}$ corresponds to the intersection of sets of compatible value functions for all $d_h \in D^*$ (each one being non-empty)
- This means that pairwise comparisons of two (or more) DMs are contradictory
- Identifying which are these contradictory comparisons amounts at solving inconsistency
- This leads to know which comparisons to remove in order to obtain a consistent collective model
- Performing these computations $\forall D^* \subset D$ allows to indentify coalitions of convergent DMs, for which a necessary and possible consensus rankings exist

Ordinal regression for group ranking: UTA-GROUP^{GMS}

- Reasoning in terms of pairwise comparisons decomposes elicitation of preference information into small natural pieces
- UTA-GROUP^{GMS} avoids discussions of DMs on technical parameters (tradeoffs, weights, …)
- Taking into account all compatible value functions permits to reason in terms of necessary and possible rankings and coalitions

UTA-GROUP^{GMS} : an illustrative example

Ranking problem: 3 DMs (d_1, d_2 and d_3), 20 actions evaluated on 5 criteria

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}	s_{17}	s_{18}	s_{19}	s_{20}
g_1	2	1	3	0	1	3	0	4	3	3	3	3	3	1	1	4	1	3	3	3
g_2	0	3	1	2	1	3	0	4	0	4	1	2	3	0	3	1	2	1	2	2
g_3	0	0	1	1	4	2	3	1	3	3	3	3	3	1	1	4	1	3	3	3
g_4	5	5	4	4	4	3	3	2	2	3	1	1	1	3	1	1	2	4	0	1
g_5	3	2	3	2	2	3	3	0	3	0	4	2	1	3	4	2	3	2	3	1

Evaluation matrix

Empty dominance relation !

UTA-GROUP^{GMS} : an illustrative example

- Statements of DMs:
  - d_1: $s_1 \succ s_2$, $s_6 \succ s_7$, $s_{17} \succ s_{20}$
  - d_2: $s_9 \succ s_{13}$, $s_4 \succ s_5$, $s_{14} \succ s_7$
  - d_3: $s_4 \succ s_3$, $s_{15} \succ s_{11}$, $s_8 \succ s_{10}$

- $\mathcal{U}_{\{d1,d3\}} = \mathcal{U}_{\{d1,d2,d3\}} = \emptyset$, i.e., statements of d_1 and d_3 are contradictory:
  - d_1: $s_1 \succ s_2 \Rightarrow s_3 \succ s_4$
  - d_3: $s_4 \succ s_3 \Rightarrow s_2 \succ s_1$

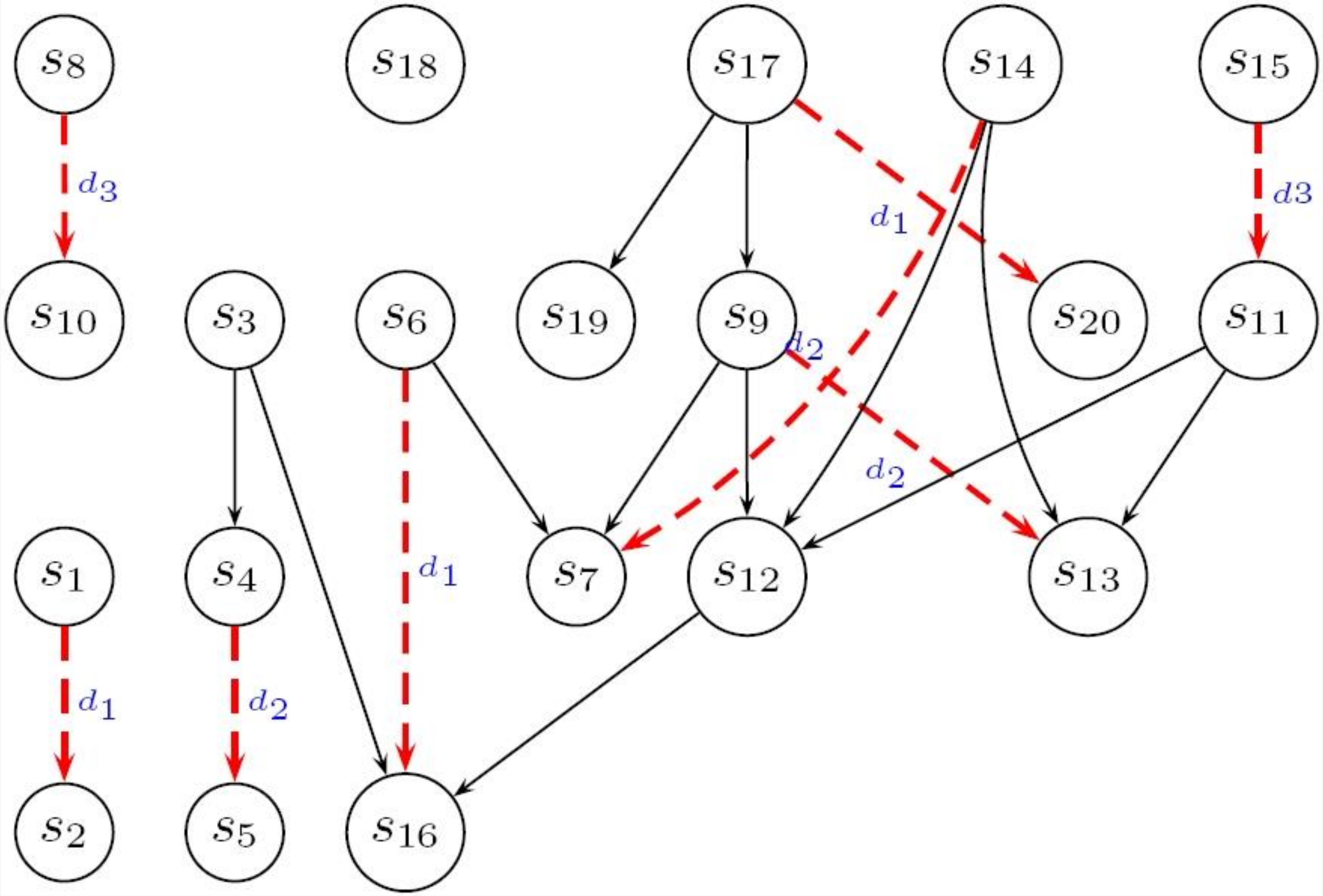
- If (d_1 removes $s_1 \succ s_2$) or (d_3 removes $s_4 \succ s_3$), then $\mathcal{U}_{\{d1,d2,d3\}} \neq \emptyset$

UTA-GROUP^{GMS} : an illustrative example

- Although $\mathcal{U}_{\{d1,d2,d3\}} = \emptyset$, the following relations are not empty:
  - $\succeq^{N,N}(\{d_1,d_2,d_3\}) = \{(s_6,s_7)\}$, i.e., $s_6 \succeq^{N} s_7$ for all d_h
  - $\succeq^{N,N}(\{d_1,d_2\}) = \{(s_6,s_7), (s_9,s_{13})\}$
  - $\succeq^{N,N}(\{d_1,d_3\}) = \{(s_6,s_7), (s_{17},s_{20})\}$
  - $\succeq^{N,N}(\{d_2,d_3\}) = \{(s_6,s_7), (s_{15},s_{11})\}$
  - $x \succeq^{N,P}(\{d_1,d_2,d_3\})\ y$: $x \succeq^{N} y$ for at least one d_h
  - $x \succeq^{P,N}(\{d_1,d_2,d_3\})\ y$: $x \succeq^{P} y$ for all d_h
  - $x \succeq^{P,P}(\{d_1,d_2,d_3\})\ y$: $x \succeq^{P} y$ for at least one d_h

UTA-GROUP^{GMS} : an illustrative example

- Suppose d_3 removes $s_4 \succ s_3$ then the collective model leads to the following collective necessary ranking:



Multicriteria group sorting with a set of additive value functions: UTA-DIS-GROUP^{GMS}

- Set of DMs: $D=\{d_1,\ldots,d_p\}$
- Preference information provided by DM d_h, $h=1,\ldots,p$:

$$a^* \rightarrow_h [\ C_{\min_h(a^*)}, C_{\max_h(a^*)}\ ]$$

for all reference actions $a^* \in A^R$

- Given a set of DMs $D' \subseteq D$,

$$a^* \rightarrow^{P}_{D'} \bigcup_{h \in D'} \left[C_{\min_h(a^*)}, C_{\max_h(a^*)} \right]$$

$$a^* \rightarrow^{N}_{D'} \bigcap_{h \in D'} \left[C_{\min_h(a^*)}, C_{\max_h(a^*)} \right]$$

Multicriteria group sorting with a set of additive value functions: UTA-DIS-GROUP^{GMS} (necessary and possible assignment)

$$a \rightarrow \left[C_{\min_{D'}^{N}(a)}, C_{\max_{D'}^{N}(a)} \right]$$

means that all DMs in D′ agree that action a can be assigned to one class from the interval $\left[C_{\min_{D'}^{N}(a)}, C_{\max_{D'}^{N}(a)} \right]$

$$a \rightarrow \left[C_{\min_{D'}^{P}(a)}, C_{\max_{D'}^{P}(a)} \right]$$

means that that there is at least one DM in D′ who believes that action a can be assigned to one class in the interval $\left[C_{\min_{D'}^{P}(a)}, C_{\max_{D'}^{P}(a)} \right]$

$$\left[C_{\min_{D'}^{N}(a)}, C_{\max_{D'}^{N}(a)} \right] \subseteq \left[C_{\min_{D'}^{P}(a)}, C_{\max_{D'}^{P}(a)} \right]$$

Robust Ordinal Regression approach for outranking methods

- Preference information provided by DM:

$$aSb \quad \text{or} \quad aS^{c}b, \quad \text{for} \quad a,b \in A^{R}$$

- Concordance function, for $a,b \in A$:

$$C(a,b)=[w_1c'_1(a,b)+\ldots+w_nc'_n(a,b)]/(w_1+\ldots+w_n)$$

  - since $(w_1+\ldots+w_n)=1$, we can consider $C(a,b)=c_1(a,b)+\ldots+c_n(a,b)$, where $c_i(a,b)=w_ic'_i(a,b)$, $i=1,\ldots,n$
  - $c_i(a,b)$ is a monotone, non-decreasing function w.r.t. $g_i(a)-g_i(b)$, such that $c_i(a,b)\geq 0$ for all $a,b \in A$ (alt. for $g_i(a)-g_i(b)\geq q_i \geq 0$), $i=1,\ldots,n$, and

    $c_1(a,b)+\ldots+c_n(a,b)=1$ in case $g_i(a)-g_i(b)=\beta_i-\alpha_i$ for all $i=1,\ldots,n$

Robust Ordinal Regression approach for outranking methods

- Ordinal regression constraints, for $a,b \in A^R$:

$E(A^R)$

$c_1(a,b)+\ldots+c_n(a,b) \geq \lambda$ and $g_i(b)-g_i(a) \leq v_i$, $i=1,\ldots,n$, if aSb

$c_1(a,b)+\ldots+c_n(a,b) \leq \lambda+\varepsilon+M_0(a,b)$ and $g_i(b)-g_i(a) \geq v_i+\varepsilon-\delta M_i(a,b)$,
$M_i(a,b) \in \{0,1\}$, $i=1,\ldots,n$, $M_0(a,b)+M_1(a,b) \ldots+M_n(a,b) \leq n$, if aS^cb

$\lambda \geq 0.5$, $v_i \geq 0$ (alt. $v_i \geq p_i \geq q_i \geq 0$), $i=1,\ldots,n$,

$c_i(a,b) \geq 0$ for all $a,b \in A^R$ and $i=1,\ldots,n$,
$c_1(a,b)+\ldots+c_n(a,b)=1$ for $g_i(a)-g_i(b)=\beta_i-\alpha_i$, $i=1,\ldots,n$

$c_i(a,b) \geq c_i(c,d)$ if $g_i(a)-g_i(b) \geq g_i(c)-g_i(d)$, for all $a,b,c,d \in A^R$, $i=1,\ldots,n$

where ε is a small positive value and δ is a big positive value
(if specified, preference and indifference thresholds p_i,q_i are given)

Robust Ordinal Regression approach for outranking methods

- Given a pair of actions $x,y \in A$, x necessarily outranks y:

$$xS^{N}y \;\Leftrightarrow\; d(x,y) \geq 0$$

$d(x,y)$ = Min$\{c_1(x,y)+\ldots+c_n(x,y) - \lambda\}$, s.t. $E(A^{\mathrm{R}})$, where

$c_i(a,b) \geq c_i(c,d)$ if $g_i(a)-g_i(b) \geq g_i(c)-g_i(d)$, for all $a,b,c,d \in A^R \cup \{x,y\}$, $i=1,\ldots,n$, and

$g_i(y)-g_i(x) \leq v_i$, $i=1,\ldots,n$

- $d(x,y) \geq 0$ means that **for all** compatible outranking models x outranks y

- For $x,y \in A^R$:

$$xSy \Rightarrow xS^{N}y$$

Robust Ordinal Regression approach for outranking methods

- Given a pair of actions $x,y \in A$, x possibly outranks y:

$$xS^P y \Leftrightarrow D(x,y) \geq 0$$

$D(x,y)$ = $\text{Max}\{c_1(x,y)+\ldots+c_n(x,y) - \lambda\}$, s.t. $E(A^\text{R})$, where

$c_i(a,b) \geq c_i(c,d)$ if $g_i(a)-g_i(b) \geq g_i(c)-g_i(d)$, for all $a,b,c,d \in A^R \cup \{x,y\}$, $i=1,\ldots,n$, and

$g_i(y)-g_i(x) \leq v_i$, $i=1,\ldots,n$

- $D(x,y) \geq 0$ means that **for at least one** compatible outranking model x outranks y

- For $x,y \in A^R$:

$$xSy \Rightarrow not\ yS^P x$$

Robust Ordinal Regression approach for outranking methods

- For any pair of actions $x,y \in A$:

$$xS^{N}y \Leftrightarrow \text{not } xS^{CP}y$$

$$xS^{P}y \Leftrightarrow \text{not } xS^{CN}y$$

so, only $xS^{N}y$ and $xS^{P}y$ are to be checked

Robust Ordinal Regression approach for outranking methods: the case of group decision

- Generalization for group decision
  is analogical to UTA-GROUP^{GMS} and UTA-DIS-GROUP^{GMS}
- For each DM $d_h \in D' \subseteq D$ we consider all compatible outranking models
- Four situations are interesting for $x,y \in A$:
  - x $S^{N,N}(D')$ y: x S^N y for all $d_h \in D'$
  - x $S^{N,P}(D')$ y: x S^N y for at least one $d_h \in D'$
  - x $S^{P,N}(D')$ y: x S^P y for all $d_h \in D'$
  - x $S^{P,P}(D')$ y: x S^P y for at least one $d_h \in D'$

ROR and Interaction among criteria

Basic concepts

Setting

- N={1,2,…,n} set of criteria
- X_i : set of possible values of the i-th criterion
- $\mathbf{X}=\prod_{i=1}^{n} X_i = X_1 \times X_2 \times \ldots \times X_n=\{(x_1,\ldots, x_n):x_1\in X_1,\ldots, x_n\in X_n\}$: set of all **conceivable** alternatives
  - X include the alternatives under study. . . and many others!
- In this case we suppose that $X_1 = X_2 =\ldots =X_n =X\subseteq \mathbf{R}_+$ such that

$$\mathbf{X}=X^n$$

- $\succeq$: weak preference realtion on **X** such that for all $x,y\in X$

$$x \succeq y$$

means

«x is at least as good as y»

Marginal preferences

- $\succeq_i$: weak **marginal preference** relation on X_i, $i \in N$, such that for all $x_i, y_i \in X_i$

$$x_i \succeq_i y_i \text{ means } «x_i \text{ is at least as good as } y_i»$$

- We suppose also that

$$x_i \geq y_i \Leftrightarrow x_i \succeq y_i$$

- $\succeq°$: weak **marginal preference** relation on $\bigcup_{i=1}^{n} X_i$, such that for all $x_i \in X_i, y_j \in X_j$, $i,j \in N$

$$x_i \succeq° y_j \text{ means } «x_i \text{ is at least as good as } y_i»$$

- We suppose also that

$$x_i \geq y_j \Leftrightarrow x_i \succeq° y_j$$

Weighted sum model

- For all all x,y∈**X**

$$x \succeq y \Leftrightarrow \sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i y_i$$

with w_i non negative for all i ∈ N.

- In this case w_i can be interpreted as the importance of criterion i∈N.
- The importance of couple of criteria {i,j}⊆N is given by $w_i + w_j$.
- The importance of set of criteria A⊆N, denoted by μ(A), is given by

$$\mu(A) = \sum_{i \in A} w_i$$

- Observe that for A,B⊆N such that A∩B=∅

$$\mu(A \cup B) = \sum_{i \in A} w_i + \sum_{i \in B} w_i = \sum_{i \in A \cup B} w_i = \mu(A) + \mu(B)$$

Introductory example

Illustrative example (Grabisch 1996)

Students	Mathematics	Physics	Literature
S1	*18*	*16*	*10*
S2	*10*	*12*	*18*
S3	*14*	*15*	*15*

Illustrative example (Grabisch 1996)

Students	Mathematics	Physics	Literature	Global evaluation (weighted sum)
S1	*18*	*16*	*10*	*15.25*
S2	*10*	*12*	*18*	*17.25*
S3	*14*	*15*	*15*	*14.62*

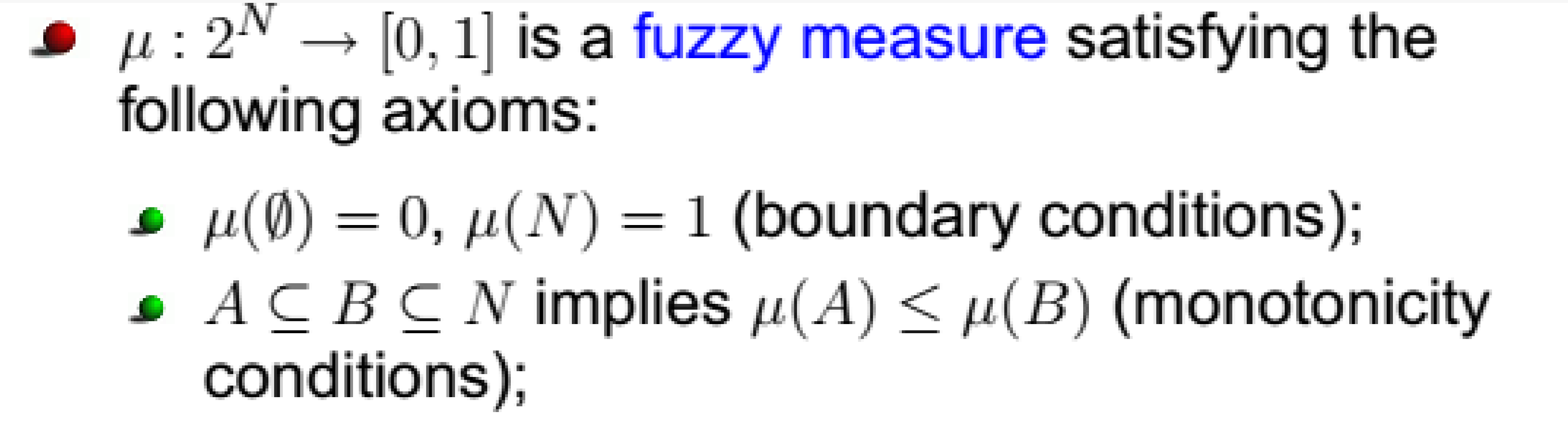
Suppose that the school is more scientifically than literary oriented, so that weights could be for example 3, 3 and 2 respectively.

Illustrative example (Grabisch 1996)

“If the school wants to favor well equilibrated students without weak points, **the above ranking is not fully satisfactory**, since student S1 has a severe weakness in literature, but has been considered better than student S3, who has no weak point. The reason is that **too much importance is given to mathematics and physics, which are in a sense redundant** since, usually, students good at mathematics are also good at physics (and vice versa), so that the evaluation is overestimated (resp. underestimated) for students good (resp. bad) at mathematics and/or physics.”

How to solve the problem?

Representing importance of criteria by means of a fuzzy measure (or capacity)



- $\mu : 2^N \rightarrow [0,1]$ is a fuzzy measure satisfying the following axioms:
    - $\mu(\emptyset) = 0$, $\mu(N) = 1$ (boundary conditions);
    - $A \subseteq B \subseteq N$ implies $\mu(A) \leq \mu(B)$ (monotonicity conditions);

For any $A \subseteq N$, $\mu(A)$ represent the importance of the set of criteria A.

It is **no more true** that for any $A,B \subseteq N$ such that $A \cap B = \emptyset$ we have

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

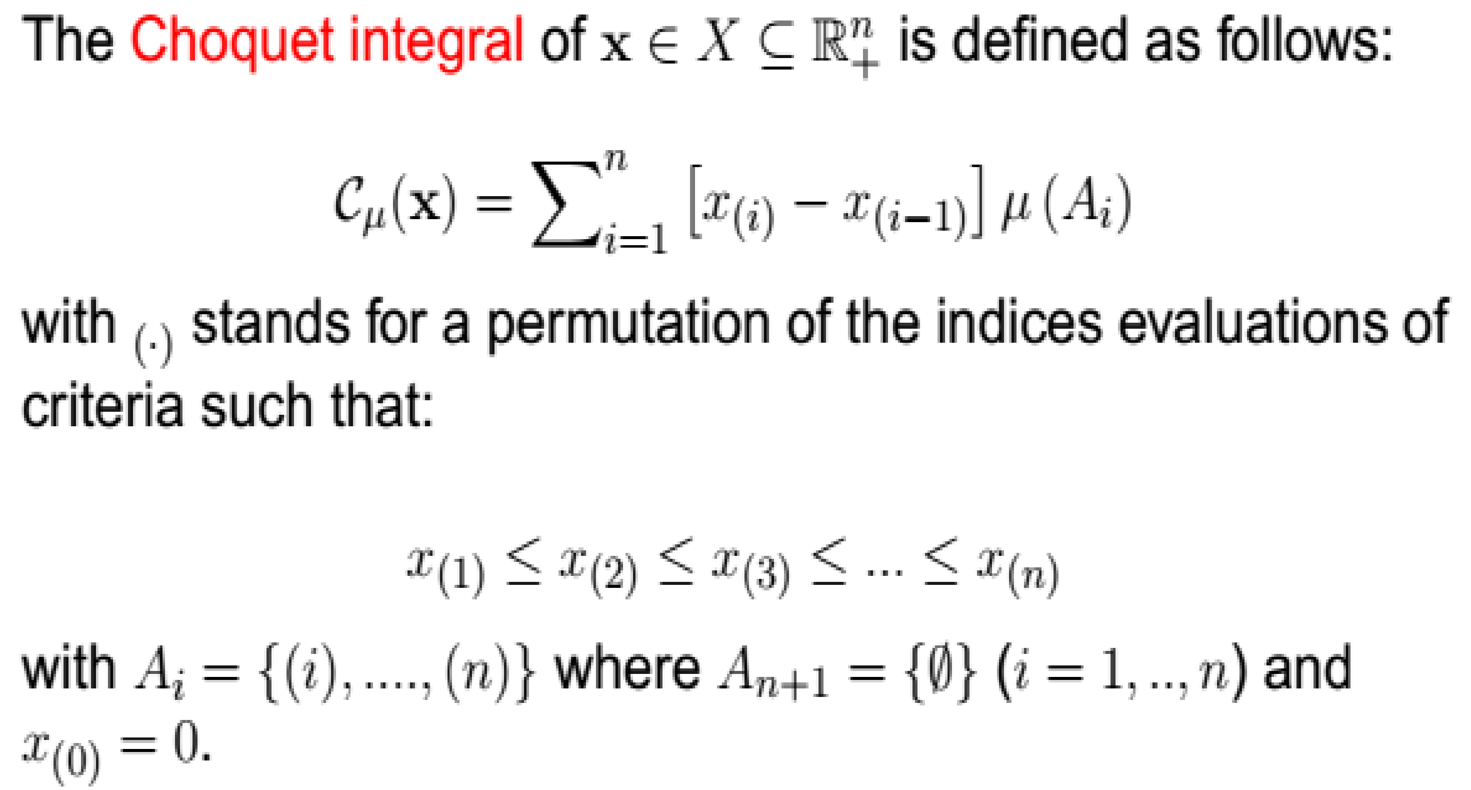
Representing importance of criteria by means of a fuzzy measure in the illustrative example (Grabisch 1996)

Set of subjects A	**μ(A)**
$\varnothing$	0
{Mathematics}	0.45
{Physics}	0.45
{Literature}	0.3
{Mathematics, Physics}	0. 5
{Mathematics, Literature}	0. 9
{Physics, Literature}	0. 9
{Mathematics, Physics,Literature}	1

$\mu(\{\text{Mathematics},\text{Physics}\}) \leq \mu(\{\text{Mathematics}\}) + \mu(\{\text{ Physics}\})$
(**redundancy** between Mathematics and Physics)

$\mu(\{\text{Mathematics},\text{Literature}\}) \leq \mu(\{\text{Mathematics}\}) + \mu(\{\text{ Literature}\})$
(**synergy** between Mathematics and Literature)
$\mu(\{\text{Physics},\text{Literature}\}) \leq \mu(\{\text{Physics}\}) + \mu(\{\text{ Literature}\})$
(**synergy** between Physiccs and Literature)

The **Choquet integral** (1952): computing a “weighted sum” using the non additive weights given by the fuzzy measure



The Choquet integral of $\mathbf{x} \in X \subseteq \mathbb{R}_+^n$ is defined as follows:

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} \left[x_{(i)} - x_{(i-1)}\right] \mu\left(A_i\right)$$

with $_{(\cdot)}$ stands for a permutation of the indices evaluations of criteria such that:

$$x_{(1)} \leq x_{(2)} \leq x_{(3)} \leq ... \leq x_{(n)}$$

with $A_i = \{(i), ..., (n)\}$ where $A_{n+1} = \{\emptyset\}$ $(i = 1, .., n)$ and $x_{(0)} = 0$.

Observe that the Choquet integral can be written also as follows:

$$C_\mu(\mathbf{x}) = \int_0^{\max_{i \in N} x_i} \mu\left(\left\{j \in N : x_j \geq x_i\right\}\right) dt$$

Illustrative example (Grabisch 1996): computing the Choquet integral for students S1

Students	Mathematics	Physics	Literature
S1	*18*	*16*	*10*

$$x_{Lit} \leq x_{Phys} \leq x_{Math}$$

A_1={Mathematics,Physics,Literature}

A_2={Mathematics,Physics}

A_3={Mathematics}

$C_\mu(18,16,10)=(10-0)\times\mu(A_1)+(16-10)\times\mu(A_2)+(18-16)\times\mu(A_2)=$

$(10-0)\times1+(16-10)\times0.5+(18-16)\times0.45=13.9$

Illustrative example (Grabisch 1996): computing the Choquet integral for students S2

Students	Mathematics	Physics	Literature
S2	*10*	*12*	*18*

$$x_{\text{Math}} \leq x_{\text{Phys}} \leq x_{\text{Lit}}$$

A_1={Mathematics,Physics,Literature}

A_2={Physics,Literature}

A_3={Literature}

$$C_\mu(10,12,18)=(10-0)\times\mu(A_1)+(12-10)\times\mu(A_2)+(18-12)\times\mu(A_2)=$$

$$(10-0)\times 1+(12-10)\times 0.9+(18-12)\times 0.3=13.6$$

Illustrative example (Grabisch 1996): computing the Choquet integral for students S3

Students	Mathematics	Physics	Literature
S3	*14*	*15*	*15*

$$x_{Math} \leq x_{Phys} \leq x_{Lit}$$

A_1={Mathematics,Physics,Literature}

A_2={Physics,Literature}

A_3={Physics}

$C_\mu(18,16,10)=(14\text{-}0)\times\mu(A_1)+(15\text{-}14)\times\mu(A_2)+(15\text{-}15)\times\mu(A_2)=$

$(14\text{-}0)\times 1+(15\text{-}14)\times 0.9+(15\text{-}15)\times 0.45=14.9$

Illustrative example (Grabisch 1996)

Students	Mathematics	Physics	Literature	Global evaluation (Choquet integral)
S1	*18*	*16*	*10*	*13.9*
S2	*10*	*12*	*18*	*13.6*
S3	*14*	*15*	*15*	*14.9*

Choquet integral ranks student S1, that has a severe weakness in literature, worse than student S3, that has no weak point.

Specific cases of Choquet integral

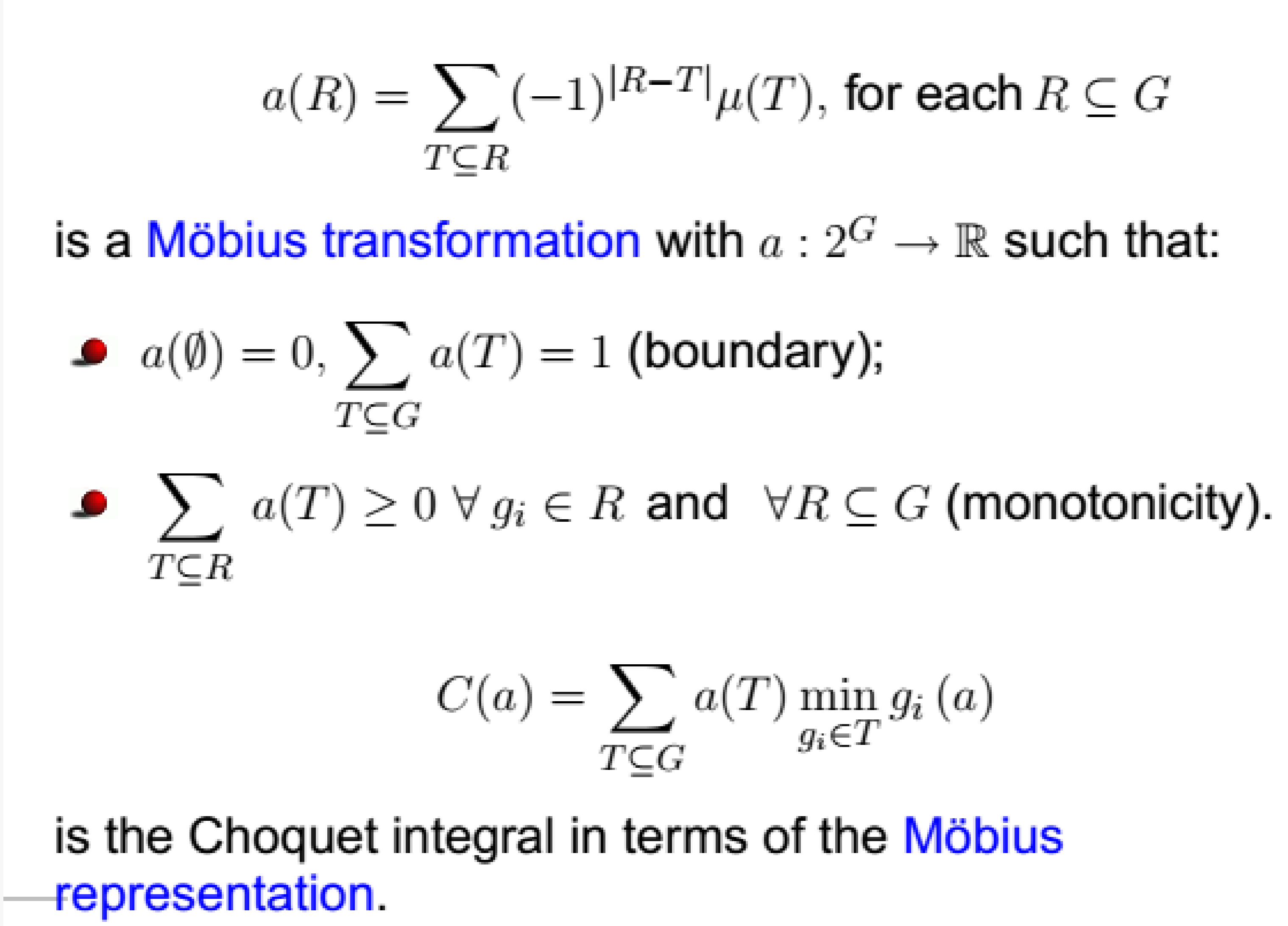
- $C_\mu(x_1,\ldots,x_n)= \mathrm{Max}(x_1,\ldots,x_n)$ if $\mu(A)=1$ for all $\emptyset \subset A \subseteq N$ (and, of course, $\mu(\emptyset)=0$)
- $C_\mu(x_1,\ldots,x_n)= \min(x_1,\ldots,x_n)$ if $\mu(A)=0$ for all $\emptyset \subseteq A \subset N$ (and, of course, $\mu(N)=1$)
- $C_\mu(x_1,\ldots,x_n)= \mathrm{OWA}(w_1,\ldots,w_n;x_1,\ldots,x_n)$ if $\mu(A)=\mu(B)$ when $|A|=|B|$, for all $A,B \subseteq N$ with

$$\mathrm{OWA}(w_1,\ldots,w_n;x_1,\ldots,x_n)=w_1x_{(1)}+\ldots+w_nx_{(n)}$$ (Yager 1988)

and $w_i= \mu(A)-\mu(B)$ with $A,B \subseteq N$ such that $|A|=i$ and $|B|=i-1$;

- $C_\mu(x_1,\ldots,x_n)= w_{(k)}$ (k-th order statistic, $0<k\leq n$) if $\mu(A)=0$ for $|A|<k$ and $\mu(A)=1$ for $|A|\geq k$ for all $A\subseteq N$.

The Möbius transformation of a fuzzy measure (or capacity)



$$a(R) = \sum_{T \subseteq R} (-1)^{|R-T|} \mu(T), \text{ for each } R \subseteq G$$

is a Möbius transformation with $a : 2^G \rightarrow \mathbb{R}$ such that:

- $a(\emptyset) = 0, \sum_{T \subseteq G} a(T) = 1$ (boundary);
- $\sum_{T \subseteq R} a(T) \geq 0 \; \forall \, g_i \in R$ and $\forall R \subseteq G$ (monotonicity).

$$C(a) = \sum_{T \subseteq G} a(T) \min_{g_i \in T} g_i\,(a)$$

is the Choquet integral in terms of the Möbius representation.

The Shapley value

- The global importance of a criterion i∈N is not solely determined by the value μ({i}), but also by all μ(A) with A⊆N such that i∈A.
- But how to extract from these values the contribution of i alone?
- By the Shapley value

$$\varphi(\{i\}) = \sum_{A\subseteq N : i\in A} \frac{a(A)}{|A|} =$$

$$\sum_{A\subseteq N : i\in A} \frac{(|A|-1)!\,(n-|A|)!}{n!}\left(\mu(A)-\mu(A-\{i\})\right)$$

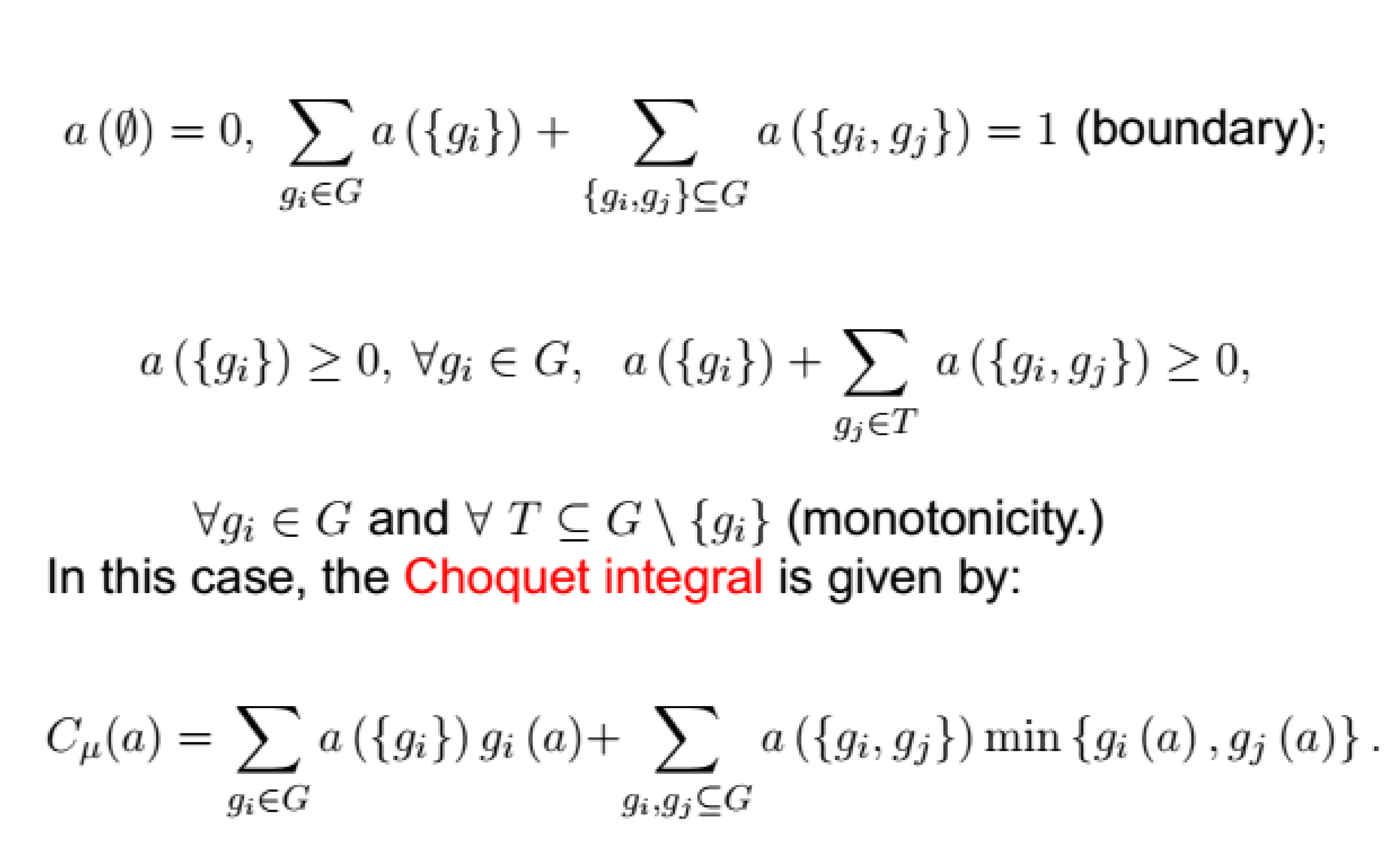
The interaction index

- The interaction between i and j is not only determined by the difference μ({i, j}) - μ({i}) - μ({j}) but also by all the coefficients μ(A) such that {i, j} ⊆ A. Then,

  how to compute a degree of interaction which is meaningful?
- By the interaction index (Murofushi 1993)

$$\varphi(\{i, j\}) = \sum_{A \subseteq N : \{i, j\} \subseteq A} \frac{a(A)}{|A| - 1}$$

2-additive fuzzy measures



$$a(\emptyset) = 0, \ \sum_{g_i \in G} a(\{g_i\}) + \sum_{\{g_i, g_j\} \subseteq G} a(\{g_i, g_j\}) = 1 \text{ (boundary);}$$

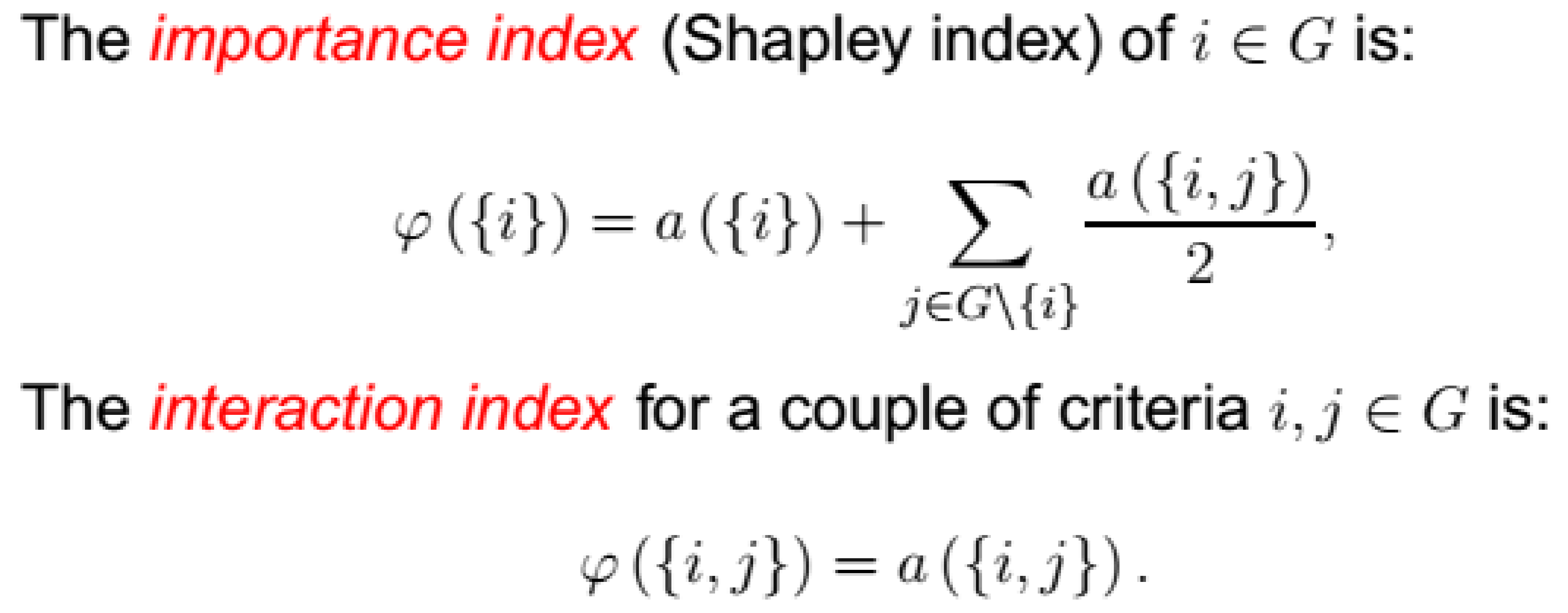
$$a(\{g_i\}) \geq 0, \forall g_i \in G, \quad a(\{g_i\}) + \sum_{g_j \in T} a(\{g_i, g_j\}) \geq 0,$$

$\forall g_i \in G$ and $\forall \ T \subseteq G \setminus \{g_i\}$ (monotonicity.)

In this case, the Choquet integral is given by:

$$C_\mu(a) = \sum_{g_i \in G} a(\{g_i\}) \, g_i(a) + \sum_{g_i, g_j \subseteq G} a(\{g_i, g_j\}) \min\{g_i(a), g_j(a)\}.$$

Shapley value and interaction indices in case of 2-additive fuzzy measures



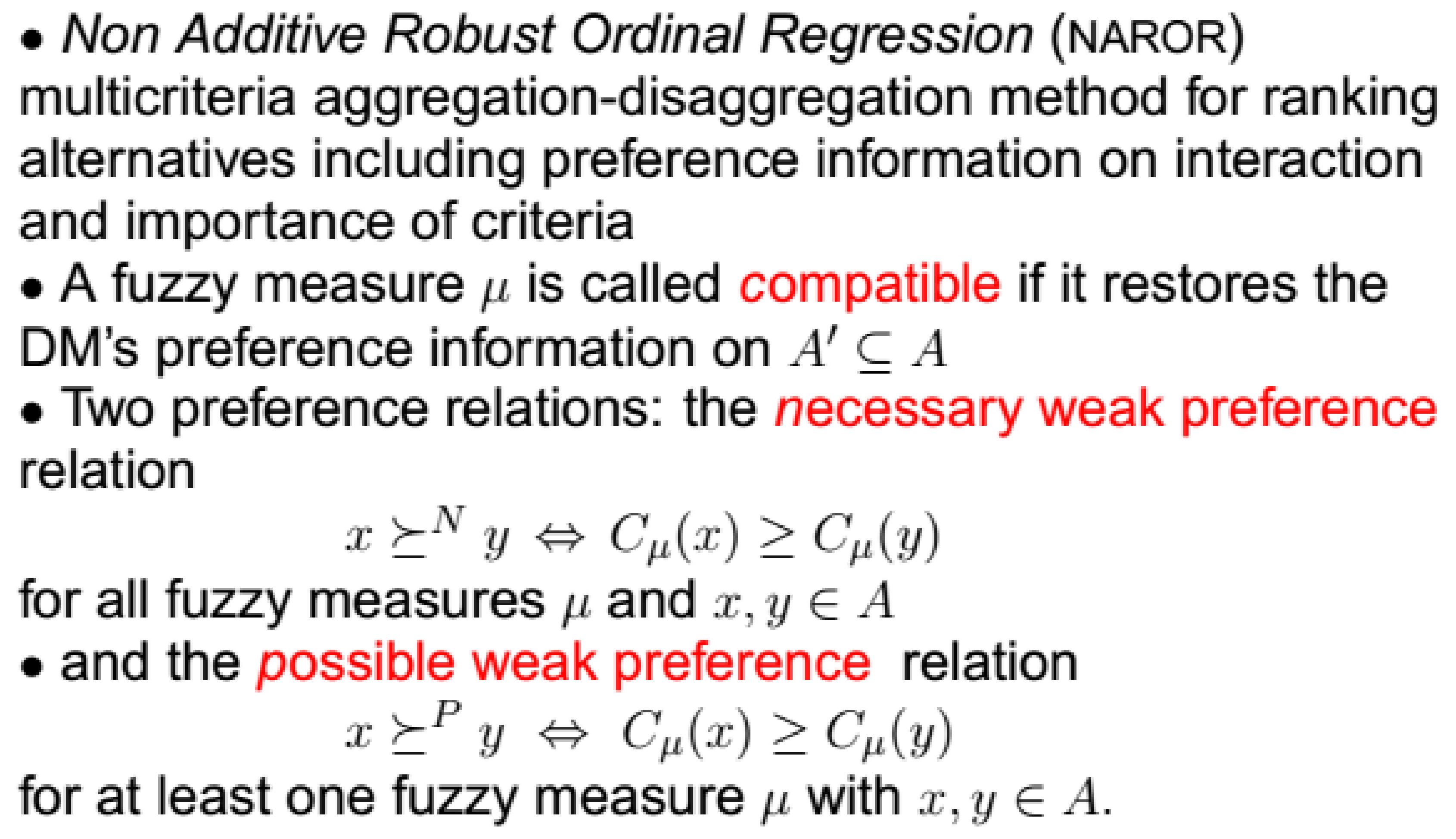
The *importance index* (Shapley index) of $i \in G$ is:

$$\varphi(\{i\}) = a(\{i\}) + \sum_{j \in G \setminus \{i\}} \frac{a(\{i,j\})}{2},$$

The *interaction index* for a couple of criteria $i, j \in G$ is:

$$\varphi(\{i,j\}) = a(\{i,j\}).$$

Non Additive Robust Ordinal Regression (Angilella, Greco, Matarazzo 2010)



- *Non Additive Robust Ordinal Regression* (NAROR) multicriteria aggregation-disaggregation method for ranking alternatives including preference information on interaction and importance of criteria
- A fuzzy measure μ is called compatible if it restores the DM's preference information on $A' \subseteq A$
- Two preference relations: the *necessary weak preference* relation

$$x \succsim^N y \Leftrightarrow C_\mu(x) \geq C_\mu(y)$$

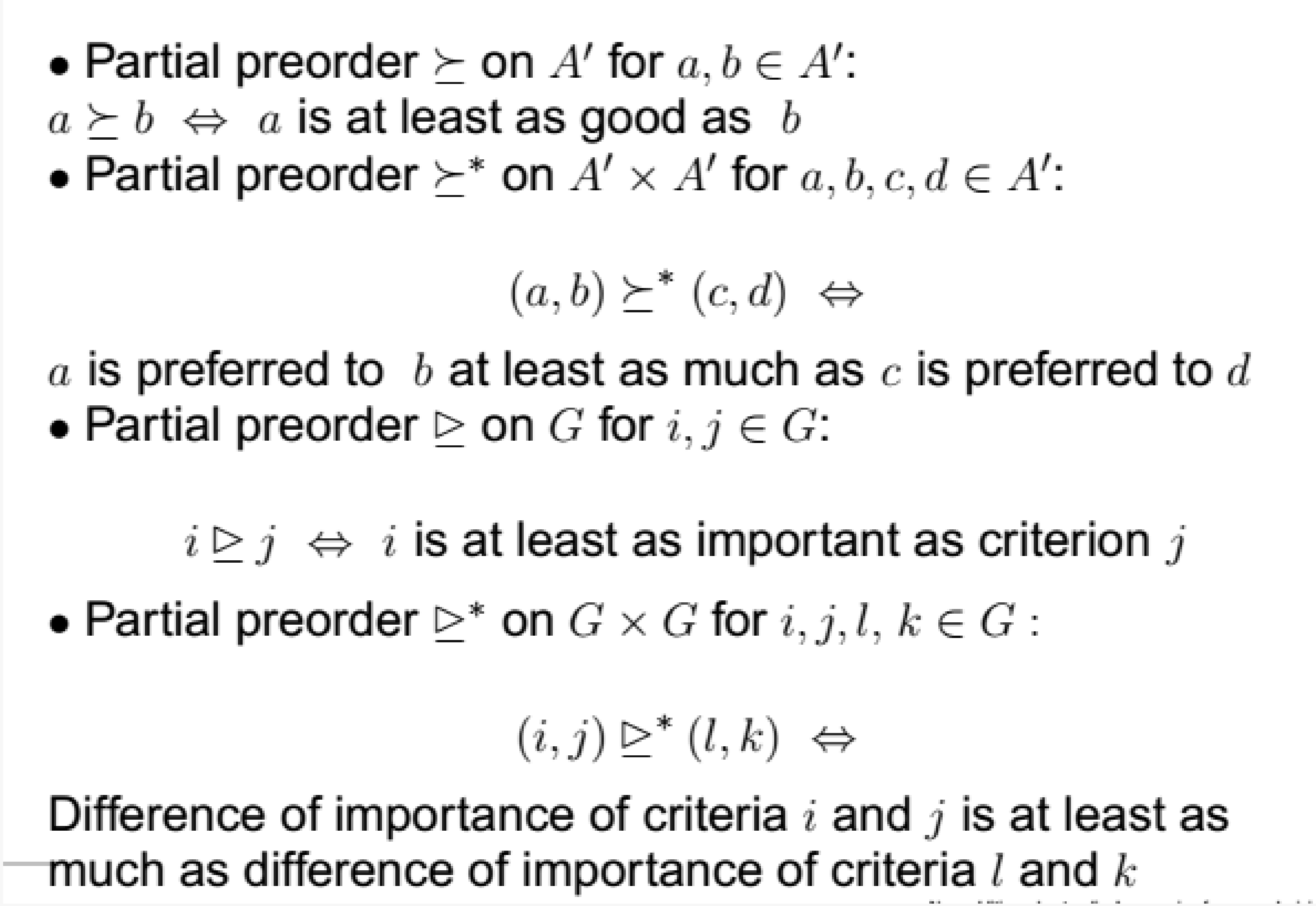
for all fuzzy measures μ and $x, y \in A$

- and the possible weak preference relation

$$x \succsim^P y \Leftrightarrow C_\mu(x) \geq C_\mu(y)$$

for at least one fuzzy measure μ with $x, y \in A$.

The Decision Maker’s preference information



- Partial preorder $\succsim$ on A' for $a, b \in A'$:

$a \succsim b \;\Leftrightarrow\; a$ is at least as good as b

- Partial preorder $\succsim^*$ on $A' \times A'$ for $a, b, c, d \in A'$:

$$(a, b) \succsim^* (c, d) \;\Leftrightarrow$$

a is preferred to b at least as much as c is preferred to d

- Partial preorder $\trianglerighteq$ on G for $i, j \in G$:

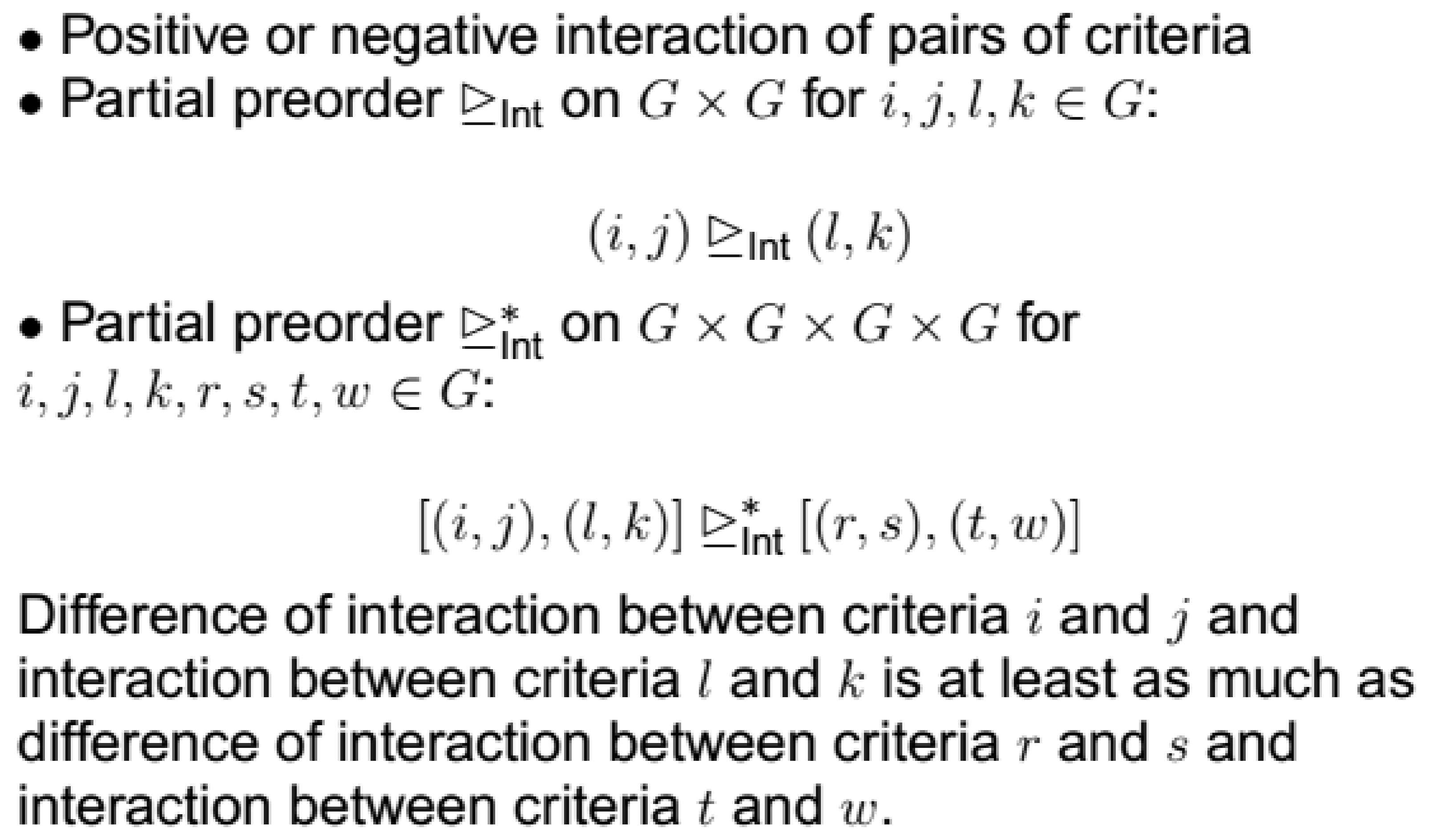
$$i \trianglerighteq j \;\Leftrightarrow\; i \text{ is at least as important as criterion } j$$

- Partial preorder $\trianglerighteq^*$ on $G \times G$ for $i, j, l, k \in G$:

$$(i, j) \trianglerighteq^* (l, k) \;\Leftrightarrow$$

Difference of importance of criteria i and j is at least as much as difference of importance of criteria l and k

The Decision Maker’s preference information



- Positive or negative interaction of pairs of criteria
- Partial preorder $\trianglerighteq_{\text{Int}}$ on $G \times G$ for $i, j, l, k \in G$:

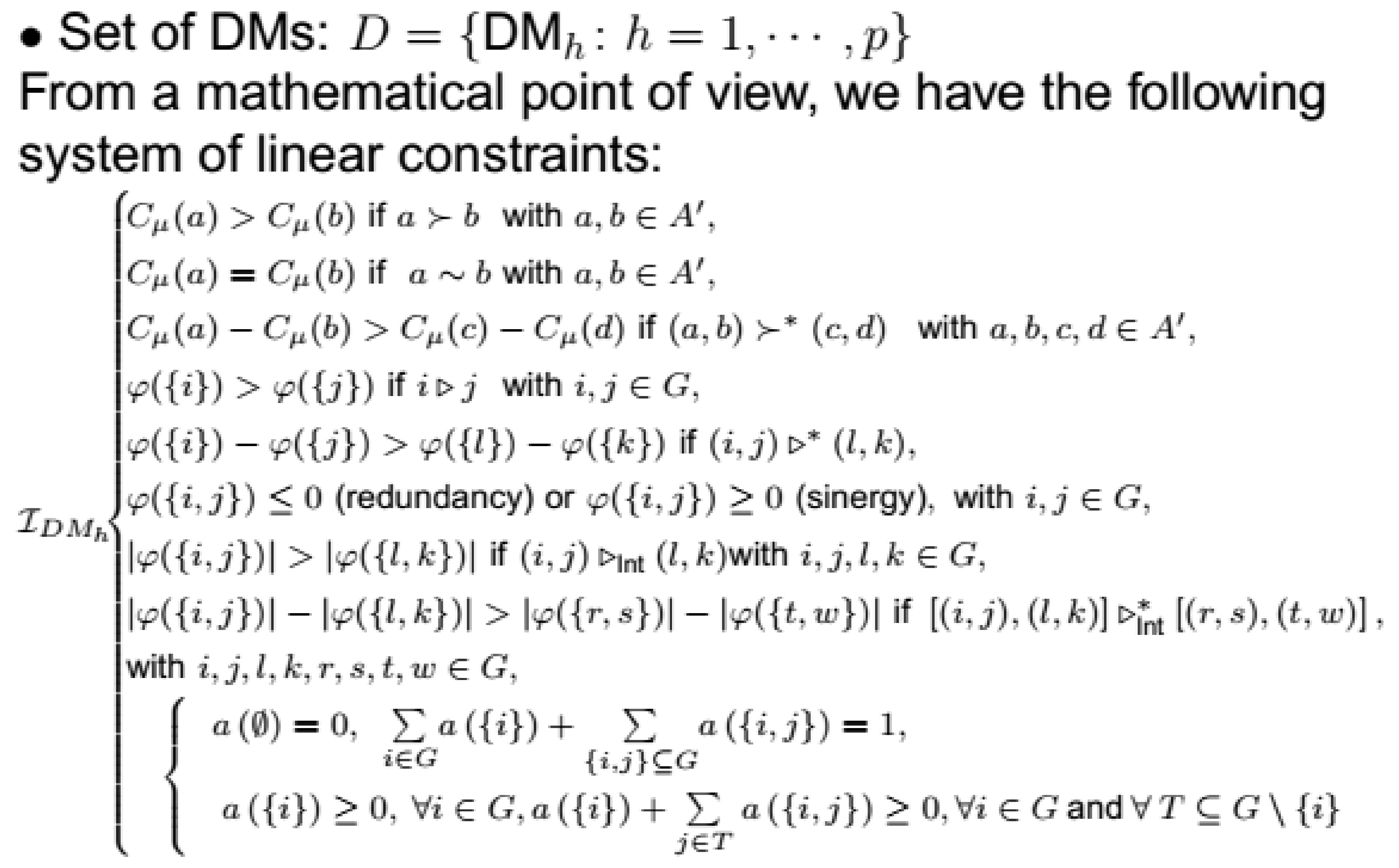
$$(i, j) \trianglerighteq_{\text{Int}} (l, k)$$

- Partial preorder $\trianglerighteq^*_{\text{Int}}$ on $G \times G \times G \times G$ for $i, j, l, k, r, s, t, w \in G$:

$$[(i, j), (l, k)] \trianglerighteq^*_{\text{Int}} [(r, s), (t, w)]$$

Difference of interaction between criteria i and j and interaction between criteria l and k is at least as much as difference of interaction between criteria r and s and interaction between criteria t and w.

The Decision Maker’s preference information

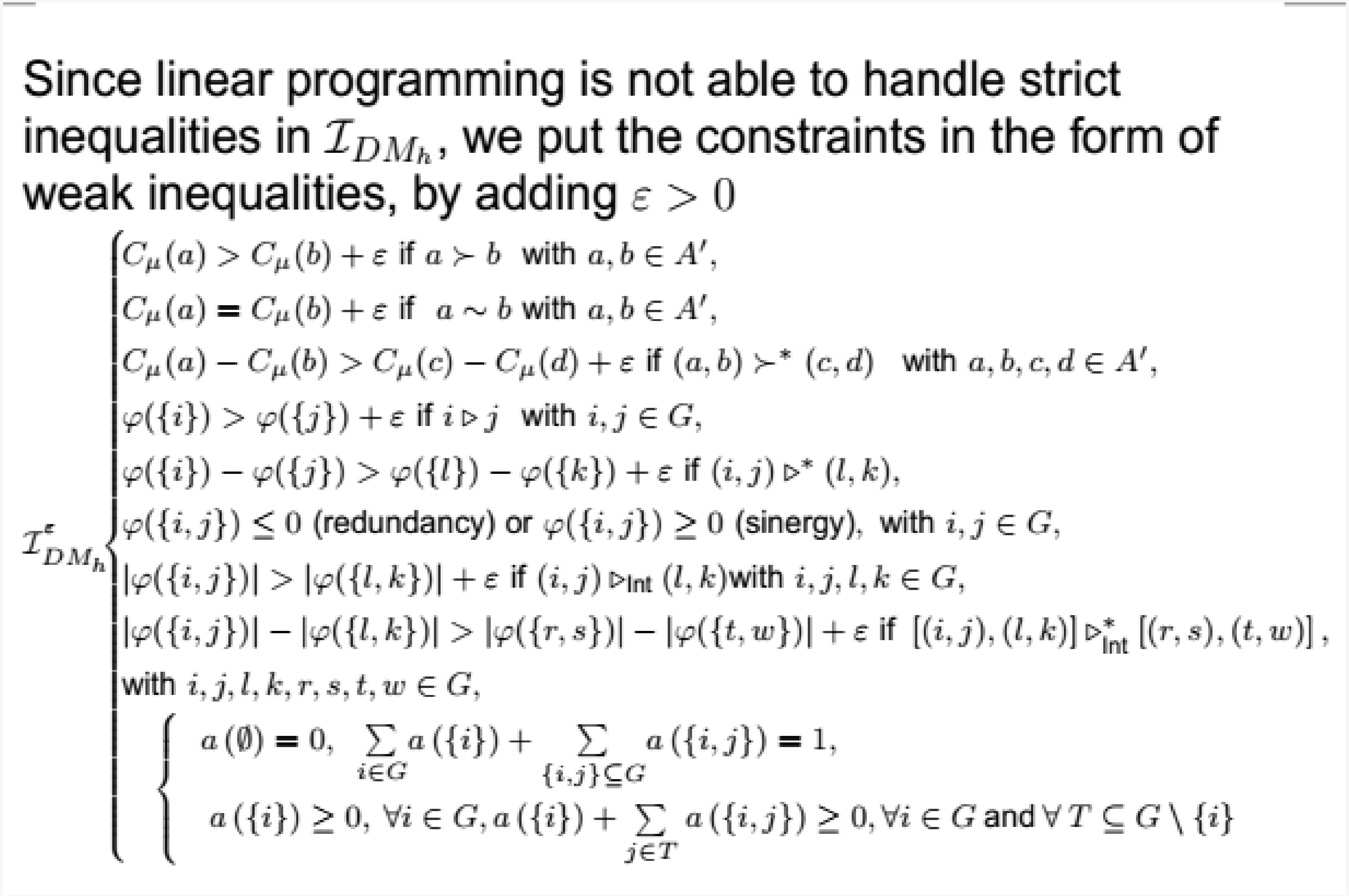


- Set of DMs: $D = \{\mathrm{DM}_h : h = 1, \cdots, p\}$

From a mathematical point of view, we have the following system of linear constraints:

$$\mathcal{I}_{DM_h}\begin{cases} C_\mu(a) > C_\mu(b) \text{ if } a \succ b \text{ with } a, b \in A', \\ C_\mu(a) = C_\mu(b) \text{ if } a \sim b \text{ with } a, b \in A', \\ C_\mu(a) - C_\mu(b) > C_\mu(c) - C_\mu(d) \text{ if } (a,b) \succ^* (c,d) \text{ with } a,b,c,d \in A', \\ \varphi(\{i\}) > \varphi(\{j\}) \text{ if } i \rhd j \text{ with } i, j \in G, \\ \varphi(\{i\}) - \varphi(\{j\}) > \varphi(\{l\}) - \varphi(\{k\}) \text{ if } (i,j) \rhd^* (l,k), \\ \varphi(\{i,j\}) \leq 0 \text{ (redundancy) or } \varphi(\{i,j\}) \geq 0 \text{ (sinergy), with } i,j \in G, \\ |\varphi(\{i,j\})| > |\varphi(\{l,k\})| \text{ if } (i,j) \rhd_{\text{Int}} (l,k) \text{with } i,j,l,k \in G, \\ |\varphi(\{i,j\})| - |\varphi(\{l,k\})| > |\varphi(\{r,s\})| - |\varphi(\{t,w\})| \text{ if } [(i,j),(l,k)] \rhd^*_{\text{Int}} [(r,s),(t,w)], \\ \text{with } i,j,l,k,r,s,t,w \in G, \\ \quad \begin{cases} a(\emptyset) = 0, \ \sum\limits_{i \in G} a(\{i\}) + \sum\limits_{\{i,j\} \subseteq G} a(\{i,j\}) = 1, \\ a(\{i\}) \geq 0, \ \forall i \in G, a(\{i\}) + \sum\limits_{j \in T} a(\{i,j\}) \geq 0, \forall i \in G \text{ and } \forall\, T \subseteq G \setminus \{i\} \end{cases} \end{cases}$$

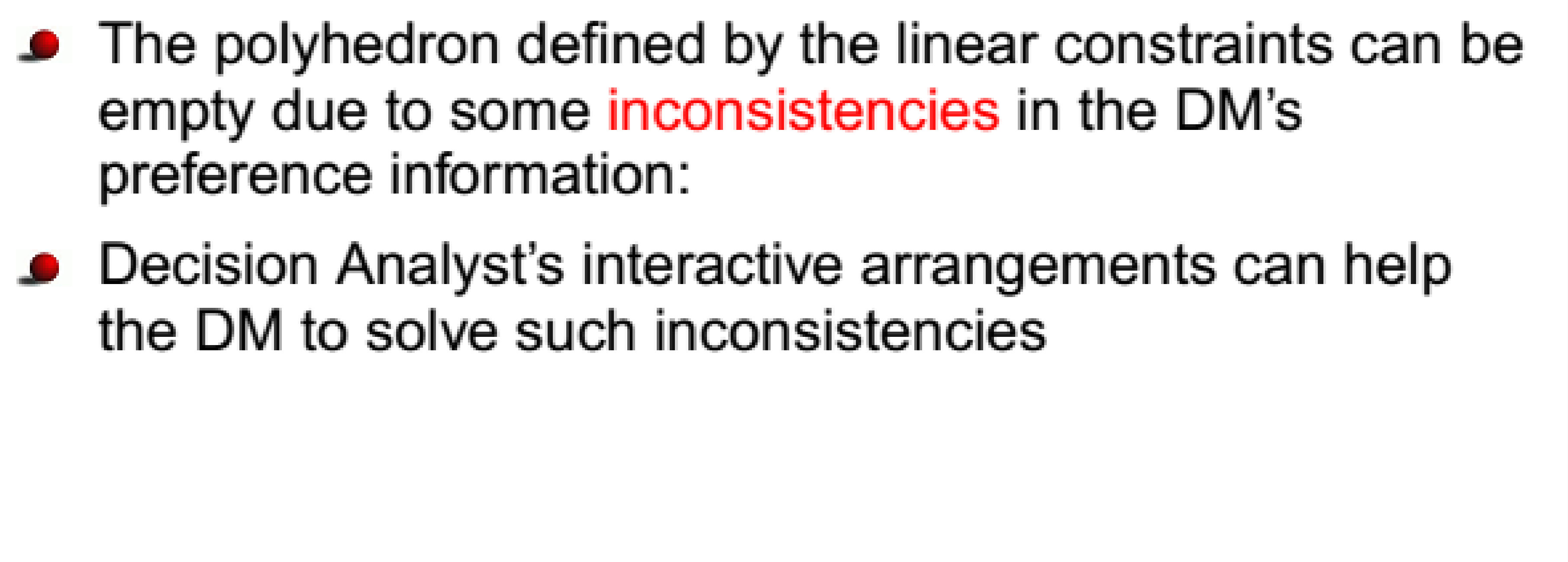
The Decision Maker’s preference information



Since linear programming is not able to handle strict inequalities in $\mathcal{I}_{DM_h}$, we put the constraints in the form of weak inequalities, by adding $\varepsilon > 0$

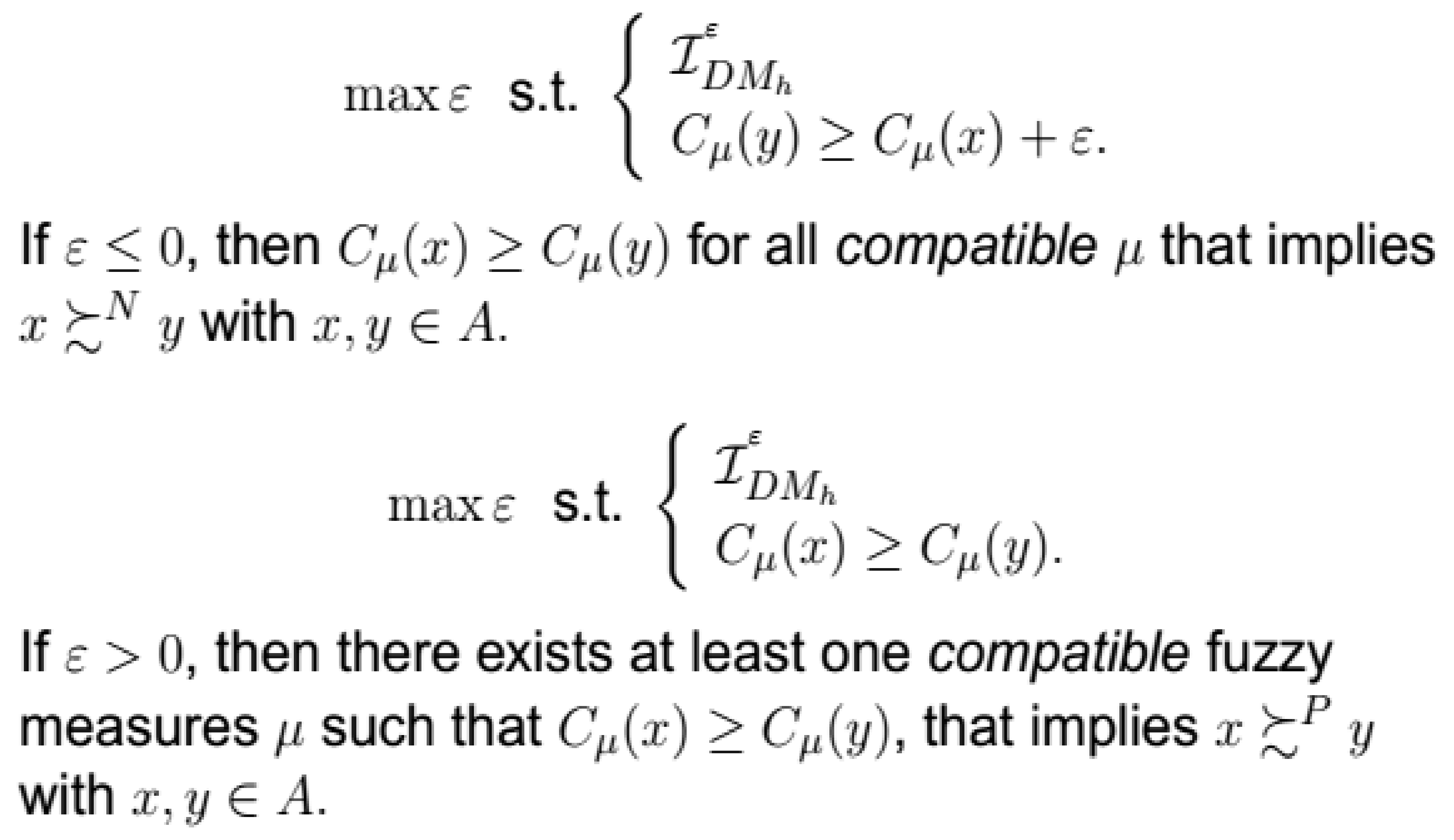
$$\mathcal{I}^{\varepsilon}_{DM_h}\begin{cases} C_\mu(a) > C_\mu(b) + \varepsilon \text{ if } a \succ b \text{ with } a, b \in A', \\ C_\mu(a) = C_\mu(b) + \varepsilon \text{ if } a \sim b \text{ with } a, b \in A', \\ C_\mu(a) - C_\mu(b) > C_\mu(c) - C_\mu(d) + \varepsilon \text{ if } (a,b) \succ^{*} (c,d) \text{ with } a, b, c, d \in A', \\ \varphi(\{i\}) > \varphi(\{j\}) + \varepsilon \text{ if } i \rhd j \text{ with } i, j \in G, \\ \varphi(\{i\}) - \varphi(\{j\}) > \varphi(\{l\}) - \varphi(\{k\}) + \varepsilon \text{ if } (i,j) \rhd^{*} (l,k), \\ \varphi(\{i,j\}) \leq 0 \text{ (redundancy) or } \varphi(\{i,j\}) \geq 0 \text{ (sinergy), with } i, j \in G, \\ |\varphi(\{i,j\})| > |\varphi(\{l,k\})| + \varepsilon \text{ if } (i,j) \rhd_{\text{Int}} (l,k) \text{with } i, j, l, k \in G, \\ |\varphi(\{i,j\})| - |\varphi(\{l,k\})| > |\varphi(\{r,s\})| - |\varphi(\{t,w\})| + \varepsilon \text{ if } \left[(i,j),(l,k)\right] \rhd^{*}_{\text{Int}} \left[(r,s),(t,w)\right], \\ \text{with } i, j, l, k, r, s, t, w \in G, \\ \begin{cases} a(\emptyset) = 0, \ \sum\limits_{i \in G} a(\{i\}) + \sum\limits_{\{i,j\} \subseteq G} a(\{i,j\}) = 1, \\ a(\{i\}) \geq 0, \forall i \in G, a(\{i\}) + \sum\limits_{j \in T} a(\{i,j\}) \geq 0, \forall i \in G \text{ and } \forall\, T \subseteq G \setminus \{i\} \end{cases} \end{cases}$$

The Decision Maker’s preference information



- The polyhedron defined by the linear constraints can be empty due to some inconsistencies in the DM’s preference information:
- Decision Analyst’s interactive arrangements can help the DM to solve such inconsistencies

NAROR methodology



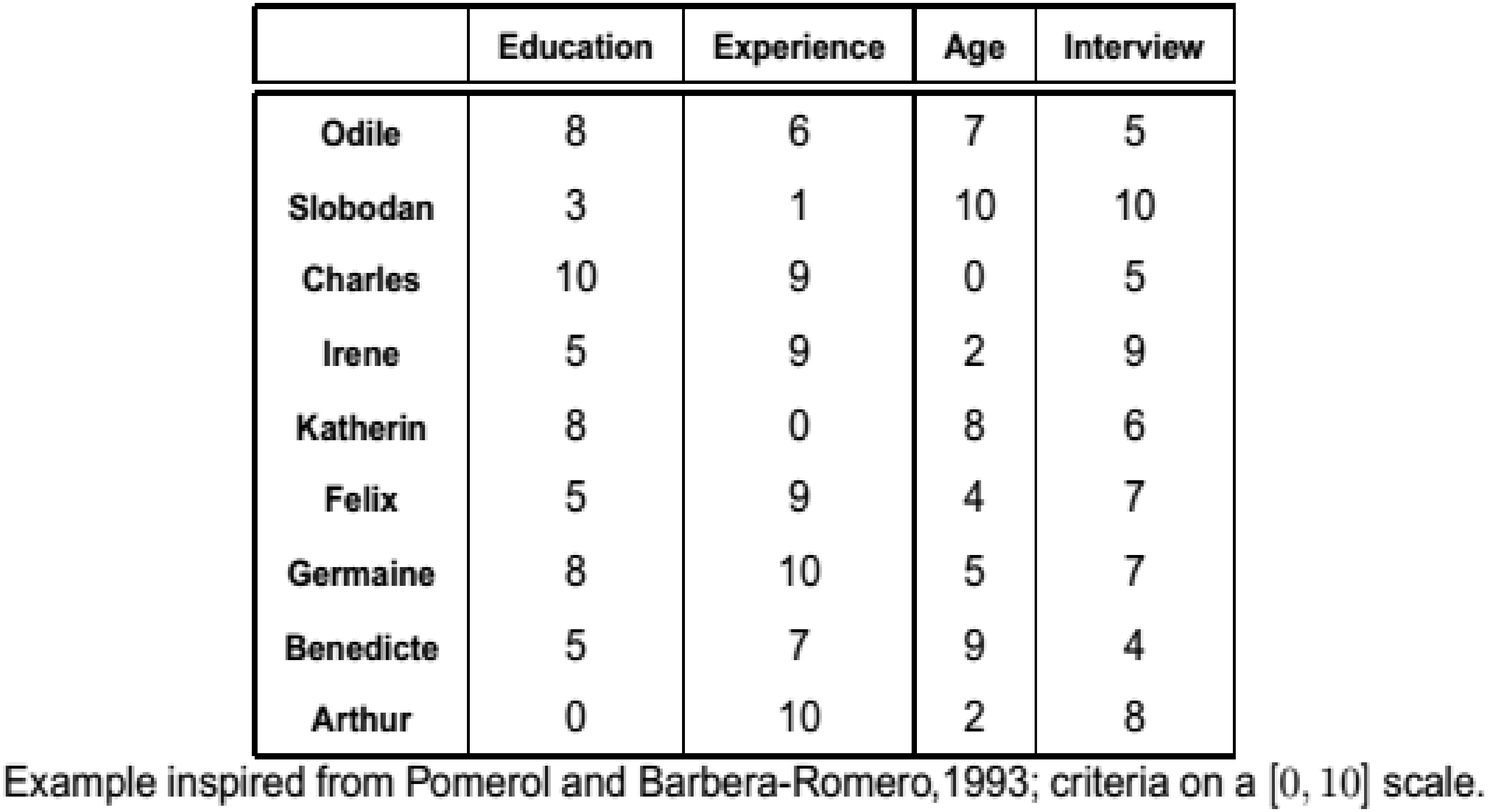
$$\max \varepsilon \quad \text{s.t.} \begin{cases} \mathcal{I}^{\varepsilon}_{DM_h} \\ C_\mu(y) \geq C_\mu(x) + \varepsilon. \end{cases}$$

If $\varepsilon \leq 0$, then $C_\mu(x) \geq C_\mu(y)$ for all *compatible* μ that implies $x \succsim^N y$ with $x, y \in A$.

$$\max \varepsilon \quad \text{s.t.} \begin{cases} \mathcal{I}^{\varepsilon}_{DM_h} \\ C_\mu(x) \geq C_\mu(y). \end{cases}$$

If $\varepsilon > 0$, then there exists at least one *compatible* fuzzy measures μ such that $C_\mu(x) \geq C_\mu(y)$, that implies $x \succsim^P y$ with $x, y \in A$.

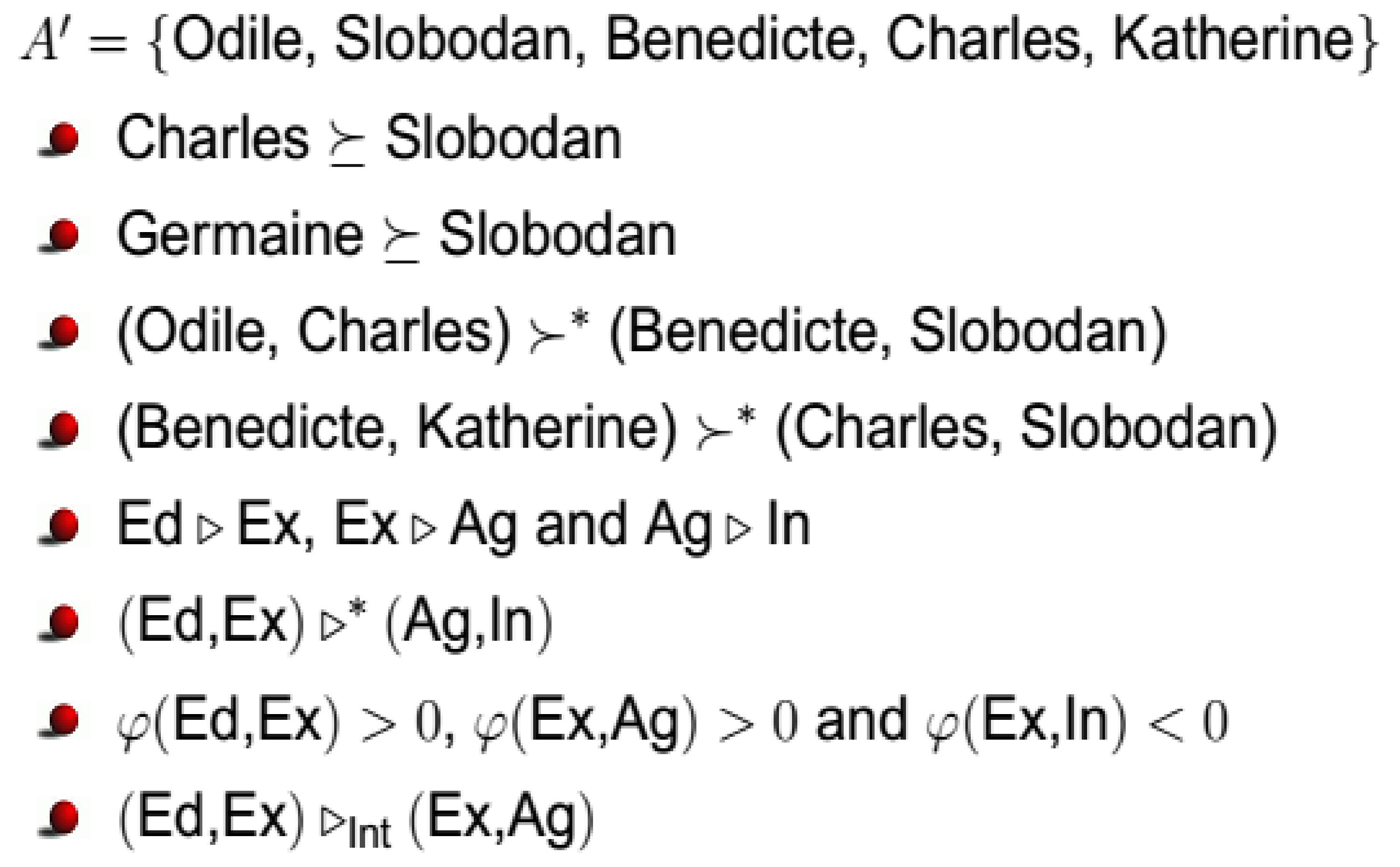
A recruitement problem



	Education	Experience	Age	Interview
Odile	8	6	7	5
Slobodan	3	1	10	10
Charles	10	9	0	5
Irene	5	9	2	9
Katherin	8	0	8	6
Felix	5	9	4	7
Germaine	8	10	5	7
Benedicte	5	7	9	4
Arthur	0	10	2	8

Example inspired from Pomerol and Barbera-Romero,1993; criteria on a $[0, 10]$ scale.

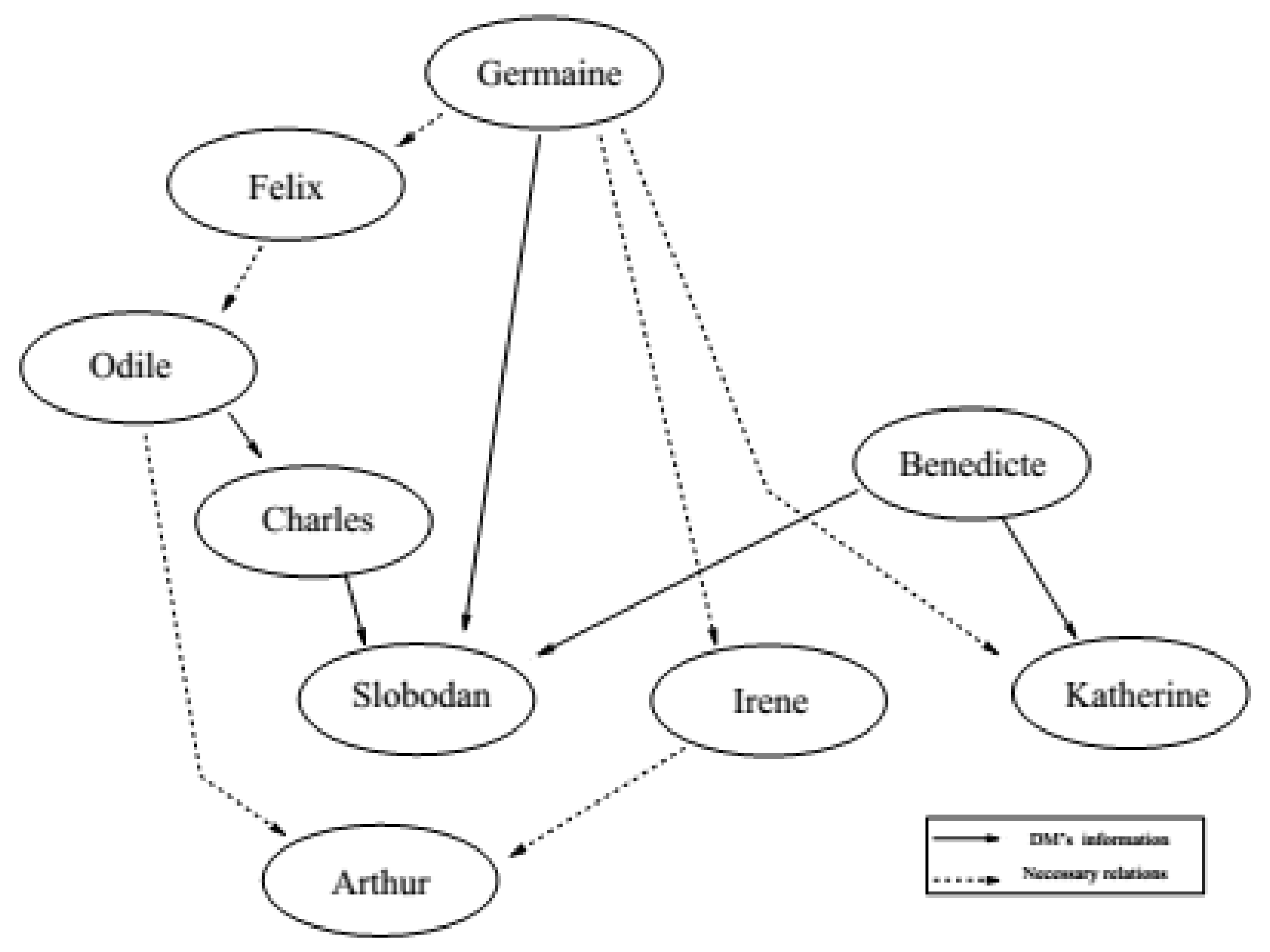
The Decision Maker’s preference information



$A' = \{$Odile, Slobodan, Benedicte, Charles, Katherine$\}$

- Charles $\succeq$ Slobodan
- Germaine $\succeq$ Slobodan
- (Odile, Charles) $\succ^*$ (Benedicte, Slobodan)
- (Benedicte, Katherine) $\succ^*$ (Charles, Slobodan)
- Ed $\triangleright$ Ex, Ex $\triangleright$ Ag and Ag $\triangleright$ In
- (Ed,Ex) $\triangleright^*$ (Ag,In)
- φ(Ed,Ex) > 0, φ(Ex,Ag) > 0 and φ(Ex,In) < 0
- (Ed,Ex) $\triangleright_{\text{Int}}$ (Ex,Ag)

Necessary and possible preference relations



ROR and the enriched additive value functions: the UTA^{GMS}-INT method

Plan

- Interaction among criteria explained on an example
- Incapacity of additive value function and Choquet integral
- Robust Ordinal Regression dealing with interactions – *UTA^{GMS}-INT*
  - Input preference information
  - Discovering the need of handling interactions
  - Identifying the pairs of interacting criteria
  - Calculating the necessary and possible preference relations
  - Didactic example
- Interaction on bipolar scales - *UTA^{GSS}*
- Conclusions

Interaction among criteria explained on an example

Interactions between two criteria

- Positive interactions (e.g., maximum speed & price of a car):

$$u_{i_1,i_2}\left(x_{i_1},x_{i_2}\right)>u_{i_1}\left(x_{i_1}\right)+u_{i_2}\left(x_{i_2}\right)$$

- Negative interactions (e.g., maximum speed & acceleration of a car):

$$u_{i_1,i_2}\left(x_{i_1},x_{i_2}\right)<u_{i_1}\left(x_{i_1}\right)+u_{i_2}\left(x_{i_2}\right)$$

Illustrative example

Students	Mathematics	Physics	Literature
S1	*Good*	*Medium*	*Bad*
S2	*Good*	*Bad*	*Medium*
S3	*Medium*	*Medium*	*Bad*
S4	*Medium*	*Bad*	*Medium*

What preference model would be able to represent the preference:

$$S2 \succ S1 \quad \text{and} \quad S3 \succ S4$$

?

Additive value function

- Consider an additive value function

$$U(x) = \sum_{i=1}^{n} u_i\left[g_i(x)\right]$$

- Does there exist an additive value function representing the preferences $S2 \succ S1$ and $S3 \succ S4$?

Additive value function

$$S2 \succ S1$$

$$U(\mathrm{S2})=\mathrm{U}_{\mathrm{Math}}(\mathrm{GOOD})+ U_{\mathrm{Phys}}(\mathrm{BAD})+U_{\mathrm{Lit}}(\mathrm{MEDIUM})$$

$$>$$

$$U_{\mathrm{Math}}(\mathrm{GOOD})+U_{\mathrm{Phys}}(\mathrm{MEDIUM})+U_{\mathrm{Lit}}(\mathrm{BAD})=U(\mathrm{S1})$$

$$S3 \succ S4$$

$$U(\mathrm{S3})=U_{\mathrm{Math}}(\mathrm{MEDIUM})+U_{\mathrm{Phys}}(\mathrm{MEDIUM})+U_{\mathrm{Lit}}(\mathrm{BAD})$$

$$>$$

$$U_{\mathrm{Math}}(\mathrm{MEDIUM})+U_{\mathrm{Phys}}(\mathrm{BAD})+U_{\mathrm{Lit}}(\mathrm{MEDIUM})=U(\mathrm{S4})$$

Additive value function

$$S2 \succ S1$$

$$U(\text{S2})=\cancel{U_{\text{Math}}(\text{GOOD})}+U_{\text{Phys}}(\text{BAD})+U_{\text{Lit}}(\text{MEDIUM})$$

$$>$$

$$\cancel{U_{\text{Math}}(\text{GOOD})}+U_{\text{Phys}}(\text{MEDIUM})+U_{\text{Lit}}(\text{BAD})=U(\text{S1})$$

$$S3 \succ S4$$

$$U(\text{S3})=\cancel{U_{\text{Math}}(\text{MEDIUM})}+U_{\text{Phys}}(\text{MEDIUM})+U_{\text{Lit}}(\text{BAD})$$

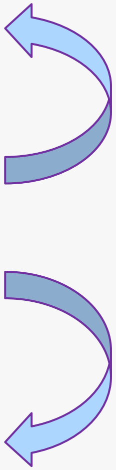
$$>$$

$$\cancel{U_{\text{Math}}(\text{MEDIUM})}+U_{\text{Phys}}(\text{BAD})+U_{\text{Lit}}(\text{MEDIUM})=U(\text{S4})$$

contradiction !

Additive value function: violation of preferential independence

Students	Mathematics	Physics	Literature
S1	~~*Good*~~	*Medium*	*Bad*
S2	~~*Good*~~	*Bad*	*Medium*
S3	~~*Medium*~~	*Medium*	*Bad*
S4	~~*Medium*~~	*Bad*	*Medium*



$S2 \succ S1$ and $S3 \succ S4$

Choquet integral

- Numerical encoding on a unique scale of the evaluations on each criterion:

Students	Mathematics	Physics	Literature
S1	*Good*	*Medium*	*Bad*
S2	*Good*	*Bad*	*Medium*
S3	*Medium*	*Medium*	*Bad*
S4	*Medium*	*Bad*	*Medium*

Students	Mathematics	Physics	Literature
S1	1	0.5	0
S2	1	0	0.5
S3	0.5	0.5	0
S4	0.5	0	0.5

Choquet integral

- Definition

$$C_\mu(a) = \sum_{i=1}^{n}\left[u\left(g_{(i)}(a)\right) - u\left(g_{(i-1)}(a)\right)\right] \times \mu(R_i)$$

where $u(g_{(i)})$ is encoding $g_{(i)}$ on a common numerical scale,

$_{(\cdot)}$ stands for the permutation of the indices of criteria:

$$g_{(n)}(a) \succeq^{I} g_{(n\text{-}1)}(a) \succeq^{I} \ldots \succeq^{I} g_{(1)}(a)$$

$\mu(R_i)$ is called capacity of R_i, i.e. weight for subset of criteria R_i

$R_i = \{(i),\ldots,(n)\}$, $i=1,\ldots,n$, $u(g_{(0)}) = 0$

Choquet integral

- The Choquet integral is not able to represent the dean's preferences $S2 \succ S3$ and $S3 \succ S4$ for any order-preserving numerical encoding u of the evaluations and for any values of capacity μ

- $S2 \succ S1$ means $C_\mu(S2) > C_\mu(S1)$, which implies

$$\cancel{[u(\text{Good})-u(\text{Medium})]}\times\mu(\{\text{Math}\}) + [u(\text{Medium})-u(\text{Bad})]\times\mu(\{\text{Math},\text{Lit}\})$$

$$>$$

$$\cancel{[u(\text{Good})-u(\text{Medium})]}\times\mu(\{\text{Math}\}) + [u(\text{Medium})-u(\text{Bad})]\times\mu(\{\text{Math},\text{Phys}\})$$

- $S3 \succ S4$ means $C_\mu(S3) > C_\mu(S4)$, which implies

$$[u(\text{Medium})-u(\text{Bad})]\times\mu(\{\text{Math},\text{Phys}\}) > [u(\text{Medium})-u(\text{Bad})]\times\mu(\{\text{Math},\text{Lit}\})$$

contradiction !

Choquet integral

- Bipolar Choquet integral could handle this interaction but it is yet less intuitive
- Problems with respect to numerical encoding. How to transform performances on criteria into numerical values of a common scale? Questions, like: „is maximum speed of 180km/h worth a fuel consumption of 12 l/100km”?
- Problems with respect to non-additive weights (capacity). How to translate the possible interaction among criteria into the capacities? Is there an intelligible relation between the preference information provided by the DM and the obtained value of the capacity? (see Mayag, Grabisch, Labreuche, 2008; Gonzales, Perny, 2005)
- Problems with respect to interpretation of the Choquet integral. Is it possible to clearly justify preference of alternative *a* over alternative *b* in terms of values of the integral’s components? (see Roy, 2009)

We propose to enrich the additive value function...

- We consider a value function of the type

$$U^{int}(a) = \sum_{i=1}^{n} u_i(g_i(a)) + $$

$$+ \underbrace{\sum_{(i_1,i_2)\in Syn^+} syn^+_{i_1,i_2}\left(g_{i_1}(a), g_{i_2}(a)\right)}_{\text{„bonus”}} - \underbrace{\sum_{(i_1,i_2)\in Syn^-} syn^-_{i_1,i_2}\left(g_{i_1}(a), g_{i_2}(a)\right)}_{\text{„malus”}}$$

Syn^+ is the set of pairs of criteria in a positive interaction

Syn^- is the set of pairs of criteria in a negative interaction

$Syn^+ \cap Syn^- = \varnothing$

$syn^+_{i_1,i_2}(\cdot,\cdot), syn^-_{i_1,i_2}(\cdot,\cdot)$ are non-decreasing functions in the two arguments

Illustrative example: value function U^{int}

Students	Mathematics	Physics	Literature
S1	*Good*	*Medium*	*Bad*
S2	*Good*	*Bad*	*Medium*
S3	*Medium*	*Medium*	*Bad*
S4	*Medium*	*Bad*	*Medium*

Preferences of the dean: $S2 \succ S1$ and $S3 \succ S4$

violate the principle of preferential independence

Illustrative example: value function U^{int}

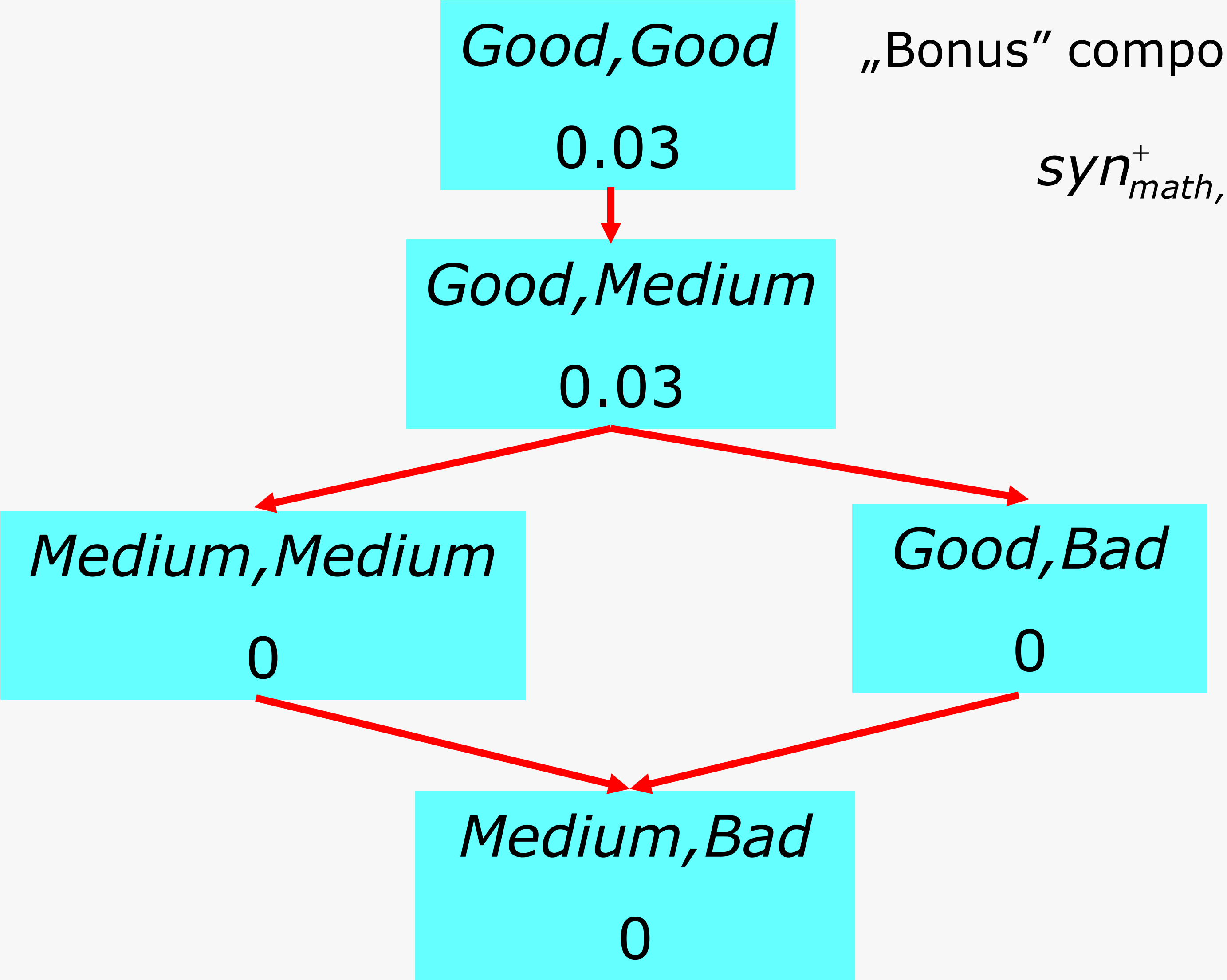
	Mathematics	Physics	Literature
Good	0.03	0.62	0.32
Medium	0.02	0.28	0.26
Bad	0	0.01	0

Non-interacting part of U^{int}

Assuming Mathematics & Literature in positive interaction

„Bonus" component of U^{int}

$syn^{+}_{math,lit}(\cdot,\cdot)$



Good,Good

0.03

Good,Medium

0.03

Medium,Medium

0

Good,Bad

0

Medium,Bad

0

Illustrative example

Students	*math*	*phys*	*lit*	$syn^{+}_{math,lit}(\cdot,\cdot)$	Total score
S1	*Good* 0.03	*Medium* 0.28	*Bad* 0	*Good,Bad* 0	0.31
S2	*Good* 0.03	*Bad* 0	*Medium* 0.26	*Good,Medium* 0.03	0.32
S3	*Medium* 0.02	*Medium* 0.28	*Bad* 0	*Medium,Bad* 0	0.30
S4	*Medium* 0.02	*Bad* 0	*Medium* 0.26	*Medium,Medium* 0	0.28

S2 $\succ$ S1 and S3 $\succ$ S4

(0.32 > 0.31 and 0.30 > 0.28)

Advantages of the new value function when used within ROR

- With respect to numerical encoding: we do not need an a priori expression of all the evaluations on a common numerical scale; i.e. the marginal value functions are not supposed to be known

- Problems with respect to non-addtive weights (capacity): we do not need non-additive weights; the value function is computed using ROR and even does not need to be shown to the Decision Maker

- Problems with interpretation of the interaction components: interpretation of „bonus” and „malus” with respect to the sum of marginal values is cognitively simple

Interactions between two criteria

- Positive interactions (e.g., maximum speed & price of a car):

$$u_{i_1,i_2}\left(x_{i_1},x_{i_2}\right) > u_{i_1}\left(x_{i_1}\right) + u_{i_2}\left(x_{i_2}\right)$$

- Negative interactions (e.g., maximum speed & acceleration of a car):

$$u_{i_1,i_2}\left(x_{i_1},x_{i_2}\right) < u_{i_1}\left(x_{i_1}\right) + u_{i_2}\left(x_{i_2}\right)$$

- $I^{(2)} = \{\{i1, i2\}: i1, i2 \in I\}$, $x_{i1} \in X_{i1}$, $x_{i2} \in X_{i2}$
- $Syn^{+} \subseteq I^{(2)}$, set of pairs of criteria for which there is a positive synergy
- $Syn^{-} \subseteq I^{(2)}$, set of pairs of criteria for which there is a negative synergy
- Synergy strength is measured by functions $syn^{+}_{i_1,i_2} : X_{i_1} \times X_{i_2} \rightarrow [0,1]$, $syn^{-}_{i_1,i_2} : X_{i_1} \times X_{i_2} \rightarrow [0,1]$ not decreasing in both arguments, called „bonus" and „malus"

Interactions between two criteria

- Positive interactions (e.g., maximum speed & price of a car):

$$u_{i_1,i_2}\left(x_{i_1},x_{i_2}\right)=u_{i_1}\left(x_{i_1}\right)+u_{i_2}\left(x_{i_2}\right)+syn^{+}_{i_1,i_2}\left(x_{i_1},x_{i_2}\right)$$

- Negative interactions (e.g., maximum speed & acceleration of a car):

$$u_{i_1,i_2}\left(x_{i_1},x_{i_2}\right)=u_{i_1}\left(x_{i_1}\right)+u_{i_2}\left(x_{i_2}\right)-syn^{-}_{i_1,i_2}\left(x_{i_1},x_{i_2}\right)$$

- $I^{(2)}=\{\{i1, i2\}: i1,i2\in I\}$, $x_{i1}\in X_{i1}$, $x_{i2}\in X_{i2}$
- $Syn^{+}\subseteq I^2$, set of pairs of criteria for which there is a positive synergy
- $Syn^{-}\subseteq I^2$, set of pairs of criteria for which there is a negative synergy
- Synergy strength is measured by
  functions $syn^{+}_{i_1,i_2} : X_{i_1}\times X_{i_2}\rightarrow[0,1]$, $syn^{-}_{i_1,i_2} : X_{i_1}\times X_{i_2}\rightarrow[0,1]$
  not decreasing in both arguments, called „bonus” and „malus”
  $syn^{+}_{i_1,i_2}\left(x_{i_1},x_{i_2}\right)\times syn^{-}_{i_1,i_2}\left(x_{i_1},x_{i_2}\right)=0,\ \forall\left(i_1,i_2\right)\in I^2,\ \forall\left(x_{i_1},x_{i_2}\right)\in X_{i_1}\times X_{i_2}$

UTA^{GMS}–*INT*

- We consider a value function of the type

$$U^{int}(a)=\sum_{i=1}^{n}u_i(g_i(a))+$$

$$+\underbrace{\sum_{(i_1,i_2)\in Syn^+} syn^+_{i_1,i_2}(g_{i_1}(a),g_{i_2}(a))}_{\text{„bonus”}}\;\text{GMS}\;-\underbrace{\sum_{(i_1,i_2)\in Syn^-} syn^-_{i_1,i_2}(g_{i_1}(a),g_{i_2}(a))}_{\text{„malus”}}$$

- Preference information elicited by the DM is the same as in the *GRIP* method:
    - pairwise comparisons of some reference alternatives $a',b'\in A'$ (partial preorder $\succeq$ on A' – set of reference alternatives)
    - ordinal intensity of preference for quadruples of reference alternatives $a',b',c',d'\in A'$, comprehensively or on specific criteria (partial preorder $\succeq^*$ or $\succeq_i^*$ on $A'\times A'$)

Three options to consider interaction

- Considering a value function of the type

$$U^{int}(a)=\sum_{i=1}^{n}u_i(g_i(a))+$$

$$+\underbrace{\sum_{(i_1,i_2)\in Syn^+} syn^+_{i_1,i_2}\left(g_{i_1}(a),g_{i_2}(a)\right)}_{\text{„bonus”}} - \underbrace{\sum_{(i_1,i_2)\in Syn^-} syn^-_{i_1,i_2}\left(g_{i_1}(a),g_{i_2}(a)\right)}_{\text{„malus”}}$$

GMS

- α) bonus and malus are not mutually exclusive, so that positive and negative synergies interplay,
- β) bonus and malus are mutually exclusive,
- γ) only one of the two synergies is considered, either the positive, or the negative

Compatible value functions

preference statements of the DM

$$\left.\begin{array}{l} U^{int}(a') \geq U^{int}(b') + \varepsilon \ \text{ if } \ a' \succ b' \\ U^{int}(a') = U^{int}(b') \ \text{ if } \ a' \sim b' \\ U^{int}(a') - U^{int}(b') \geq U^{int}(c') - U^{int}(d') \ \text{ if } \ (a',b') \succsim^{*} (c',d') \\ u_i(g_i(a')) - u_i(g_i(b')) \geq u_i(g_i(c')) - u_i(g_i(d')), \\ \quad \text{if } \ (a',b') \succsim_i^{*} (c',d'), \ i \in I \end{array}\right\} a',b',c',d' \in A' \qquad (28)$$

$$U^{int}(a') \geq U^{int}(b') \ \text{ if } \ g_i(a') \geq g_i(b') \text{ for } i = 1, \ldots, n, \ \text{ and for all } a', b' \in A' \qquad (29)$$

monotonicity of U^{int}

$$u_i(g_i(a'_{\tau_i(j)})) - u_i(g_i(a'_{\tau_i(j-1)})) \geq 0, \ i = 1, ..., n, \ j = 2, ..., m' \qquad (30)$$

$$u_i(g_i(a'_{\tau_i(j)})) - u_i(g_i(a'_{\tau_i(j-1)})) = 0, \ \text{if } g_i(a'_{\tau_i(j)})) = g_i(a'_{\tau_i(j-1)}), \ i = 1, ..., n, \ j = 2, ..., m' \qquad (31)$$

$$u_i(g_i(a'_{\tau_i(1)})) \geq 0, \ i = 1, ..., n \qquad (32)$$

monotonicity of the non-interacting part of U^{int}

$$u_i(x_{i*}) = 0, \ i = 1, ..., n \qquad (33)$$

$$U^{int}(x_1^*, \ldots, x_n^*) = 1 \qquad (34)$$

$$U^{int}(a') \geq 0, \ \text{ for all } a' \in A' \qquad (35)$$

normalization and non-negativity of U^{int}

Compatible value functions (cont.)

monotonicity of the bonus and malus functions

$$syn^{+}_{i_1,i_2}(g_{i_1}(a'),g_{i_2}(a')) \geq syn^{+}_{i_1,i_2}(g_{i_1}(b'),g_{i_2}(b')),$$

$$(i_1,i_2) \in I^2,\ i_1 > i_2,\ \text{if } g_{i_1}(a') \geq g_{i_1}(b') \text{ and } g_{i_2}(a') \geq g_{i_2}(b'),\quad \text{for all } a',b' \in A' \tag{36}$$

$$syn^{+}_{i_1,i_2}(g_{i_1}(a'),g_{i_2}(a')) = syn^{+}_{i_1,i_2}(g_{i_1}(b'),g_{i_2}(b')),$$

$$(i_1,i_2) \in I^2,\ i_1 > i_2,\ \text{if } g_{i_1}(a') = g_{i_1}(b') \text{ and } g_{i_2}(a') = g_{i_2}(b'),\quad \text{for all } a',b' \in A' \tag{37}$$

$$syn^{-}_{i_1,i_2}(g_{i_1}(a'),g_{i_2}(a')) \geq syn^{-}_{i_1,i_2}(g_{i_1}(b'),g_{i_2}(b')),$$

$$(i_1,i_2) \in I^2,\ i_1 > i_2,\ \text{if } g_{i_1}(a') \geq g_{i_1}(b') \text{ and } g_{i_2}(a') \geq g_{i_2}(b'),\quad \text{for all } a',b' \in A' \tag{38}$$

$$syn^{-}_{i_1,i_2}(g_{i_1}(a'),g_{i_2}(a')) = syn^{-}_{i_1,i_2}(g_{i_1}(b'),g_{i_2}(b')),$$

$$(i_1,i_2) \in I^2,\ i_1 > i_2,\ \text{if } g_{i_1}(a') = g_{i_1}(b') \text{ and } g_{i_2}(a') = g_{i_2}(b'),\quad \text{for all } a',b' \in A' \tag{39}$$

$$syn^{+}_{i_1,i_2}(x_{i_1*},x_{i_2*}) = 0,\quad syn^{-}_{i_1,i_2}(x_{i_1*},x_{i_2*}) = 0,\quad (i_1,i_2) \in I \times I,\ i_1 > i_2 \tag{40}$$

$$u_{i_1}(g_{i_1}(a')) + u_{i_2}(g_{i_2}(a')) - syn^{-}_{i_1,i_2}(g_{i_1}(a'),g_{i_2}(a')) \geq u_{i_1}(g_{i_1}(b')) + u_{i_2}(g_{i_2}(b'))$$

$$-syn^{-}_{i_1,i_2}(g_{i_1}(b'),g_{i_2}(b')),\ (i_1,i_2) \in I^2,\quad i1>i2,\ \forall a',b' \in A',\ \text{if } g_{i1}(a') \underline{\geq} g_{i1}(b'),\ g_{i2}(a') \underline{\geq} g_{i2}(b') \tag{41}$$

$$x_{i*} \leq g_i(a_{\tau_i(1)}) \leq g_i(a_{\tau_i(2)}) \leq \ldots \leq g_i(a_{\tau_i(m'-1)}) \leq g_i(a_{\tau_i(m')}) \leq x_i^*$$

Eliciting a minimal set of pairs of interacting criteria

- Let us introduce two binary variables:

$$\left\{\begin{array}{l}\left\{\begin{array}{l}\delta^{+}_{i_1 i_2} = 1 \text{ iff } (i_1, i_2) \text{ are positively interacting,}\\ \delta^{+}_{i_1 i_2} = 0 \text{ otherwise,}\end{array}\right.\\ \left\{\begin{array}{l}\delta^{-}_{i_1 i_2} = 1 \text{ iff } (i_1, i_2) \text{ are negatively interacting,}\\ \delta^{-}_{i_1 i_2} = 0 \text{ otherwise.}\end{array}\right.\end{array}\right\}$$

- Any pair of criteria can be either in positive or in negative interaction:

$$\boxed{\delta^{+}_{i_1 i_2} + \delta^{-}_{i_1 i_2} \leq 1, \quad \text{for all } (i_1, i_2) \in I \times I,\ i_1 > i_2}$$

Eliciting a minimal set of pairs of interacting criteria

- In order to find a minimal set of pairs of criteria with either positive or negative interactions, one has to solve the following program (*P*)

$$\text{Minimize } z = \sum_{i_1<i_2, i_1, i_2 \in I} (\delta^+_{i_1 i_2} + \delta^-_{i_1 i_2})$$

subject to

$$\text{constraints } (28) - (41)$$

$$\delta^+_{i_1 i_2} + \delta^-_{i_1 i_2} \leq 1, \quad \text{for all } (i_1, i_2) \in I \times I,\ i_1 > i_2 \tag{44}$$

$$syn^+_{i_1, i_2}(x^*_{i_1}, x^*_{i_2}) \leq \delta^+_{i_1 i_2}, \quad \text{for all } (i_1, i_2) \in I \times I,\ i_1 > i_2 \tag{45}$$

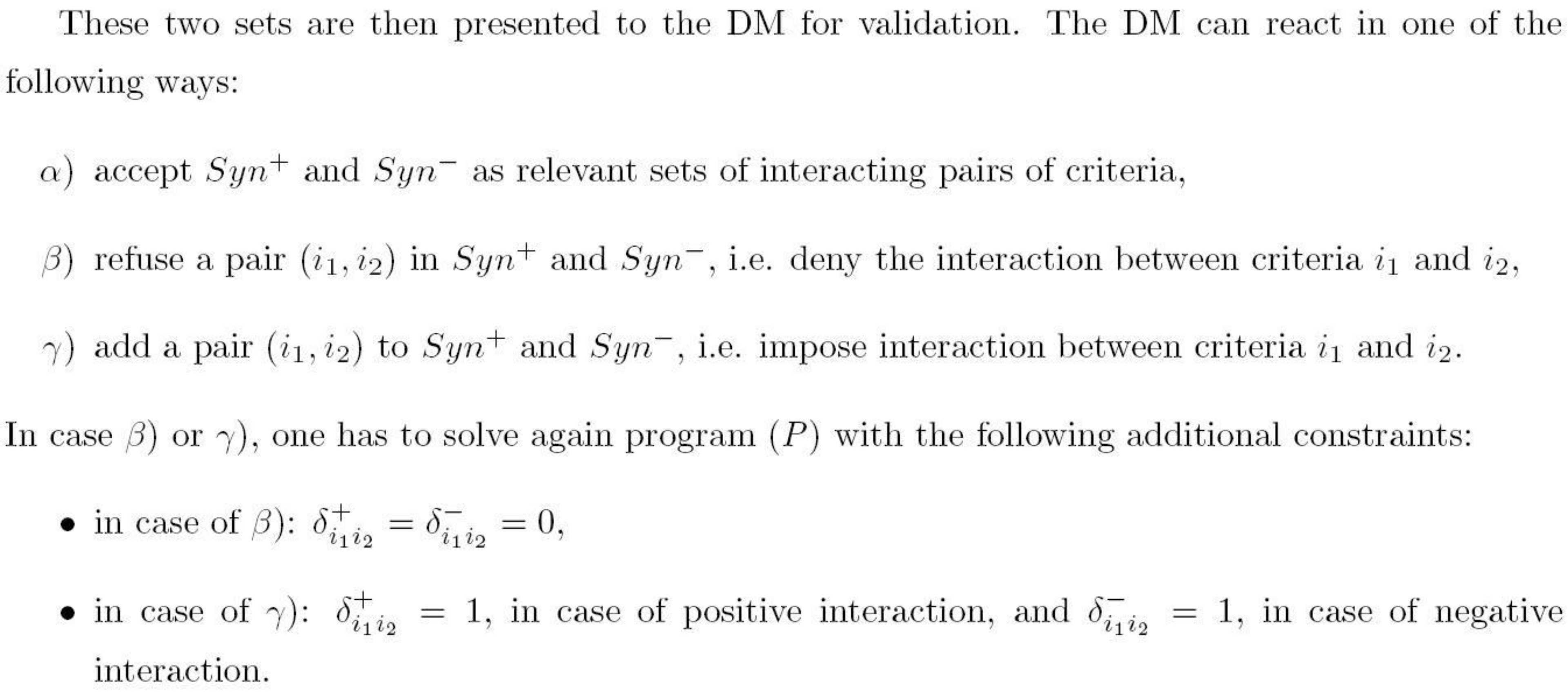
$$syn^-_{i_1, i_2}(x^*_{i_1}, x^*_{i_2}) \leq \delta^-_{i_1 i_2}, \quad \text{for all } (i_1, i_2) \in I \times I,\ i_1 > i_2 \tag{46}$$

$$\delta^+_{i_1 i_2}, \delta^-_{i_1 i_2} \in \{0, 1\}, \quad \text{for all } (i_1, i_2) \in I \times I,\ i_1 > i_2, \tag{47}$$

- (*P*) yields a value function U^{int} involving a minimal set of pairs of interacting criteria

$$Syn^+ = \{(i_1, i_2) \in I \times I,\ i_1 > i_2 : \delta^+_{i_1 i_2} = 1\} \text{ and } Syn^- = \{(i_1, i_2) \in I \times I,\ i_1 > i_2 : \delta^-_{i_1 i_2} = 1\}$$

Eliciting a minimal set of pairs of interacting criteria



These two sets are then presented to the DM for validation. The DM can react in one of the following ways:

$\alpha)$ accept Syn^+ and Syn^- as relevant sets of interacting pairs of criteria,

$\beta)$ refuse a pair (i_1, i_2) in Syn^+ and Syn^-, i.e. deny the interaction between criteria i_1 and i_2,

$\gamma)$ add a pair (i_1, i_2) to Syn^+ and Syn^-, i.e. impose interaction between criteria i_1 and i_2.

In case $\beta)$ or $\gamma)$, one has to solve again program (P) with the following additional constraints:

- in case of $\beta)$: $\delta^+_{i_1 i_2} = \delta^-_{i_1 i_2} = 0$,
- in case of $\gamma)$: $\delta^+_{i_1 i_2} = 1$, in case of positive interaction, and $\delta^-_{i_1 i_2} = 1$, in case of negative interaction.

Computing necessary and possible preference relations

- To confirm relation $a \succsim^N b$ it is enough to check if $\varepsilon^* \leq 0$ or the set of constraints is infeasible, where ε^*:

Maximize: ε

subject to

$$U^{int}(b) \geq U^{int}(a) + \varepsilon, \qquad (48)$$

can *b* be preferred to *a* for some compatible U^{int} ?

constraints (28), (33), (34)

$$U^{int}(a') \geq U^{int}(b') \ \text{ if } \ g_i(a') \geq g_i(b'), \ \text{for } i = 1, \ldots, n, \ \text{ and for all } a', b' \in A' \cup \{a, b\} \qquad (49)$$

$$u_i(g_i(a'_{\pi_i(j)})) - u_i(g_i(a'_{\pi_i(j-1)})) \geq 0, \ i = 1, ..., n, \ j = 2, ..., m' + \omega \qquad (50)$$

$$u_i(g_i(a'_{\pi_i(j)})) - u_i(g_i(a'_{\pi_i(j-1)})) = 0, \ \text{if } g_i(a'_{\pi_i(j)})) = g_i(a'_{\pi_i(j-1)}), \ i \in I, \ j = 2, ..., m' + \omega \qquad (51)$$

$$u_i(g_i(a'_{\pi_i(1)})) \geq 0, \ i = 1, ..., n \qquad (52)$$

$$U^{int}(a') \geq 0, \ \text{ for all } a' \in A' \cup \{a, b\} \qquad (53)$$

monotonicity of of U^{int}, of the non-interacting part of U^{int} and non-negativity of U^{int} for *A′* augmented by {*a*,*b*}

Computing necessary and possible preference relations (cont.)

$$syn^{+}_{i_1,i_2}(g_{i_1}(a'), g_{i_2}(a')) \geq syn^{+}_{i_1,i_2}(g_{i_1}(b'), g_{i_2}(b')), \text{ for all } (i_1, i_2) \in Syn^{+}, \tag{54}$$

$$\text{if } g_{i_1}(a') \geq g_{i_1}(b') \text{ and } g_{i_2}(a') \geq g_{i_2}(b'), \quad \text{for all } a', b' \in A' \cup \{a, b\}$$

$$syn^{+}_{i_1,i_2}(g_{i_1}(a'), g_{i_2}(a')) = syn^{+}_{i_1,i_2}(g_{i_1}(b'), g_{i_2}(b')), \text{ for all } (i_1, i_2) \in Syn^{+}, \tag{55}$$

$$\text{if } g_{i_1}(a') = g_{i_1}(b') \text{ and } g_{i_2}(a') = g_{i_2}(b'), \quad \text{for all } a', b' \in A' \cup \{a, b\}$$

$$syn^{-}_{i_1,i_2}(g_{i_1}(a'), g_{i_2}(a')) \geq syn^{-}_{i_1,i_2}(g_{i_1}(b'), g_{i_2}(b')), \text{ for all } (i_1, i_2) \in Syn^{-}, \tag{56}$$

$$\text{if } g_{i_1}(a') \geq g_{i_1}(b') \text{ and } g_{i_2}(a') \geq g_{i_2}(b'), \quad \text{for all } a', b' \in A' \cup \{a, b\}$$

$$syn^{-}_{i_1,i_2}(g_{i_1}(a'), g_{i_2}(a')) = syn^{-}_{i_1,i_2}(g_{i_1}(b'), g_{i_2}(b')), \text{ for all } (i_1, i_2) \in Syn^{-}, \tag{57}$$

$$\text{if } g_{i_1}(a') = g_{i_1}(b') \text{ and } g_{i_2}(a') = g_{i_2}(b'), \quad \text{for all } a', b' \in A' \cup \{a, b\}$$

$$syn^{+}_{i_1,i_2}(x_{i_1*}, x_{i_2*}) = 0, \quad syn^{-}_{i_1,i_2}(x_{i_1*}, x_{i_2*}) = 0, \quad \text{for all } (i_1, i_2) \in Syn^{+} \cup Syn^{-} \tag{58}$$

$$u_{i_1}(g_{i_1}(a')) + u_{i_2}(g_{i_2}(a')) - syn^{-}_{i_1,i_2}(g_{i_1}(a'), g_{i_2}(a')) \geq \tag{59}$$

$$u_{i_1}(g_{i_1}(b')) + u_{i_2}(g_{i_2}(b')) - syn^{-}_{i_1,i_2}(g_{i_1}(b'), g_{i_2}(b')),$$

monotonicity of the interacting part of U^{int} for A' augmented by $\{a,b\}$

$\forall(i1,i2) \in Syn^{-}$, $\forall a',b' \in A' \cup \{a,b\}$,
if $g_{i1}(a') \geq g_{i1}(b')$, $g_{i2}(a') \geq g_{i2}(b')$

$$U^{int}(a) = \sum_{i=1}^{n} u_i(a) + \sum_{(i_1,i_2) \in Syn+} syn^{+}_{i_1,i_2}(g_{i_1}(a), g_{i_2}(a)) - \sum_{(i_1,i_2) \in Syn-} syn^{-}_{i_1,i_2}(g_{i_1}(a), g_{i_2}(a))$$

Computing necessary and possible preference relations

- To confirm relation $a \succsim^P b$ one has to check if the set of constraints is feasible and $\varepsilon^*>0$ in the previous problem, where constraint (48) is replaced by $U^{int}(a) \geq U^{int}(b)+ \varepsilon$
- Using $\succsim^N$ and $\succsim^P$ one can compute:
  - necessary ranking (partial preorder in A)
  - possible ranking (strongly complete and negatively transitive relation in A)
- If a score is needed to assign to the different alternatives, one can calculate a „representative” value function among all the compatible ones.

Example of application of *UTA^{GMS}-INT*

Illustrative example

- Performance matrix

Students	Mathematics (*math*)	Physics (*phys*)	Literature (*lit*)
S1	*Good*	*Medium*	*Bad*
S2	*Good*	*Bad*	*Medium*
S3	*Medium*	*Medium*	*Bad*
S4	*Medium*	*Bad*	*Medium*
S5	*Medium*	*Medium*	*Medium*
S6	*Medium*	*Good*	*Medium*
S7	*Good*	*Good*	*Bad*

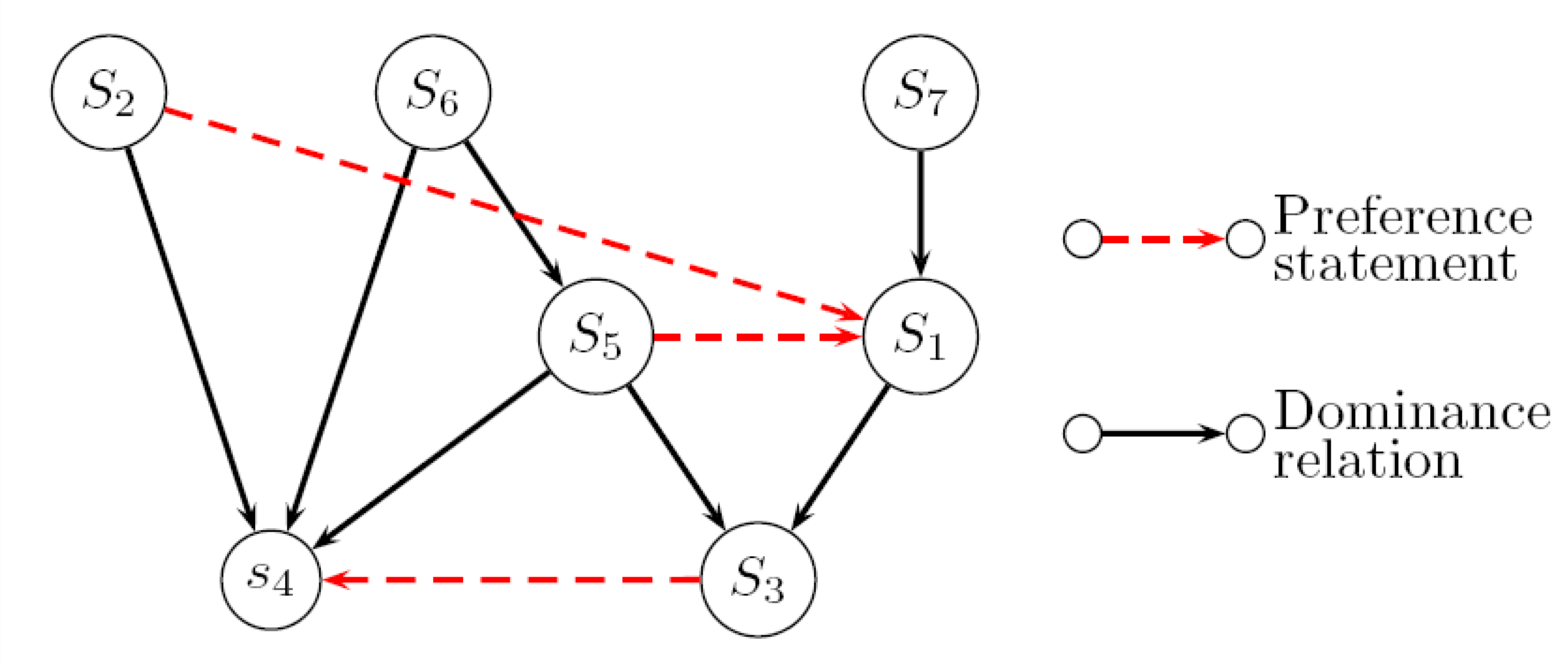
Preference information elicited by the DM

- Pairwise comarisons of (reference) students
  - S2 $\succ$ S1
  - S3 $\succ$ S4
  - S5 $\succ$ S1
- Overall intensity of preferences
  - (S3,S4) $\succ^*$ (S2,S1)
- Intensity of preference relative to single criteria
  - $(Medium, Bad) \succ_i^* (Good, Medium), \quad i=math, lit$
  - $(Good, Medium) \succ_i^* (Medium, Bad), \quad i=phys$

- $A' = \{S1, S2, S3, S4, S5\}$

Preference information given by the DM

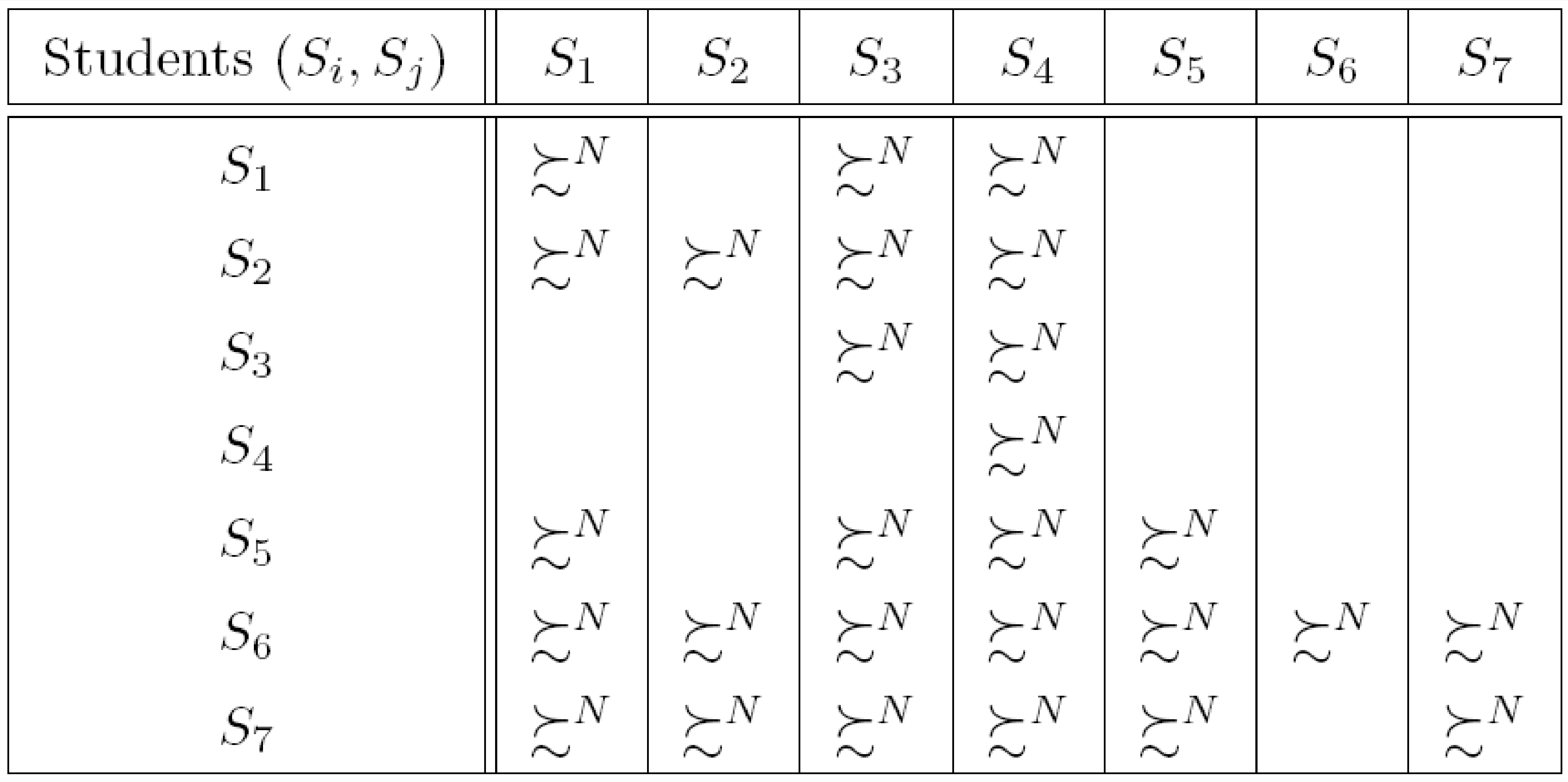
- Dominance relation and pairwise comparisons of reference students



Decision aiding procedure

- First, we solve program (*P*), and obtain at the optimum
  $\delta^+_{i1,i2} = \delta^-_{i1,i2} = 0$, for all $(i1,i2) \in I \times I$, except $\delta^-_{math,lit} = 1$
- This means there exists U^{int} compatible with the dean's preferences, involving a negative interaction between Mathematics and Literature
- Suppose that the dean is not willing to consider a negative interaction between these two criteria → $\delta^-_{math,lit} = 0$ enters (*P*) → (*P*′)
- Solving (*P*′), we get
  $\delta^+_{i1,i2} = \delta^-_{i1,i2} = 0$, for all $(i1,i2) \in I \times I$, except $\delta^-_{math,phys} = 1$
- Suppose that the dean is not willing to consider a negative interaction between these two criteria → $\delta^-_{math,phys} = 0$ enters (*P*′) → (*P*′′)
- Solving (*P*′′), we get
  $\delta^+_{i1,i2} = \delta^-_{i1,i2} = 0$, for all $(i1,i2) \in I \times I$, except $\delta^+_{math,lit} = 1$
- Suppose the dean accepts to consider this pos. interaction (*math*, *lit*)

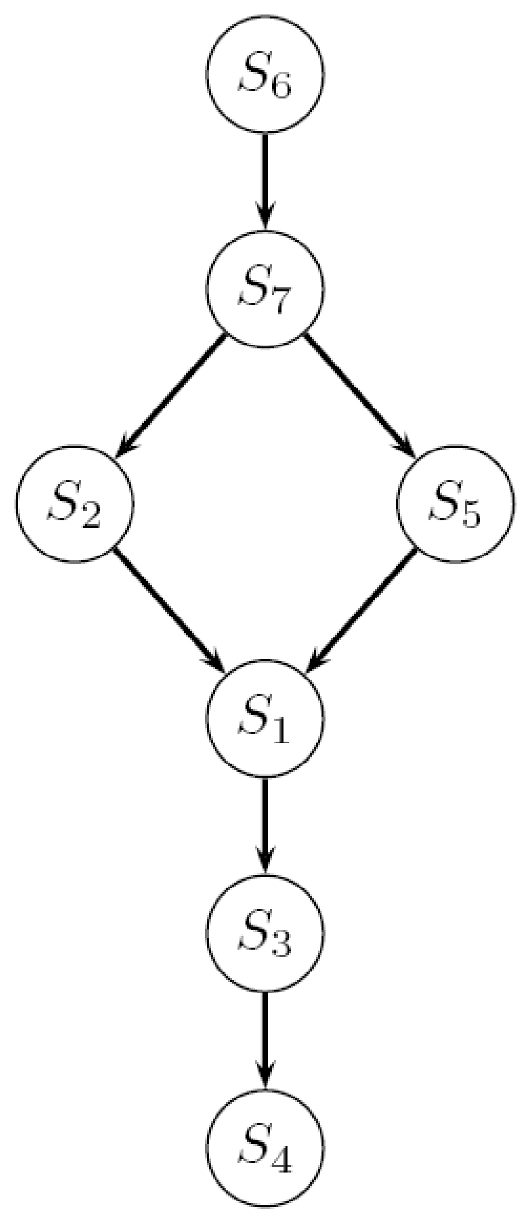
Necessary preference relation $\succsim^N$



Students (S_i, S_j)	S_1	S_2	S_3	S_4	S_5	S_6	S_7
S_1	$\succsim^N$		$\succsim^N$	$\succsim^N$			
S_2	$\succsim^N$	$\succsim^N$	$\succsim^N$	$\succsim^N$			
S_3			$\succsim^N$	$\succsim^N$			
S_4				$\succsim^N$			
S_5	$\succsim^N$		$\succsim^N$	$\succsim^N$	$\succsim^N$		
S_6	$\succsim^N$	$\succsim^N$	$\succsim^N$	$\succsim^N$	$\succsim^N$	$\succsim^N$	$\succsim^N$
S_7	$\succsim^N$	$\succsim^N$	$\succsim^N$	$\succsim^N$	$\succsim^N$		$\succsim^N$

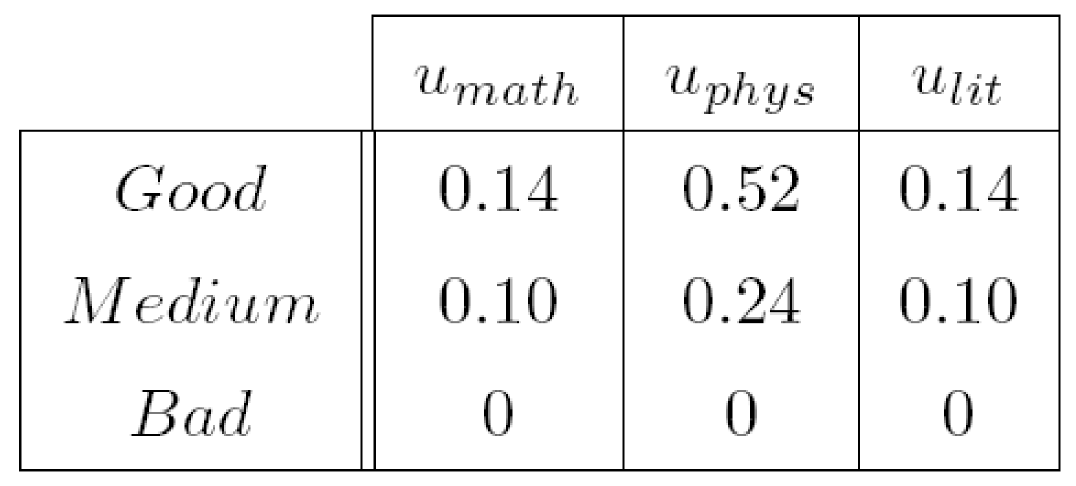
Necessary preference relation $\succsim^N$ – graphical representation

- Using this information, the dean is able to identify the best student, even if the value function U^{int} is not unique



Representative value function U^{int}

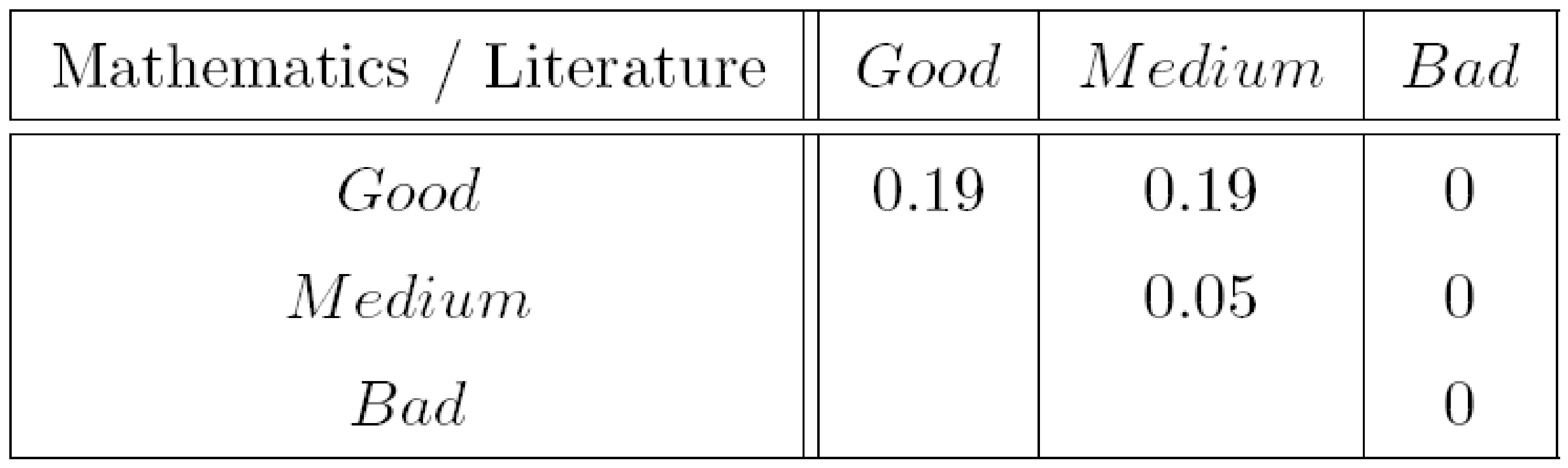
- In order to give a score to each student, we compute a representative value function
- Non-interacting part of the representative U^{int}



	u_{math}	u_{phys}	u_{lit}
Good	0.14	0.52	0.14
Medium	0.10	0.24	0.10
Bad	0	0	0

Representative value function U^{int}

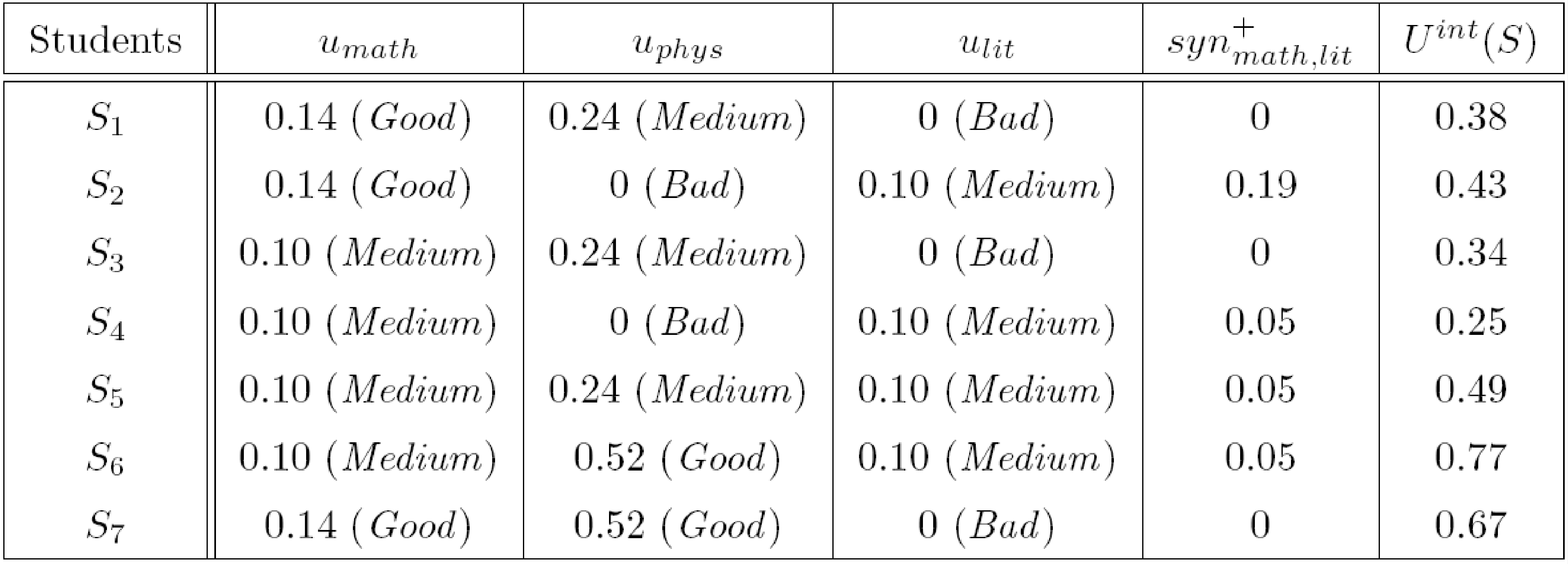
- „Bonus” component $syn^{+}_{math,lit}$ of the representative U^{int}



Mathematics / Literature	*Good*	*Medium*	*Bad*
Good	0.19	0.19	0
Medium		0.05	0
Bad			0

Representative value function U^{int} – scores and ranking

- „Bonus" component $syn^{+}_{math,lit}$ of the representative U^{int} and scores of students



Students	u_{math}	u_{phys}	u_{lit}	$syn^{+}_{math,lit}$	$U^{int}(S)$
S_1	$0.14\ (Good)$	$0.24\ (Medium)$	$0\ (Bad)$	0	0.38
S_2	$0.14\ (Good)$	$0\ (Bad)$	$0.10\ (Medium)$	0.19	0.43
S_3	$0.10\ (Medium)$	$0.24\ (Medium)$	$0\ (Bad)$	0	0.34
S_4	$0.10\ (Medium)$	$0\ (Bad)$	$0.10\ (Medium)$	0.05	0.25
S_5	$0.10\ (Medium)$	$0.24\ (Medium)$	$0.10\ (Medium)$	0.05	0.49
S_6	$0.10\ (Medium)$	$0.52\ (Good)$	$0.10\ (Medium)$	0.05	0.77
S_7	$0.14\ (Good)$	$0.52\ (Good)$	$0\ (Bad)$	0	0.67

- S6 ≻ S7 ≻ S5 ≻ S2 ≻ S1 ≻ S3 ≻ S4

SMAA and Robust Ordinal Regression

SMAA methods (Lahdelma et al. 1998)

Basic assumptions:

- Imprecision or lack of data (weights of criteria and evaluations of alternatives over criteria)
- density function $f_W(w)$ over the weight space W,
- density function $f_\chi(\xi)$ over evaluation space $\chi \subseteq \mathbb{R}^{m \times n}$,

Computations for each alternative of:

- Rank acceptability index

$$b_j^r = \int_{\xi \in \chi} f_\chi(\xi) \int_{w \in W_j^r(\xi)} f_W(w)\, dw\, d\xi$$

- Pairwise winning index:

$$p(a_h, a_k) = \int_{w \in W} f_W(w) \int_{\xi \in \chi : u(\xi_h, w) \geq u(\xi_k,, w)} f_\chi(\xi)\, d\xi dw$$

where $W_j^r(\xi) = \{w \in W : rank(j, \xi, w) = r\}$.

ROR and SMAA under a unified framework

- Very often $a \succsim^P b$ and $b \succsim^P a$. Therefore it is interesting to know which is the frequency of the preference of a over b and viceversa the frequency of the preference of b over a or the frequency with which an alternative fills the k-th position in a ranking and so on.
- The link between ROR and SMAA has been already investigated in M. Kadziński, and T. Tervonen (2013a,2013b).

Indirect preference information

- indirect elicitation based on Ordinal Regression, (E. Jacquet-Lagreze and Y. Siskos 1982, 2001; Y. Siskos and E. Grigoroudis 2010);
- robust indirect elicitation based on Robust Ordinal Regression (ROR) (S. Corrente, S. Greco, M. Kadziński, R. Słowiński 2013,2014; S. Greco, V. Mousseau and R. Słowiński 2008);
- stochastic indirect elicitation based on Stochastic Ordinal Regression (SOR) (M. Kadziński and T. Tervonen 2013a, 2013b).

SOR: ROR and SMAA under a unified framework

- Very often $a \succsim^P b$ and $b \succsim^P a$. Therefore it is interesting to know which is the frequency of the preference of a over b and viceversa the frequency of the preference of b over a or the frequency with which an alternative fills the k-th position in a ranking and so on.
- The link between ROR and SMAA has been already investigated in M. Kadziński and T. Tervonen 2013a,2013b.

SMAA-Choquet
(Angilella, Corrente, Greco 2012, 2014)

SMAA Choquet

On the DM's preference constraints, a sampling of compatible preference parameters (Möbius measures) is obtained by a *Hit-and-Run* algoritm (Smith, 1984) that is outperformed for a maximum number of iterations.

The Hit and Run algorithm

At each iteration generates a candidate point (the preference parameters) that is uniformly distributed along a randomly chosen direction within the feasible region defined by the DM's preference information.

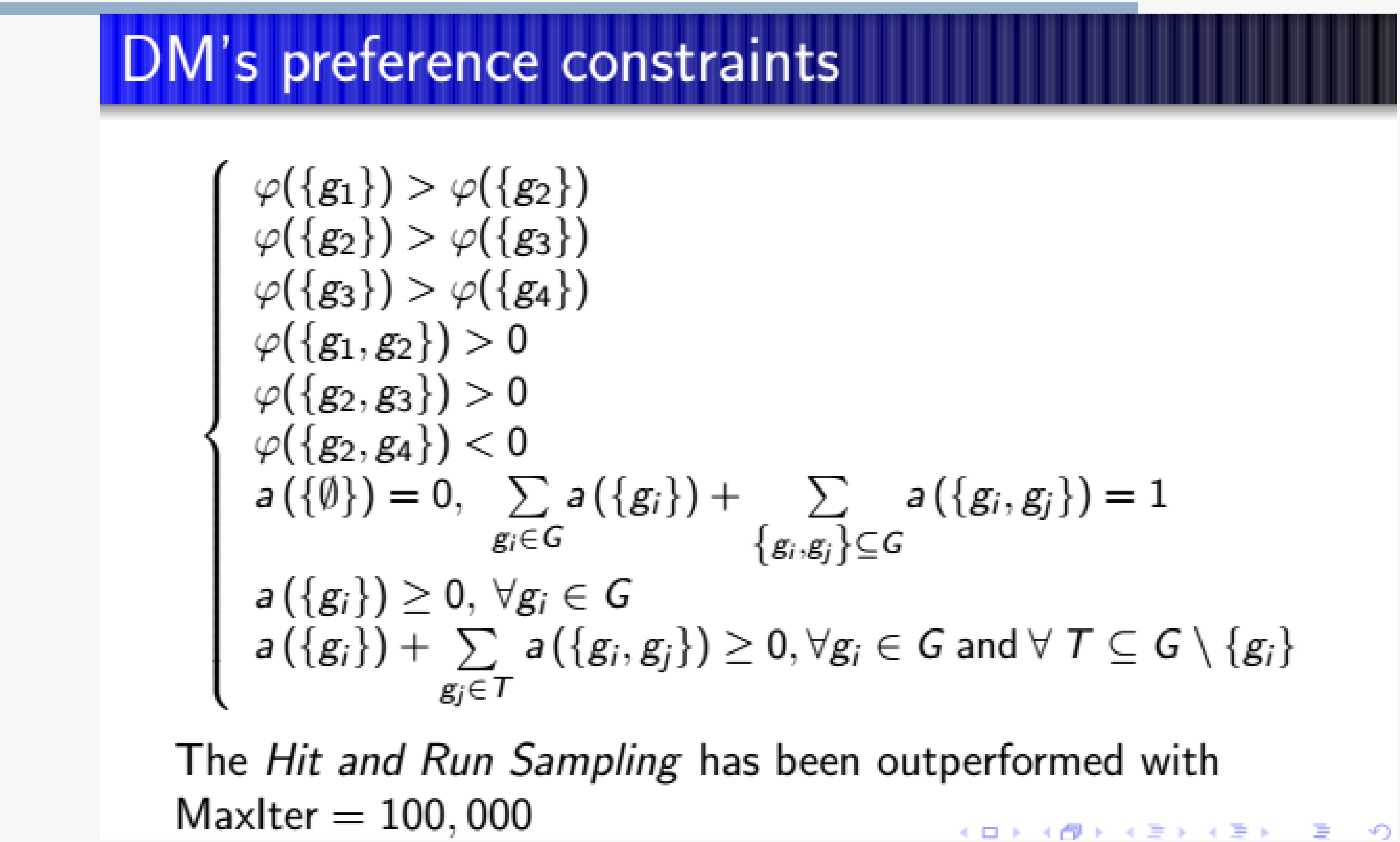
Indices of SMAA-Choquet

Computation of:

- the rank acceptability index of every alternative by considering the different compatible preference parameters (the Möbius measures obtained after each iteration) giving to alternative $a_j \in A$ the rank r on the basis of a utility function expressed in terms of a Choquet integral.
- the Möbius measures corresponding to the capacities for which the Choquet integral ranks every alternative a_j as the best.

Evaluation matrix

	Alternatives								
Criteria	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
g_1	8	3	10	5	8	5	8	5	0
g_2	6	1	9	9	0	9	10	7	10
g_3	7	10	0	2	8	4	5	9	2
g_4	5	10	5	9	6	7	7	4	8

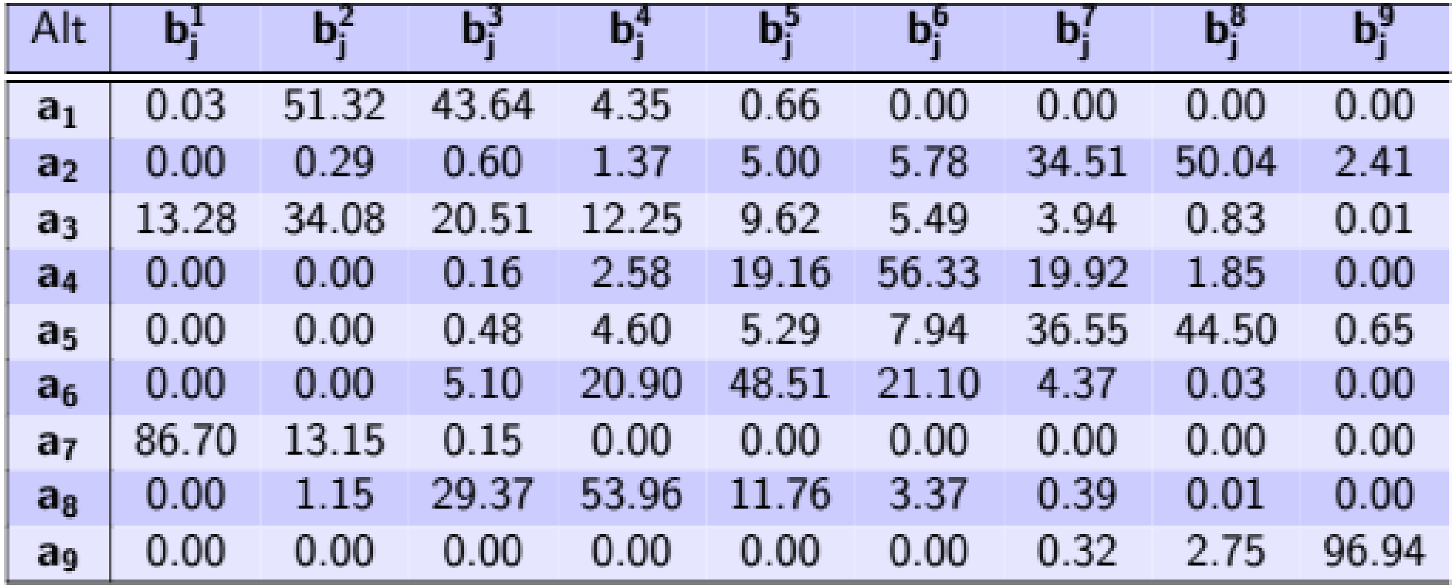


DM's preference constraints

$$\begin{cases} \varphi(\{g_1\}) > \varphi(\{g_2\}) \\ \varphi(\{g_2\}) > \varphi(\{g_3\}) \\ \varphi(\{g_3\}) > \varphi(\{g_4\}) \\ \varphi(\{g_1, g_2\}) > 0 \\ \varphi(\{g_2, g_3\}) > 0 \\ \varphi(\{g_2, g_4\}) < 0 \\ a(\{\emptyset\}) = 0, \quad \sum\limits_{g_i \in G} a(\{g_i\}) + \sum\limits_{\{g_i, g_j\} \subseteq G} a(\{g_i, g_j\}) = 1 \\ a(\{g_i\}) \geq 0, \ \forall g_i \in G \\ a(\{g_i\}) + \sum\limits_{g_j \in T} a(\{g_i, g_j\}) \geq 0, \forall g_i \in G \text{ and } \forall\, T \subseteq G \setminus \{g_i\} \end{cases}$$

The *Hit and Run Sampling* has been outperformed with MaxIter $= 100,000$

Rank acceptabilities (b_i) in percentages



Alt	b_j^1	b_j^2	b_j^3	b_j^4	b_j^5	b_j^6	b_j^7	b_j^8	b_j^9
a_1	0.03	51.32	43.64	4.35	0.66	0.00	0.00	0.00	0.00
a_2	0.00	0.29	0.60	1.37	5.00	5.78	34.51	50.04	2.41
a_3	13.28	34.08	20.51	12.25	9.62	5.49	3.94	0.83	0.01
a_4	0.00	0.00	0.16	2.58	19.16	56.33	19.92	1.85	0.00
a_5	0.00	0.00	0.48	4.60	5.29	7.94	36.55	44.50	0.65
a_6	0.00	0.00	5.10	20.90	48.51	21.10	4.37	0.03	0.00
a_7	86.70	13.15	0.15	0.00	0.00	0.00	0.00	0.00	0.00
a_8	0.00	1.15	29.37	53.96	11.76	3.37	0.39	0.01	0.00
a_9	0.00	0.00	0.00	0.00	0.00	0.00	0.32	2.75	96.94

First rank acceptability (b_1) and central weights

Alt	a_1	a_3	a_7
b_1	0.03	13.28	86.70
$a(\{1\})$	0.32799	0.4833	0.23032
$a(\{2\})$	0.1308	0.12818	0.15372
$a(\{3\})$	0.055755	0.19372	0.14022
$a(\{4\})$	0.1808	0.16042	0.1788
$a(\{1,2\})$	0.045792	0.21504	0.2164
$a(\{1,3\})$	0.1751	-0.12405	0.053909
$a(\{1,4\})$	-0.10253	-0.044048	0.0067834
$a(\{2,3\})$	0.25613	0.053126	0.092624
$a(\{2,4\})$	-0.040672	-0.049708	-0.063965
$a(\{3,4\})$	-0.029203	-0.015979	-0.0088056

Subjective Stochastic Ordinal Regression (Corrente, Greco, Kadzinski and Slowinski 2015)

Subjective Stochastic Ordinal Regression (SSOR)

- Which probability to use in the space of compatible value functions?
- We induce a probability distribution that permits to represent uncertain DM's preferences.
- Differently from SOR, the probability distribution is not given in a exogeneous way.

Uncertain preference

In addition to the certain preference information, the DM could also give some uncertain preference information as the following:

"The preference of a over b" is at least as credible as "the preference of b over a".

All value functions have to be compatible with the certain preference information provided by the DM. At the same time, we propose to induce a probability distribution on the set of compatible value functions reflecting the uncertain preference information.

The procedure

- Sampling a number s_V of value functions satisfying monotonicity and normalization constraints and being compatible with the certain preference information provided by the DM,
- Induce a probability distribution $\left\{w_t(U_t) \in [0,1] : \sum_{t=1}^{s_V} w_t(U_t) = 1\right\}$ on the set composed by these value functions on the base of the uncertain preference relation.

Denoting by $a \succsim_L b$ and $(a,b) \succsim_{L,*} (c,d)$ the uncertain preference relations between pairs and quadruples of alternatives, respectively, we shall translate these preferences as follow:

$$a \succsim_L b \Leftrightarrow \sum_{t:U_t(a)>U_t(b)} w(U_t) \geq \sum_{t:U_t(b)>U_t(a)} w(U_t)$$

$$(a,b) \succsim_{L,*} (c,d) \Leftrightarrow \sum_{t:U_t(a)>U_t(b)} w(U_t) \geq \sum_{t:U_t(c)>U_t(d)} w(U_t)$$

Inducing a probability distribution

Solving the following LP problem . . .

$$\varepsilon_L = \max \varepsilon, \quad \text{subject to}$$

$$\left.\begin{array}{l}
\displaystyle\sum_{t:U_t(a)>U_t(b)} w(U_t) \geq \sum_{t:U_t(b)>U_t(a)} w(U_t) + \varepsilon, \quad \text{if } a \succ_L b, \\
\displaystyle\sum_{t:U_t(a)>U_t(b)} w(U_t) \geq \sum_{t:U_t(c)>U_t(d)} w(U_t) + \varepsilon, \quad \text{if } (a,b) \succ_{L,*} (c,d), \\
\displaystyle\sum_{t=1}^{s_V} w(U_t) = 1 \\
w(U_t) \geq 0, t = 1, \ldots, s_V
\end{array}\right\} E_L$$

if E_L is feasible and $\varepsilon_L > 0$, then there exist at least one probability distribution on the set of value functions compatible with the uncertain preference provided by the DM.

Three possibilities

1. Considering the most discriminant probability distribution obtained by solving the previous LP problem;
2. Sampling a certain number of probability distributions compatible with the uncertain preference information provided by the DM by using the Hit-And-Run (HAR) method (R.L. Smith 1984) and computing their barycenter $\mathbf{w}^*$ called *representative probability distribution*;
3. Applying ROR on the space of probability distributions compatible with the uncertain preference information provided by the DM.

In the first two cases . . .

We can define the indices typical of SMAA:

- $b_k^r = \sum_{t:rank(k,U_t)=r} w(U_t),$
- $p(a_h, a_k) = \sum_{t:U_t(a_h)>U_t(a_k)} w(U_t).$

Based on the parwise winning index, the following preference relation can be defined:

$$a_h \succsim_L^R a_k \Leftrightarrow p(a_h, a_k) \geq p(a_k, a_h).$$

In the third case . . .

Since more than one probability distribution can be induced on the space of compatible value functions, the *probabilistic necessary* and *probabilistic possible* preference relations could be defined:

- $a_h \succsim_L^N a_k$ iff the preference of a_h over a_k is more credible than the preference of a_k over a_h for all induced probability distributions,
- $a_h \succsim_L^P a_k$ iff the preference of a_h over a_k is more credible than the preference of a_k over a_h for at least one induced probability distribution.

Proposition

- $\succsim_L^R$ *is strongly complete, that is for all* $a, b \in A$, $a \succsim_L^R b$ *or* $b \succsim_L^R a$,
- $\succsim^N \subseteq \succsim_L^N \subseteq \succsim_L^R \subseteq \succsim_L^P \subseteq \succsim^P$,
- *For all* $a, b \in A$, $a \succsim_L^N b$ *or* $b \succsim_L^P a$,
- *For all* $a, b, c \in A$, *if* $a \succsim^N b$ *and* $b \succsim_L^N c$, *then* $a \succsim_L^N c$,
- *For all* $a, b, c \in A$, *if* $a \succsim_L^N b$ *and* $b \succsim^N c$, *then* $a \succsim_L^N c$,
- *For all* $a, b, c \in A$, *if* $a \succsim^N b$ *and* $b \succsim_L^P c$, *then* $a \succsim_L^P c$,
- *For all* $a, b, c \in A$, *if* $a \succsim_L^P b$ *and* $b \succsim^N c$, *then* $a \succsim_L^P c$.

Here by $\succsim^R$ *we denoted the preference relation obtained by considering whichever probability distribution compatible with the uncertain preferences provided by the DM.*

Illustrative example

Problem definition

Let us suppose that the owner of a firm has to employ one among six sales manager evaluated on the following criteria: sales management experience (g_1), international experience (g_2) and human qualities (g_3).

Table : Evaluations of the sales managers

	g_1	g_2	g_3
Bassama	28	18	28
Calvet	26	40	44
Ferret	35	62	25
Frechet	9	62	88
Petron	6	15	100
Varlot	62	43	0

Certain preference information

The owner of the firm provided the following certain preference information:

- Varlot is preferred to Petron (Varlot $\succ$ Petron),
- Varlot is preferred to Petron more than Ferret is preferred to Calvet ((Varlot,Petron) $\succ^*$ (Ferret,Calvet)).

We sampled $s_V = 10,000$ value functions satisfying the monotonicity and normalization constraints as well as the constraints translating the certain preference information provided by the DM.

Likely preference information

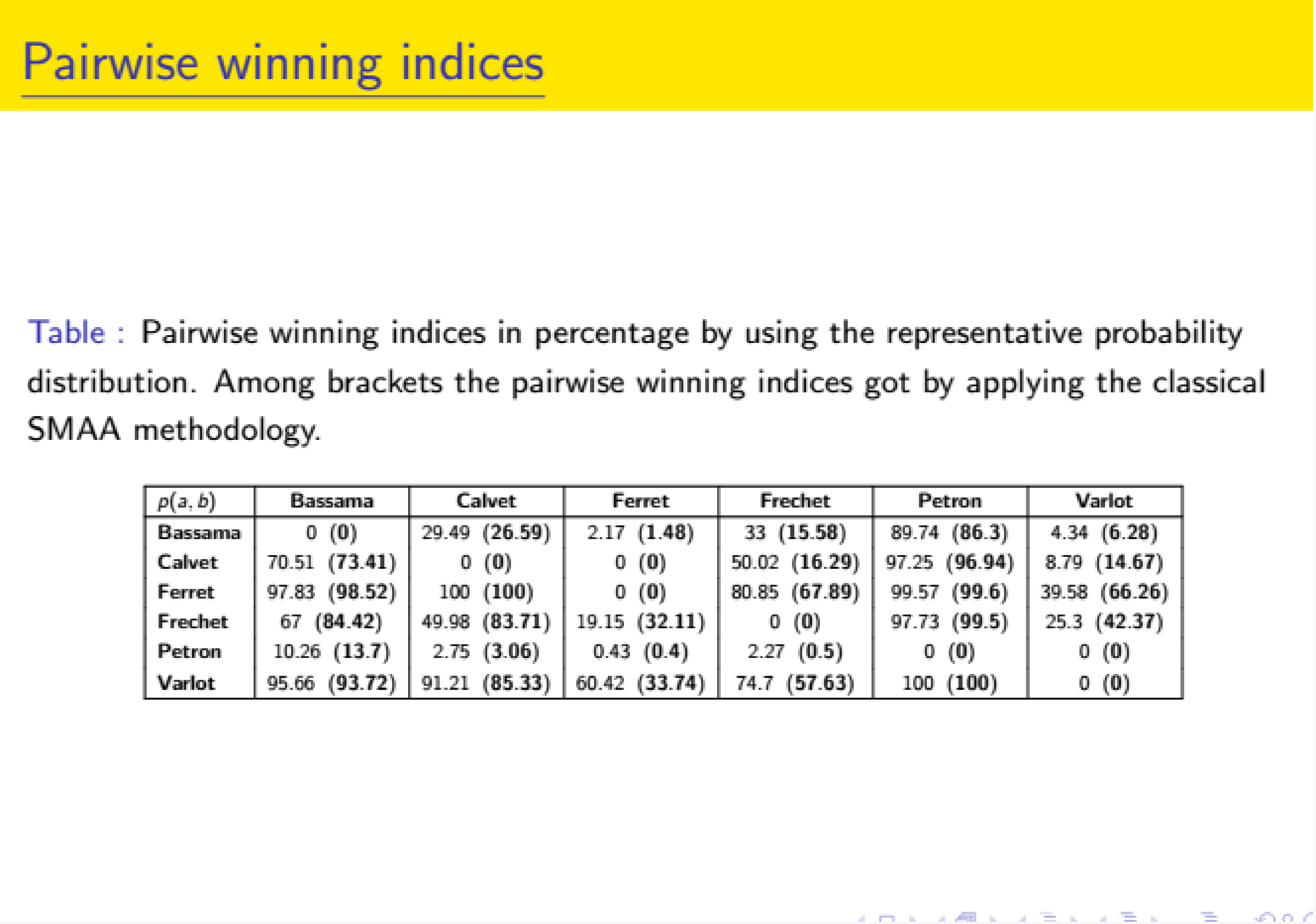
Besides, the owner of the firm wishes to provide also this further uncertain preference information:

- The preference of Calvet over Frechet is more credible than the viceversa (Calvet $\succ^L$ Frechet),
- The preference of Varlot over Frechet is more credible than the preference of Calvet over Frechet ((Varlot,Frechet) $\succ_{L,*}$ (Calvet,Frechet)).

Rank acceptability indices

Table : Rank acceptability indices in percentage by using the representative probability distribution. Among brackets the rank acceptability indices got by applying the classical SMAA methodology.

	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6
Bassama	1.02 **(0.73)**	1.89 **(1.97)**	18.62 **(10.79)**	21.37 **(19.15)**	47.5 **(54.03)**	9.6 **(13.33)**
Calvet	0 **(0)**	1.52 **(2.5)**	37.75 **(19.54)**	47.17 **(55.8)**	12.89 **(21.09)**	0.66 **(1.07)**
Ferret	26.95 **(45.13)**	64.36 **(42.77)**	8.29 **(11.39)**	0.38 **(0.66)**	0.02 **(0.05)**	0 **(0)**
Frechet	15.87 **(26.75)**	11.55 **(19.07)**	19.74 **(33.22)**	23.06 **(11.74)**	28.27 **(8.94)**	1.50 **(0.28)**
Petron	0 **(0)**	0.27 **(0.09)**	0.78 **(0.41)**	1.56 **(1.89)**	9.15 **(12.29)**	88.23 **(85.32)**
Varlot	56.16 **(27.39)**	20.41 **(33.6)**	14.82 **(24.65)**	6.45 **(10.76)**	2.15 **(3.6)**	0 **(0)**

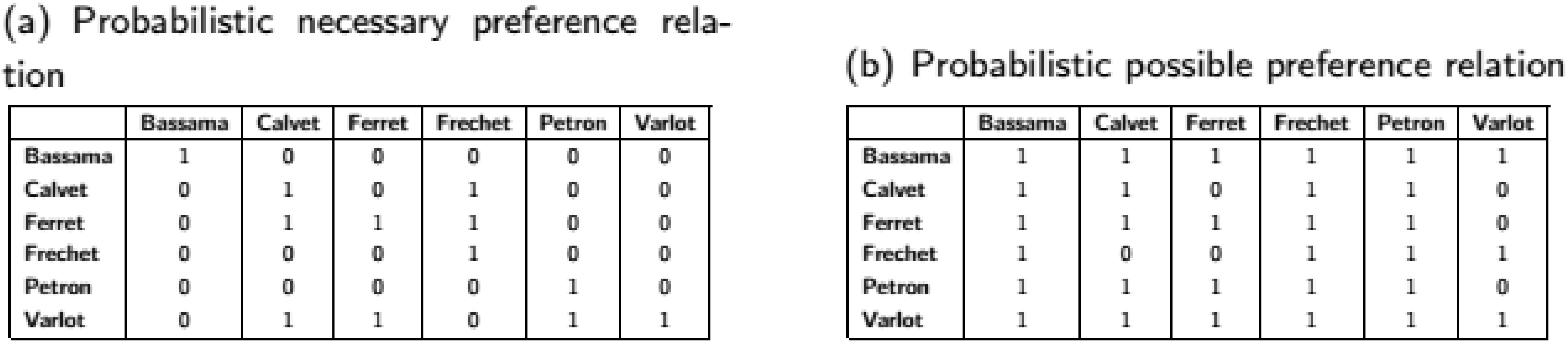


Pairwise winning indices

Table : Pairwise winning indices in percentage by using the representative probability distribution. Among brackets the pairwise winning indices got by applying the classical SMAA methodology.

$p(a,b)$	**Bassama**	**Calvet**	**Ferret**	**Frechet**	**Petron**	**Varlot**
Bassama	0 **(0)**	29.49 **(26.59)**	2.17 **(1.48)**	33 **(15.58)**	89.74 **(86.3)**	4.34 **(6.28)**
Calvet	70.51 **(73.41)**	0 **(0)**	0 **(0)**	50.02 **(16.29)**	97.25 **(96.94)**	8.79 **(14.67)**
Ferret	97.83 **(98.52)**	100 **(100)**	0 **(0)**	80.85 **(67.89)**	99.57 **(99.6)**	39.58 **(66.26)**
Frechet	67 **(84.42)**	49.98 **(83.71)**	19.15 **(32.11)**	0 **(0)**	97.73 **(99.5)**	25.3 **(42.37)**
Petron	10.26 **(13.7)**	2.75 **(3.06)**	0.43 **(0.4)**	2.27 **(0.5)**	0 **(0)**	0 **(0)**
Varlot	95.66 **(93.72)**	91.21 **(85.33)**	60.42 **(33.74)**	74.7 **(57.63)**	100 **(100)**	0 **(0)**

Probabilistic Necessary and Possible preference relations



(a) Probabilistic necessary preference relation

	Bassama	Calvet	Ferret	Frechet	Petron	Varlot
Bassama	1	0	0	0	0	0
Calvet	0	1	0	1	0	0
Ferret	0	1	1	1	0	0
Frechet	0	0	0	1	0	0
Petron	0	0	0	0	1	0
Varlot	0	1	1	0	1	1

(b) Probabilistic possible preference relation

	Bassama	Calvet	Ferret	Frechet	Petron	Varlot
Bassama	1	1	1	1	1	1
Calvet	1	1	0	1	1	0
Ferret	1	1	1	1	1	0
Frechet	1	0	0	1	1	1
Petron	1	1	1	1	1	0
Varlot	1	1	1	1	1	1

Multiple Criteria Hierarchy Process (Corrente, Greco, Slowinski 2012, 2013)

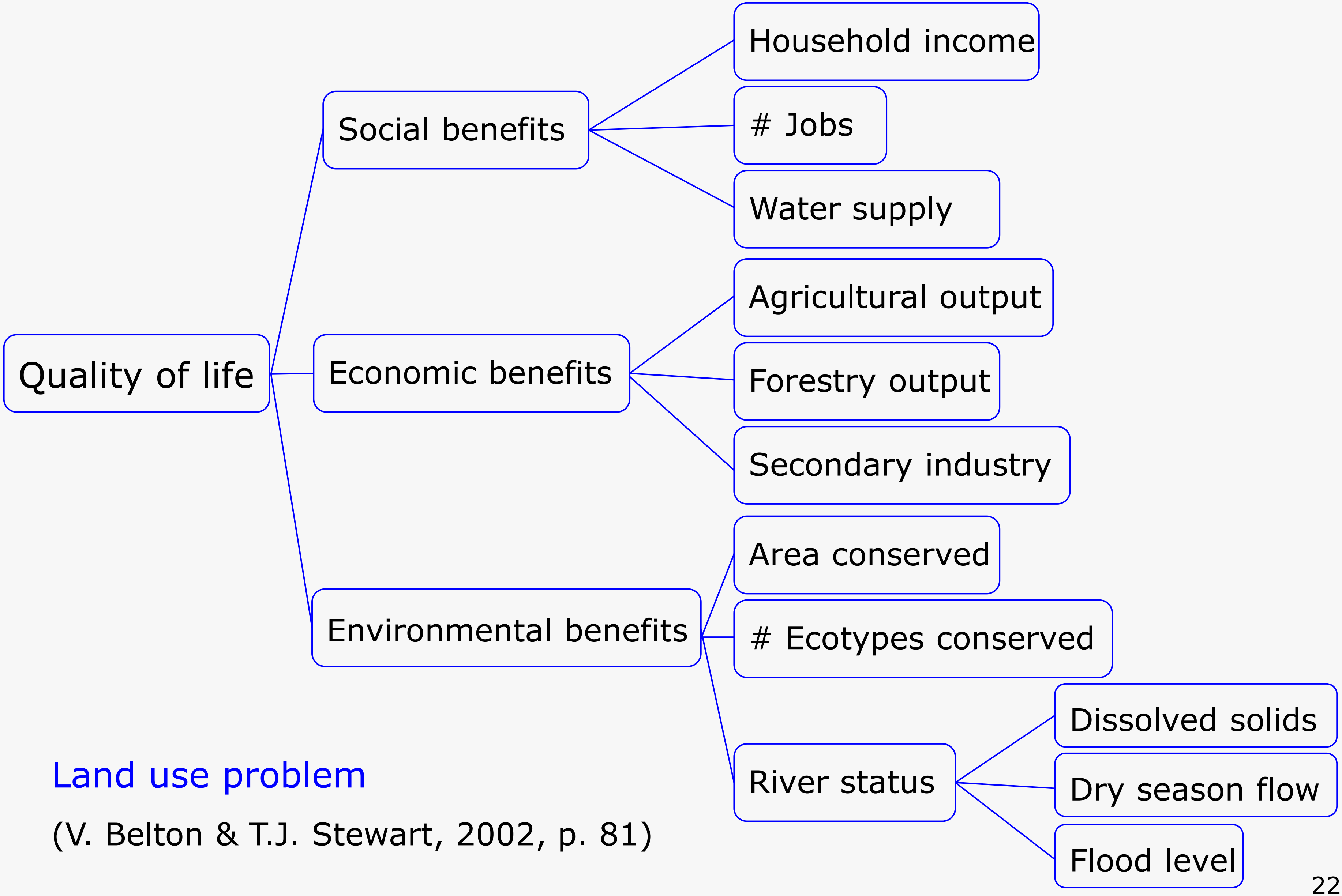
Hierarchical decomposition of complex decision problems

- „Almost everyone who has seriously thought about the objectives in a complex problem has come up with some sort of hierarchy of objectives."
  (R.L. Keeney & H. Raiffa, 1976, p. 41)

- „A hierarchy is an abstraction of the structure of a system to study the functional interactions of its components and their impacts on the entire system."
  (T.L. Saaty, 1980, p.5)

- „In the process of structuring the problem, it is possible (even likely) that the criteria may have been constructed hierarchically in terms of a value tree."
  (V. Belton & T.J. Stewart, 2002, p. 80)

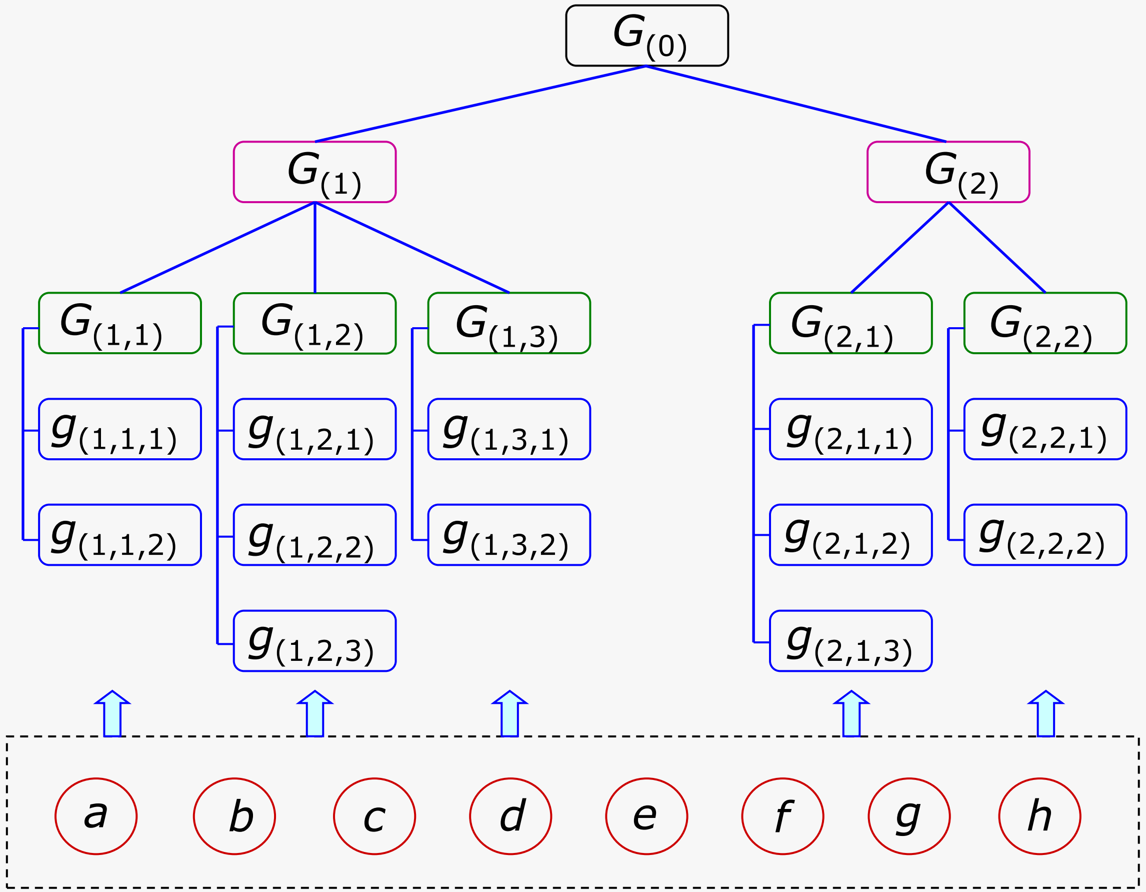
Tree structure of objectives-criteria – an example



Land use problem

(V. Belton & T.J. Stewart, 2002, p. 81)

Multiple Criteria Hierarchy Process (MCHP)



root criterion

1st level criteria

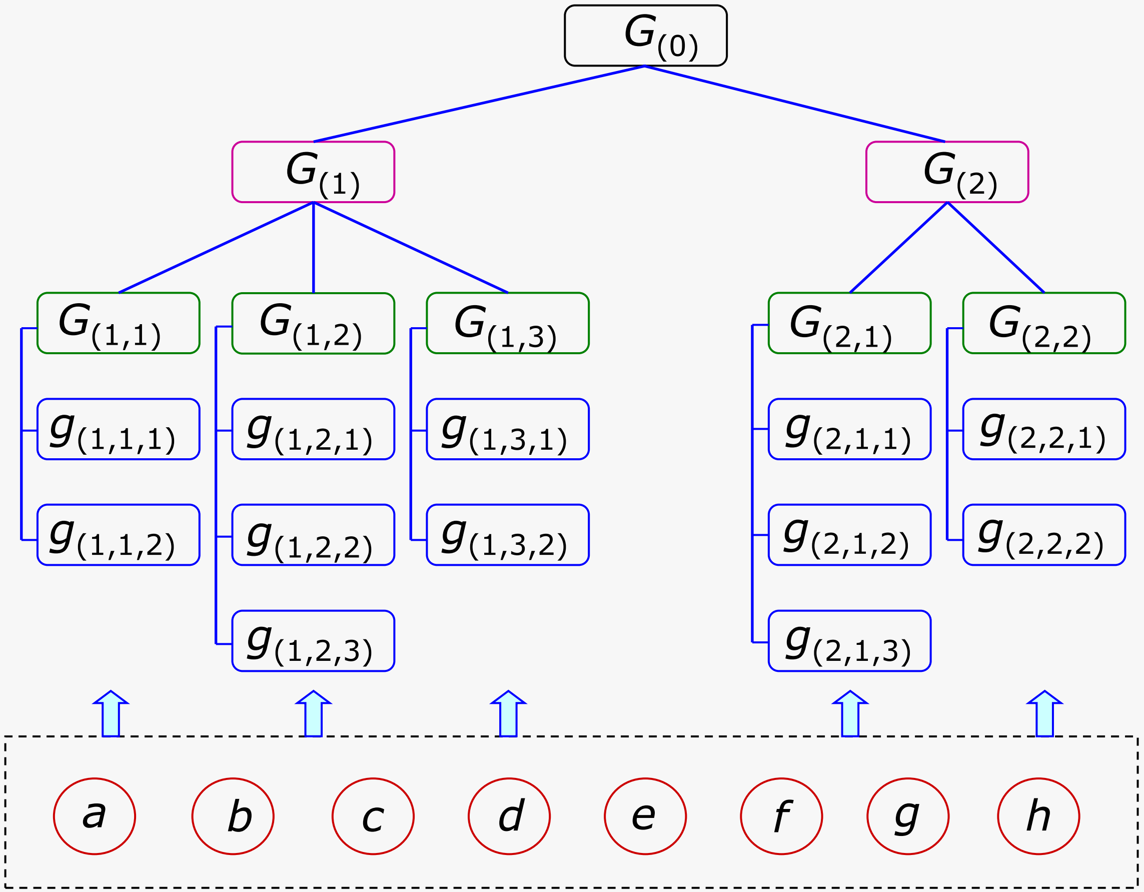
2nd level criteria

elementary criteria

alternatives

S. Corrente, S. Greco, R. Słowiński: Multiple Criteria Hierarchy Process in Robust Ordinal Regression. *Decision Support Systems*, 53 (2012) no.3, 660-674

Multiple Criteria Hierarchy Process (MCHP) - notation



$G_{(0)}$ – root criterion

Level criterion G_r

$G_{(1)},G_{(2)}$ $\{G_r, r=(1),\ldots(m)\}$

$G_{(1,1)},G_{(1,2)},\ldots,G_{(2,2)}$

$\{G_{(r,j)}, j=1,\ldots,n(r)\}$

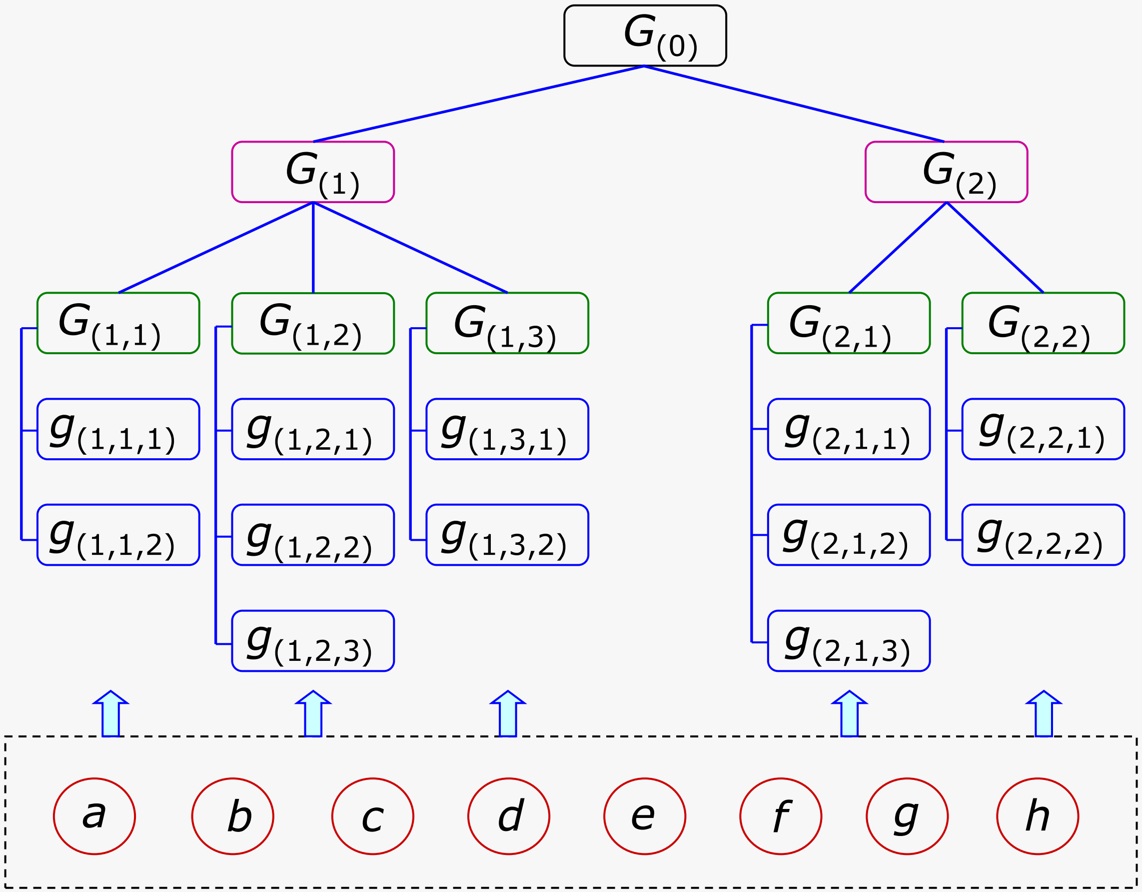
Elementary criterion g_t

EL–set of elementary criteria

$EL=\{g_{(1,1,1)}, g_{(1,1,2)},\ldots,g_{(2,2,2)}\}$

alternatives

Multiple Criteria Hierarchy Process (MCHP) - notation



$E(G_r)$ – set of elementary criteria descending from G_r

$E(G_{(1)})=\{g_{(1,1,1)},g_{(1,1,2)},$
$g_{(1,2,1)},g_{(1,2,2)},g_{(1,2,3)},$
$g_{(1,3,1)},g_{(1,3,2)}\}$

$E(G_{(2,2)})=\{g_{(2,2,1)},g_{(2,2,2)}\}$

$\mathcal{G}^k{}_r$ – set of criteria descending from G_r and located at level k

$\mathcal{G}^2{}_{(0)}=\{G_{(1,1)},G_{(1,2)},G_{(1,3)},$
$G_{(2,1)}, G_{(2,2)},\}$

$\mathcal{G}^3{}_{(2)}=\{g_{(2,1,1)},g_{(2,1,2)},g_{(2,1,3)},$
$g_{(2,2,1)},g_{(2,2,2)}\}$

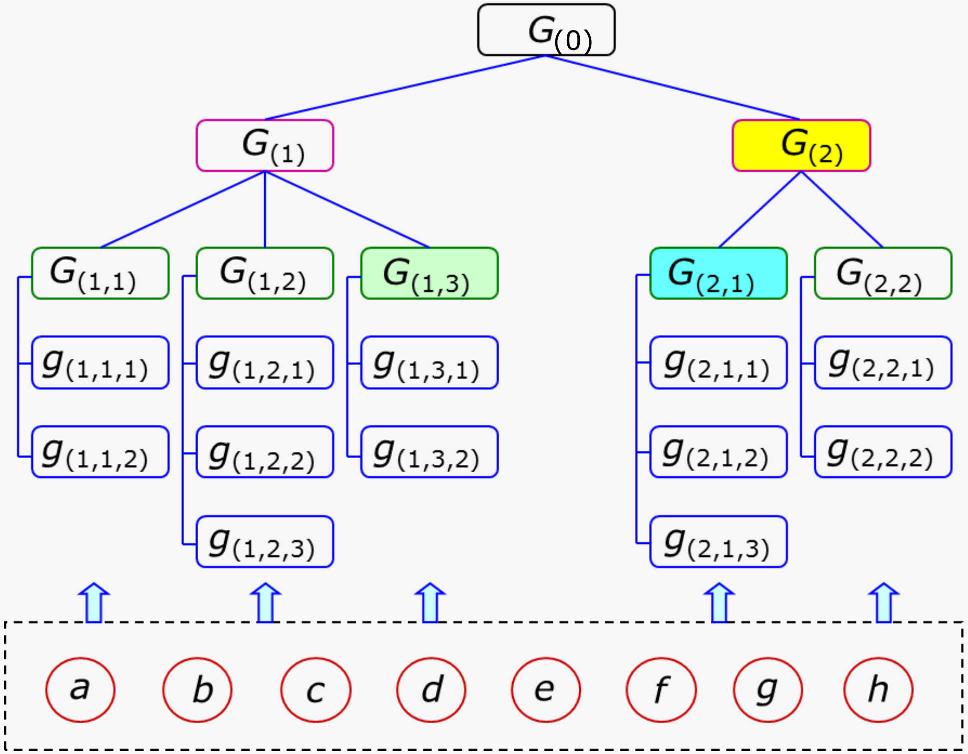
Multiple Criteria Hierarchy Process (MCHP) – main idea

- We wish to consider preference relation $\succeq_r$ in each node of the hierarchy tree, e.g.:

$a \succeq_{(2)} b$ iff $U_{(2)}(a) \geq U_{(2)}(b)$

$c \succeq_{(1,3)} d$ iff $U_{(1,3)}(c) \geq U_{(1,3)}(d)$

$e \succeq_{(2,1)} f$ iff $U_{(2,1)}(e) \geq U_{(2,1)}(f)$

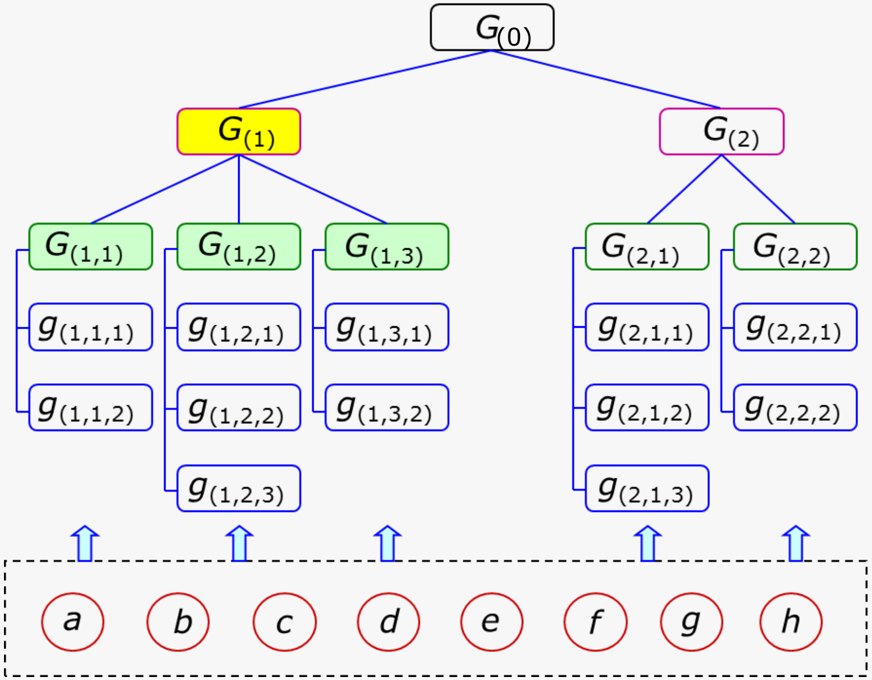


Multiple Criteria Hierarchy Process (MCHP) – main idea

- In case of preferentially independent criteria, preference relation $\succeq_r$ should enjoy some intuitive properties, e.g.:

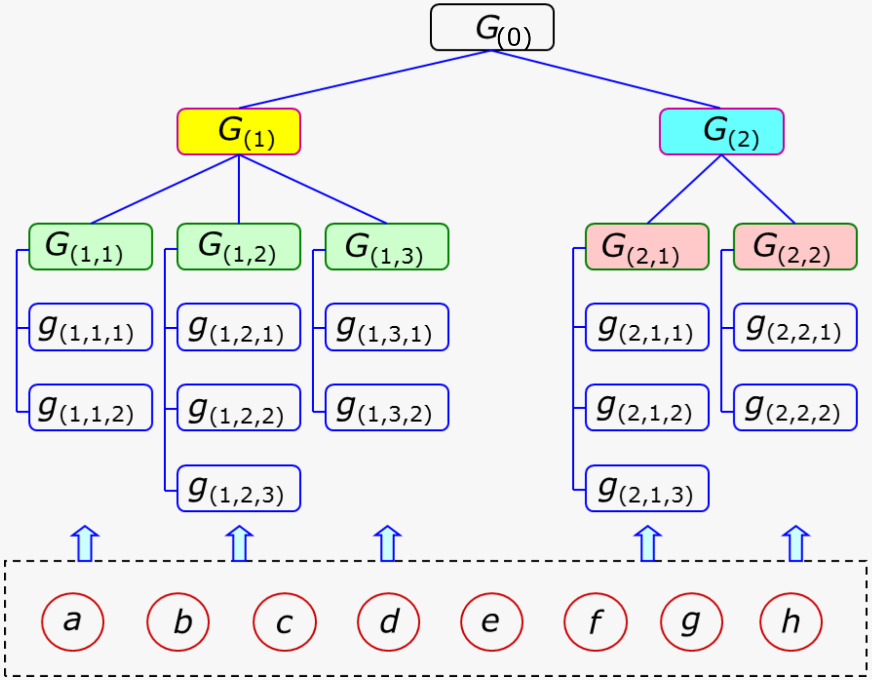
$$a \succeq_{(\mathbf{r},j)} b \ \text{ for all } j = 1,\ldots,n(\mathbf{r}) \ \Rightarrow \ a \succeq_{\mathbf{r}} b$$

$$a \succeq_{(1,1)} b, \ a \succeq_{(1,2)} b, \ a \succeq_{(1,3)} b \ \Rightarrow \ a \succeq_{(1)} b$$



Multiple Criteria Hierarchy Process (MCHP) – main idea

- In case of preferentially independent criteria, preference relation $\succeq_{\mathbf{r}}$ should enjoy some intuitive properties, e.g.:



$$a \succeq_{(\mathbf{r},j)} b \ \text{ for all } j = 1,\dots,n(\mathbf{r}) \ \Rightarrow \ a \succeq_{\mathbf{r}} b$$

$$a \succeq_{(1,1)} b,\ a \succeq_{(1,2)} b,\ a \succeq_{(1,3)} b \ \Rightarrow \ a \succeq_{(1)} b$$

$$not\left(a \succeq_{(\mathbf{r},j)} b\right) \ \text{ for all } j = 1,\dots,n(\mathbf{r}) \ \Rightarrow \ not\left(a \succeq_{\mathbf{r}} b\right)$$

$$not\left(a \succeq_{(2,1)} b\right),\ not\left(a \succeq_{(2,2)} b\right) \ \Rightarrow \ not\left(a \succeq_{(2)} b\right)$$

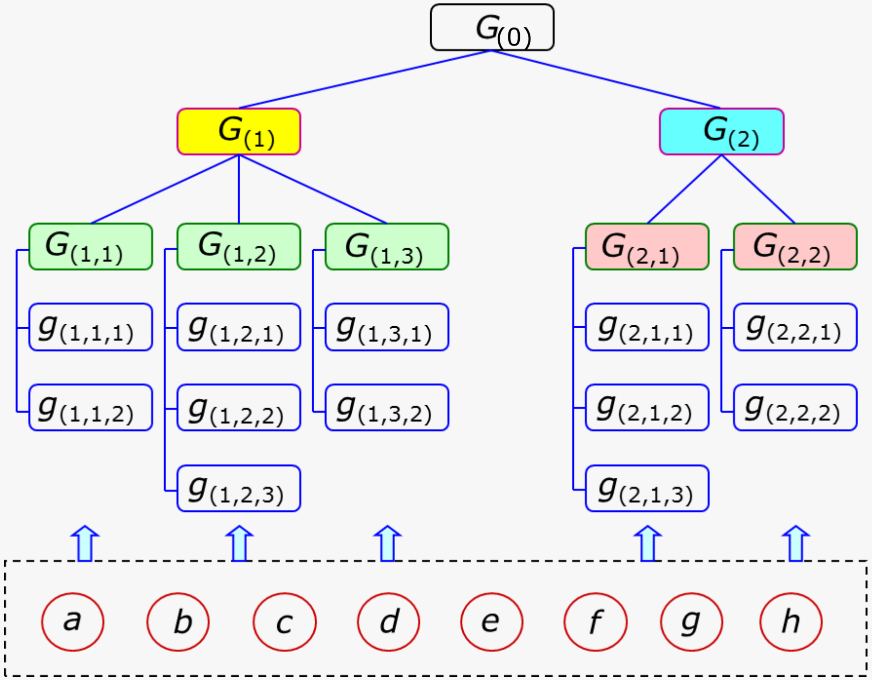
$$\Updownarrow$$

$$a \succeq_{\mathbf{r}} b \ \Rightarrow a \succeq_{(\mathbf{r},j)} b \ \text{ for at least one } \ j \in \{1,\dots,n(\mathbf{r})\}$$

$$a \succeq_{(2)} b \ \Rightarrow \ a \succeq_{(2,1)} b \ \text{ or } \ a \succeq_{(2,2)} b$$

Multiple Criteria Hierarchy Process (MCHP) – main idea

- Any MCDM method could be used to construct preference relation $\succsim_r$ in particular nodes using the available preference information:
  - MAVT
  - ELECTRE/PROMETHEE
  - UTA
  - …



- The choice depends on type of aggregation & preference information

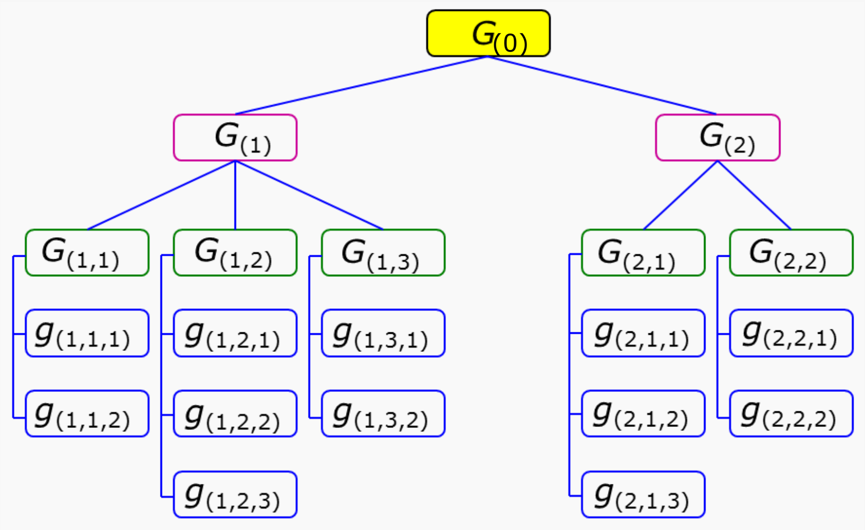
B. Roy, R. Słowiński: Questions guiding the choice of a multicriteria decision aiding method. *EURO Journal on Decision Processes*, 1 (2013) no.1, 69–97.

Multiple Criteria Hierarchy Process (MCHP) – main idea

- Consider the simplest preference model – weighted sum

- For any $a \in A$, the value of a is:

$$
\begin{aligned}
U(a) &= \\
& w_{(1,1,1)}\, g_{(1,1,1)}(a) + w_{(1,1,2)}\, g_{(1,1,2)}(a) + \\
& w_{(1,2,1)}\, g_{(1,2,1)}(a) + w_{(1,2,2)}\, g_{(1,2,2)}(a) + \\
& w_{(1,2,3)}\, g_{(1,2,3)}(a) + \\
& w_{(1,3,1)}\, g_{(1,3,1)}(a) + w_{(1,3,2)}\, g_{(1,3,2)}(a) + \\
& w_{(2,1,1)}\, g_{(2,1,1)}(a) + w_{(2,1,2)}\, g_{(2,1,2)}(a) + \\
& w_{(2,1,3)}\, g_{(2,1,3)}(a) + \\
& w_{(2,2,1)}\, g_{(2,2,1)}(a) + w_{(2,2,2)}\, g_{(2,2,2)}(a)
\end{aligned}
$$



The unknown model parameters are weights of elementary criteria only

$$w_{\mathbf{r}} = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} w_{\mathbf{t}}$$

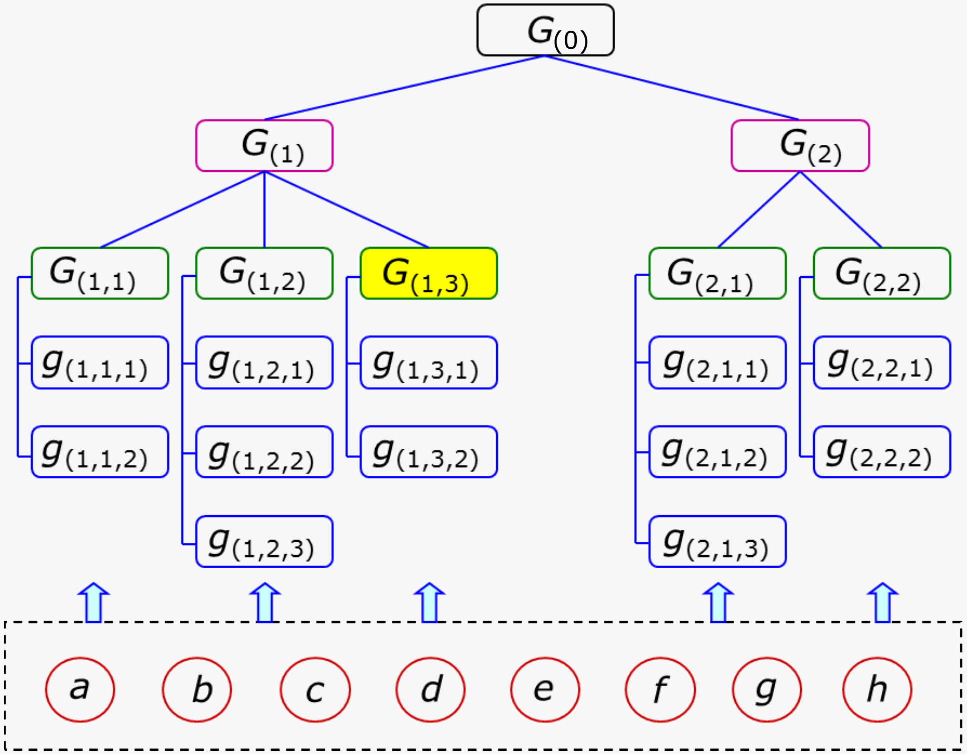
t - index of elementary criterion $g_{\mathbf{t}}$

$$w_{(1,1)} = \sum_{\mathbf{t} \in E\left(G_{(1,1)}\right)} w_{\mathbf{t}} = $$

$$= w_{(1,1,1)} + w_{(1,1,2)}$$

Multiple Criteria Hierarchy Process (MCHP) – main idea

- Each node is associated with a marginal value function
- Elicitation of preferences and analysis of recommendation in tree nodes: weights adapt to preferences & preference relations follow from weights, e.g.:



$$a \succsim_{(1,3)} b \;\Leftrightarrow\; U_{(1,3)}(a) \geq U_{(1,3)}(b)\text{ , i.e.,}$$

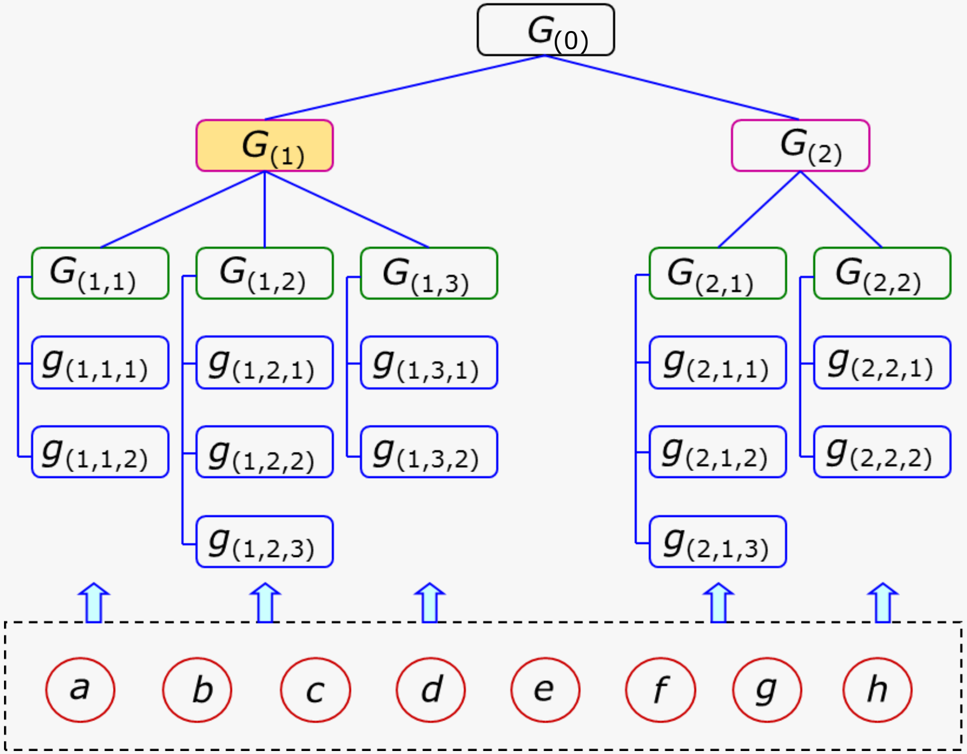
$$w_{(1,3,1)}\ g_{(1,3,1)}(a) + w_{(1,3,2)}\ g_{(1,3,2)}(a) \geq w_{(1,3,1)}\ g_{(1,3,1)}(b) + w_{(1,3,2)}\ g_{(1,3,2)}(b)$$

MCHP with additive value function

- Marginal value function associated with node r

$$U_{\mathbf{r}}(a) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} u_{\mathbf{t}}(a)$$

$u_t(\cdot)$ – marginal value function monotonically dependent on elementary criterion g_t



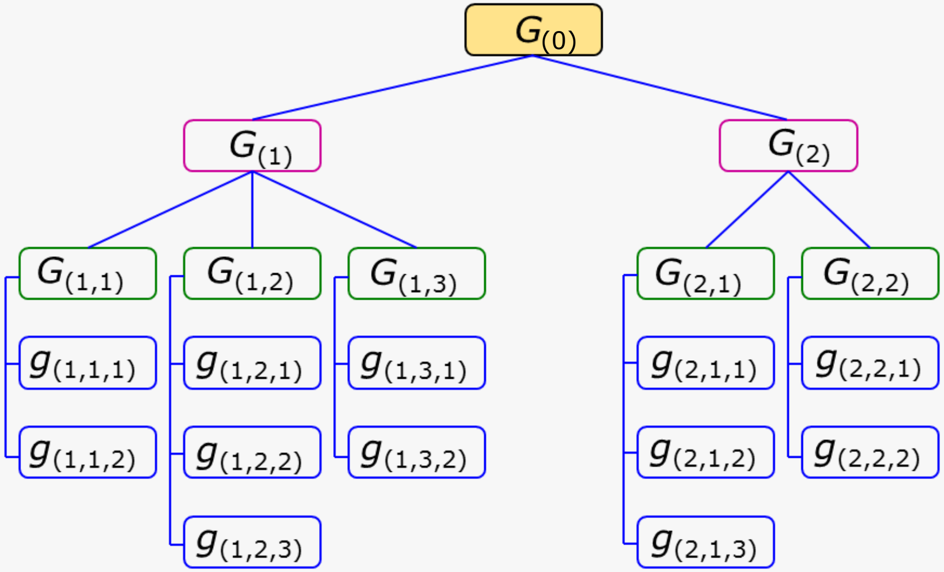
$$U_1(a) = \sum_{\mathbf{t} \in E(G_1)} u_{\mathbf{t}}(a) =$$

$$= u_{(1,1,1)}(a) + u_{(1,1,2)}(a) + u_{(1,2,1)}(a) + u_{(1,2,2)}(a) + u_{(1,2,3)}(a) + u_{(1,3,1)}(a) + u_{(1,3,2)}(a)$$

MCHP with additive value function

- Total value function associated with the root

$$U(a) = \sum_{\mathbf{t} \in EL} u_{\mathbf{t}}(a)$$



$$\begin{aligned} U(a) &= u_{(1,1,1)}(a) + u_{(1,1,2)}(a) + u_{(1,2,1)}(a) + u_{(1,2,2)}(a) + u_{(1,2,3)}(a) + u_{(1,3,1)}(a) + \\ & u_{(1,3,2)}(a) + u_{(2,1,1)}(a) + u_{(2,1,2)}(a) + u_{(2,1,3)}(a) + u_{(2,2,1)}(a) + u_{(2,2,2)}(a) = \\ &= U_{(1,1)}(a) + U_{(1,2)}(a) + U_{(1,3)}(a) + U_{(2,1)}(a) + U_{(2,2)}(a) = \\ &= U_{(1)}(a) + U_{(2)}(a) \end{aligned}$$

- How to construct marginal value functions $u_t(\cdot)$, $t \in EL$?

MCHP with additive value function - preference elicitation by DM

- Direct or indirect ?
- Direct elicitation of numerical values of model parameters by DMs demands much of their cognitive effort
- Indirect = through decision examples
- Decision aiding based on decision examples is gaining importance because:
  - Decision example is a relatively „easy” preference information
  - Decisions can also be observed without active participation of DMs
  - Psychologists confirm that DMs are more confident exercising their decisions than explaining them

MCHP with additive value function - preference elicitation by DM

- Types of indirect preference information in particular nodes of the tree:
  - Pairwise comparison: *a* is at least as good as *b* on criterion $G_\mathbf{r}$

$$a \succeq_\mathbf{r} b \;\Leftrightarrow\; U_\mathbf{r}(a) \geq U_\mathbf{r}(b)$$

  - Intensity of preference: considering criterion $G_\mathbf{r}$ or $g_\mathbf{t}$,

    a is preferred to *b* at least as much as *c* is preferred to *d*

$$(a,b) \succeq^*_\mathbf{r} (c,d) \Leftrightarrow U_\mathbf{r}(a) - U_\mathbf{r}(b) \geq U_\mathbf{r}(c) - U_\mathbf{r}(d) \geq 0$$

$$(a,b) \succeq^*_\mathbf{t} (c,d) \Leftrightarrow u_\mathbf{t}(a) - u_\mathbf{t}(b) \geq u_\mathbf{t}(c) - u_\mathbf{t}(d) \geq 0$$

J. Figueira, S. Greco, R. Słowiński: Building a set of additive value functions representing a reference preorder and intensities of preference: GRIP method. *EJOR*, 195 (2009) no.2, 460-486

Checking for the existence of a compatible value function in node **r**

$$\varepsilon^* = \max \varepsilon, \text{ subject to :}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} U_{\mathbf{r}}\left(a^*\right) \geq U_{\mathbf{r}}\left(b^*\right) + \varepsilon \quad \text{if } \; a^* \succ_{\mathbf{r}} b^* \\ U_{\mathbf{r}}\left(a^*\right) = U_{\mathbf{r}}\left(b^*\right) \quad \text{if } \; a^* \sim_{\mathbf{r}} b^* \\ U_{\mathbf{r}}\left(a^*\right) - U_{\mathbf{r}}\left(b^*\right) \geq U_{\mathbf{r}}\left(c^*\right) - U_{\mathbf{r}}\left(d^*\right) + \varepsilon \quad \text{if } \; \left(a^*, b^*\right) \succ_{\mathbf{r}}^* \left(c^*, d^*\right) \\ U_{\mathbf{r}}\left(a^*\right) - U_{\mathbf{r}}\left(b^*\right) = U_{\mathbf{r}}\left(c^*\right) - U_{\mathbf{r}}\left(d^*\right) \quad \text{if } \; \left(a^*, b^*\right) \sim_{\mathbf{r}}^* \left(c^*, d^*\right) \end{array}\right\} \forall a^*, b^*, c^*, d^* \in A^R \\ u_{\mathbf{t}}\left(x_{\mathbf{t}}^k\right) - u_{\mathbf{t}}\left(x_{\mathbf{t}}^{k-1}\right) \geq 0, \quad \mathbf{t} \in EL, \; k = 2, \ldots, m_{\mathbf{t}}\left(A^R\right) \\ u_{\mathbf{t}}\left(x_{\mathbf{t}}^0\right) = 0, \quad \mathbf{t} \in EL \quad (EL : \text{set of elementary criteria}) \\ \displaystyle\sum_{\mathbf{t} \in EL} u_{\mathbf{t}}\left(x_{\mathbf{t}}^{m_{\mathbf{t}}}\right) = 1 \end{array}\right\} \left(E^{A^R}\right)$$

Since $U_{\mathbf{r}}(a) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} u_{\mathbf{t}}(a)$, the only unknown of this LP problem are marginal value functions of elementary criteria $u_{\mathbf{t}}$ and threshold ε

Checking for the existence of a compatible value function in node **r**

$$\varepsilon^* = \max \varepsilon, \ \text{subject to:}$$

$$\left.\begin{array}{l}
\left.\begin{array}{l}
U_{\mathbf{r}}\left(a^*\right) \geq U_{\mathbf{r}}\left(b^*\right) + \varepsilon \quad \text{if } \ a^* \succ_{\mathbf{r}} b^* \\
U_{\mathbf{r}}\left(a^*\right) = U_{\mathbf{r}}\left(b^*\right) \quad \text{if } \ a^* \sim_{\mathbf{r}} b^* \\
U_{\mathbf{r}}\left(a^*\right) - U_{\mathbf{r}}\left(b^*\right) \geq U_{\mathbf{r}}\left(c^*\right) - U_{\mathbf{r}}\left(d^*\right) + \varepsilon \quad \text{if } \ \left(a^*, b^*\right) \succ^*_{\mathbf{r}} \left(c^*, d^*\right) \\
U_{\mathbf{r}}\left(a^*\right) - U_{\mathbf{r}}\left(b^*\right) = U_{\mathbf{r}}\left(c^*\right) - U_{\mathbf{r}}\left(d^*\right) \quad \text{if } \ \left(a^*, b^*\right) \sim^*_{\mathbf{r}} \left(c^*, d^*\right)
\end{array}\right\} \forall a^*, b^*, c^*, d^* \in A^R \\
u_{\mathbf{t}}\left(x_{\mathbf{t}}^k\right) - u_{\mathbf{t}}\left(x_{\mathbf{t}}^{k-1}\right) \geq 0, \quad \mathbf{t} \in EL, \ k = 2, \ldots, m_{\mathbf{t}}\left(A^R\right) \\
u_{\mathbf{t}}\left(x_{\mathbf{t}}^0\right) = 0, \quad \mathbf{t} \in EL \quad (EL: \text{set of elementary criteria}) \\
\sum_{\mathbf{t} \in EL} u_{\mathbf{t}}\left(x_{\mathbf{t}}^{m_{\mathbf{t}}}\right) = 1
\end{array}\right\} \left(E^{A^R}\right)$$

If E^{A^R} is feasible and $\varepsilon^* > 0$, then there exists at least one value function compatible with the preference information

Checking for the existence of a compatible value function in node **r**

If for the given preference information there is no compatible value function, the user can:

- identify and eliminate „troublesome” pieces of preference information (Mousseau et al. 2003),
- continue to use „not completely compatible” set of value functions with an acceptable approximation error (Jacquet-Lagrèze & Siskos 1982),
- augment the complexity of the value function, e.g., pass from additive linear to additive monotonic, or to Choquet integral

Calculating necessary and possible preference relations in node **r**

- For all pairs of alternatives $a,b \in A$, their performances on elementary criteria $g_{\mathbf{t}}(a)$, $g_{\mathbf{t}}(b)$ add to $m_{\mathbf{t}}$ characteristic points of marginal value function $u_{\mathbf{t}}$, $\mathbf{t} \in EL$; then E^{A^R} becomes $E(a,b)$
- Consider constraints:

$$\left.\begin{array}{c} U_{\mathbf{r}}(b) \geq U_{\mathbf{r}}(a) + \varepsilon \\ E(a,b) \end{array}\right\} \left(E_{\mathbf{r}}^{N}(a,b)\right) \qquad \left.\begin{array}{c} U_{\mathbf{r}}(a) \geq U_{\mathbf{r}}(b) \\ E(a,b) \end{array}\right\} \left(E_{\mathbf{r}}^{P}(a,b)\right)$$

- The necessary and the possible preference relations (LP problems):

$$a \succsim_{\mathbf{r}}^{N} b \Leftrightarrow \text{if } E_{\mathbf{r}}^{N}(a,b) \text{ infeasible or } \varepsilon_{\mathbf{r}}^{N}(a,b) = \max \varepsilon, \text{ s.t. } E_{\mathbf{r}}^{N}(a,b) \text{ is} \leq 0$$

$$a \succsim_{\mathbf{r}}^{P} b \Leftrightarrow \text{if } E_{\mathbf{r}}^{P}(a,b) \text{ feasible and } \varepsilon_{\mathbf{r}}^{P}(a,b) = \max \varepsilon, \text{ s.t. } E_{\mathbf{r}}^{P}(a,b) \text{ is} > 0$$

One can also work with a „representative” value function

- It may be desirable to have a total order and scores of alternatives
- The idea is to select among compatible value functions that value function which better highlights the necessary ranking, i.e., maximizes the difference of values for pairs of alternatives a and b, such that $a \gtrsim^N_{\mathbf{r}} b$ while not($b \gtrsim^N_{\mathbf{r}} a$)
- As secondary objective, we minimize the difference of values for pairs of alternatives for which no necessary relation holds, i.e., such that not($a \gtrsim^N_{\mathbf{r}} b$) and not($b \gtrsim^N_{\mathbf{r}} a$)
- Lexicographic sequence of $G_{\mathbf{r}}$’s may underline their relative importance ($G_{\mathbf{r}}$ is the root criterion or any level criterion, excluding those from EL)

S. Greco, M. Kadziński, R. SŁowiński:Selection of a representative value function in robust multiple criteria sorting. *Computers & Operations Research*, *38*(11), 1620-1637.

S. Corrente, S. Greco, R. Słowiński: Multiple Criteria Hierarchy Process in Robust Ordinal Regression. *Decision Support Systems*, 53 (2012) no.3, 660-674

Properties of necessary and possible preference relations in node **r**

- Given two alternatives $a,b \in A$, and any non-elementary criterion $G_{\mathbf{r}}$:

(*i*) $a \succeq^{N}_{(\mathbf{r},j)} b$ for all $j = 1,\ldots,n(\mathbf{r}) \;\Rightarrow\; a \succeq^{N}_{\mathbf{r}} b$

(*ii*) $a \succeq^{N}_{(\mathbf{r},j)} b$ for all $j = 1,\ldots,n(\mathbf{r}),\; j \neq w,$ and $a \succeq^{P}_{(\mathbf{r},w)} b \;\Rightarrow\; a \succeq^{P}_{\mathbf{r}} b$

(*iii*)

$$not\left(a \succeq^{P}_{(\mathbf{r},j)} b\right) \text{ for all } j = 1,\ldots,n(\mathbf{r}) \;\Rightarrow\; not\left(a \succeq^{P}_{\mathbf{r}} b\right)$$

$$\Updownarrow$$

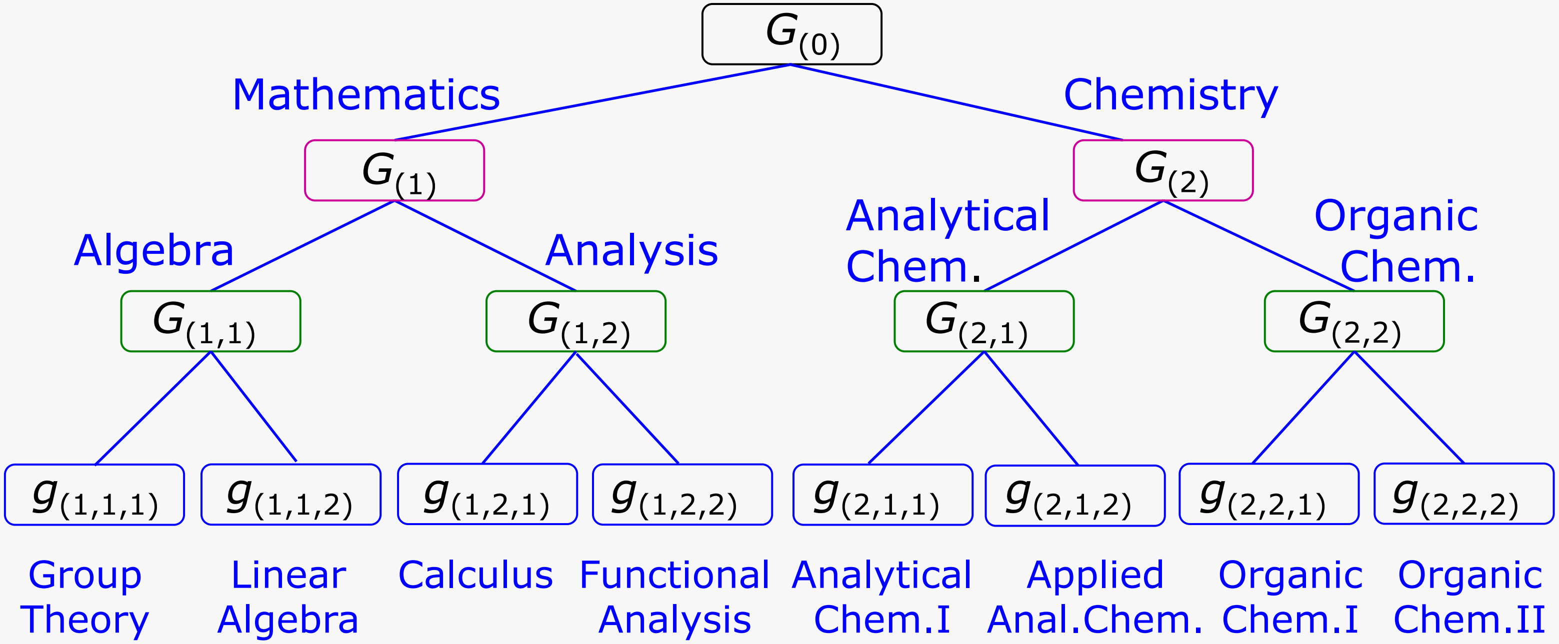
$$a \succeq^{P}_{\mathbf{r}} b \;\Rightarrow\; a \succeq^{P}_{(\mathbf{r},j)} b \text{ for at least one } j \in \{1,\ldots,n(\mathbf{r})\}$$

- Remark: hierarchical properties are expressed in terms of preference
  - necessary (*i*)
  - necessary & possible (*ii*)
  - possible (*iii*)

Multiple Criteria Hierarchy Process (MCHP) – value function model

- Example:

  Ranking of students wrt. hierarchical criteria of Mathematics & Chemistry



- 15 students: A, B, C, D, E, F, H, I, L, M, N, O, P, Q, R

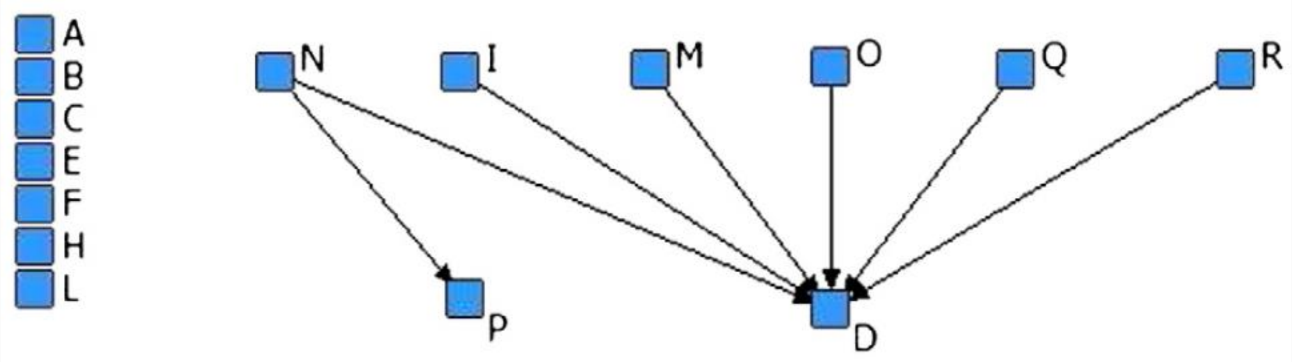
Multiple Criteria Hierarchy Process (MCHP) – value function model

- Performances of students on elementary criteria

Elementary criterion vs. Student	$g_{(1,1,1)}$	$g_{(1,1,2)}$	$g_{(1,2,1)}$	$g_{(1,2,2)}$	$g_{(2,1,1)}$	$g_{(2,1,2)}$	$g_{(2,2,1)}$	$g_{(2,2,2)}$
A	Very bad	Very good	Very bad	Good	Very good	Very good	Very bad	Bad
B	Bad	Very good	Medium	Very good	Very bad	Bad	Very bad	Very bad
C	Very good	Medium	Medium	Very bad	Very good	Good	Bad	Medium
D	Medium	Very bad	Bad	Very bad	Very bad	Bad	Medium	Very bad
E	Very good	Very good	Medium	Medium	Bad	Very good	Bad	Very bad
F	Good	Bad	Bad	Medium	Very bad	Very bad	Very good	Very good
H	Medium	Very bad	Bad	Bad	Very good	Very bad	Very bad	Very bad
I	Good	Good	Good	Medium	Medium	Bad	Good	Very bad
L	Good	Very bad	Bad	Good	Good	Very bad	Very good	Good
M	Medium	Medium	Medium	Bad	Medium	Medium	Very good	Good
N	Good	Bad	Very good	Medium	Bad	Very good	Very good	Medium
O	Good	Medium	Bad	Bad	Medium	Bad	Very good	Very bad
P	Bad	Very bad	Bad	Medium	Bad	Very good	Medium	Very bad
Q	Very good	Very good	Medium	Very bad	Bad	Medium	Medium	Bad
R	Good	Good	Bad	Very bad	Bad	Bad	Medium	Medium

Multiple Criteria Hierarchy Process (MCHP) – value function model

- Dominance relation in the set of students



Multiple Criteria Hierarchy Process (MCHP) – value function model

- On Chemistry, student I is preferred to student H

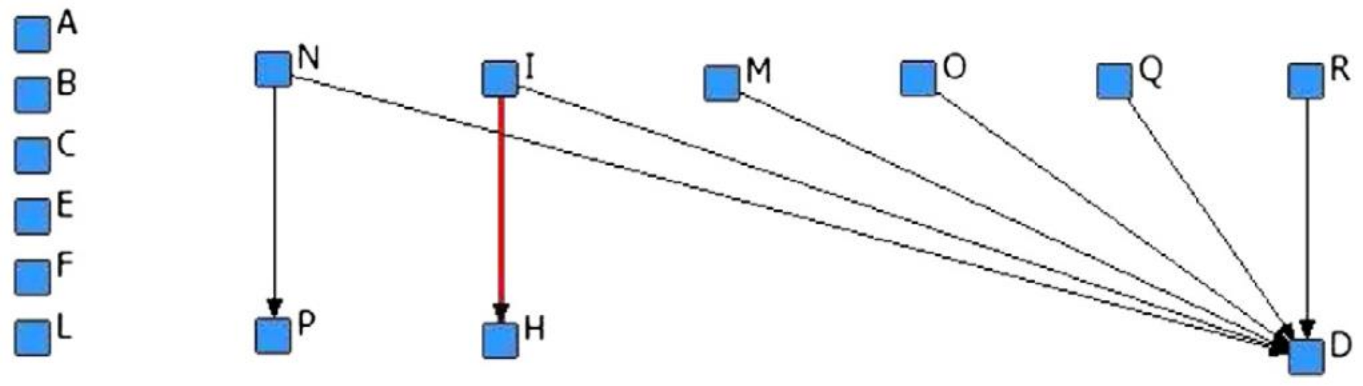
$$U_{(2)}(\mathrm{I}) > U_{(2)}(\mathrm{H}) \Leftrightarrow$$

$$\Leftrightarrow U_{(2,1)}(\mathrm{I}) + U_{(2,2)}(\mathrm{I}) > U_{(2,1)}(\mathrm{H}) + U_{(2,2)}(\mathrm{H}) \Leftrightarrow$$

$$\Leftrightarrow u_{(2,1,1)}(\mathrm{I}) + u_{(2,1,2)}(\mathrm{I}) + u_{(2,2,1)}(\mathrm{I}) + u_{(2,2,2)}(\mathrm{I}) >$$

$$> u_{(2,1,1)}(\mathrm{H}) + u_{(2,1,2)}(\mathrm{H}) + u_{(2,2,1)}(\mathrm{H}) + u_{(2,2,2)}(\mathrm{H})$$

- Necessary preference relation after the 1st piece of preference information



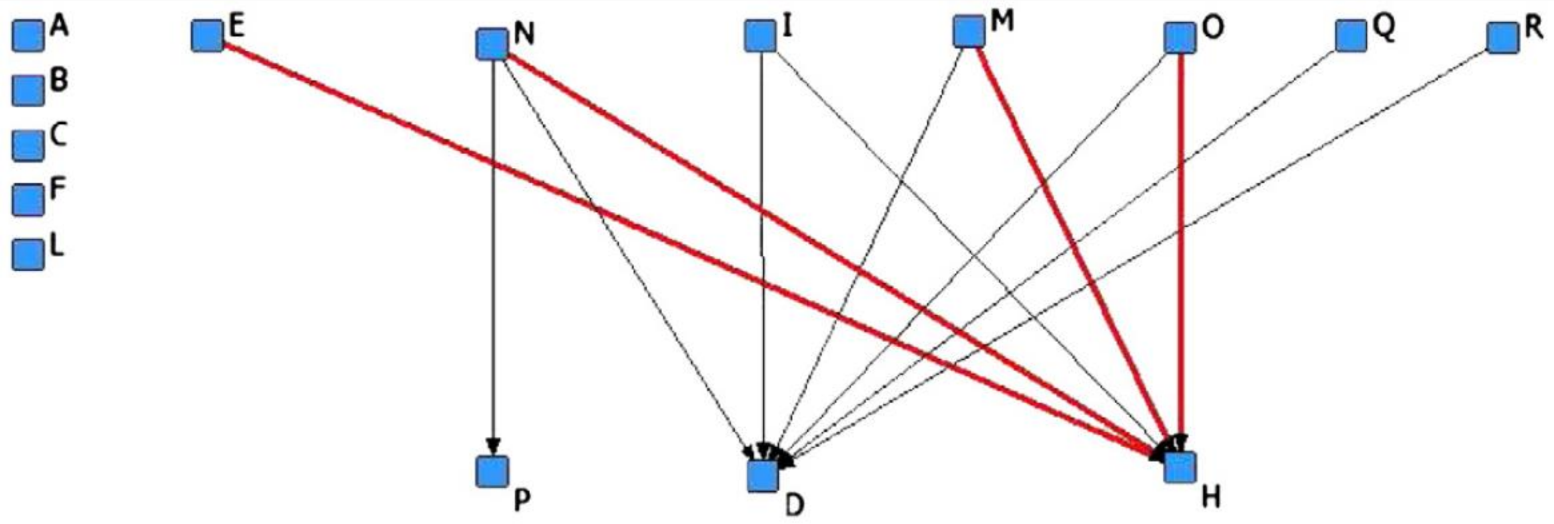
Multiple Criteria Hierarchy Process (MCHP) – value function model

- On Analytical Chemistry, student E is preferred to student H

$$U_{(2,1)}(\mathrm{E}) > U_{(2,1)}(\mathrm{H}) \Leftrightarrow$$

$$\Leftrightarrow u_{(2,1,1)}(\mathrm{E}) + u_{(2,1,2)}(\mathrm{E}) > u_{(2,11)}(\mathrm{H}) + U_{(2,1,2)}(\mathrm{H})$$

- Necessary preference relation after the 2nd piece of preference information



Multiple Criteria Hierarchy Process (MCHP) – value function model

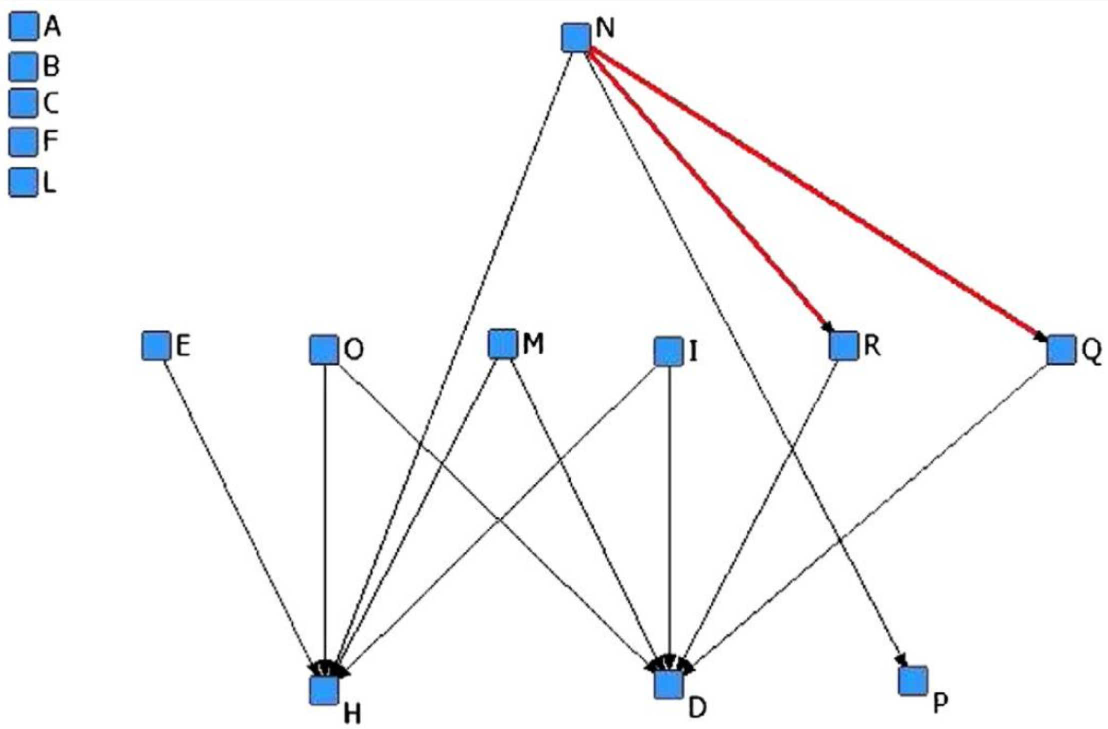
- On Mathematics, student N is preferred to student Q

$$U_{(1)}(\mathrm{N}) > U_{(1)}(\mathrm{Q}) \Leftrightarrow$$

$$\Leftrightarrow U_{(1,1)}(\mathrm{N}) + U_{(1,2)}(\mathrm{N}) > U_{(1,1)}(\mathrm{Q}) + U_{(1,2)}(\mathrm{Q}) \Leftrightarrow$$

$$\Leftrightarrow u_{(1,1,1)}(\mathrm{N}) + u_{(1,1,2)}(\mathrm{N}) + u_{(1,2,1)}(\mathrm{N}) + u_{(1,2,2)}(\mathrm{N}) > u_{(1,1,1)}(\mathrm{Q}) + u_{(1,1,2)}(\mathrm{Q}) + u_{(1,2,1)}(\mathrm{Q}) + u_{(1,2,2)}(\mathrm{Q})$$

- Necessary preference relation after the 3rd piece of preference information



Multiple Criteria Hierarchy Process (MCHP) – value function model

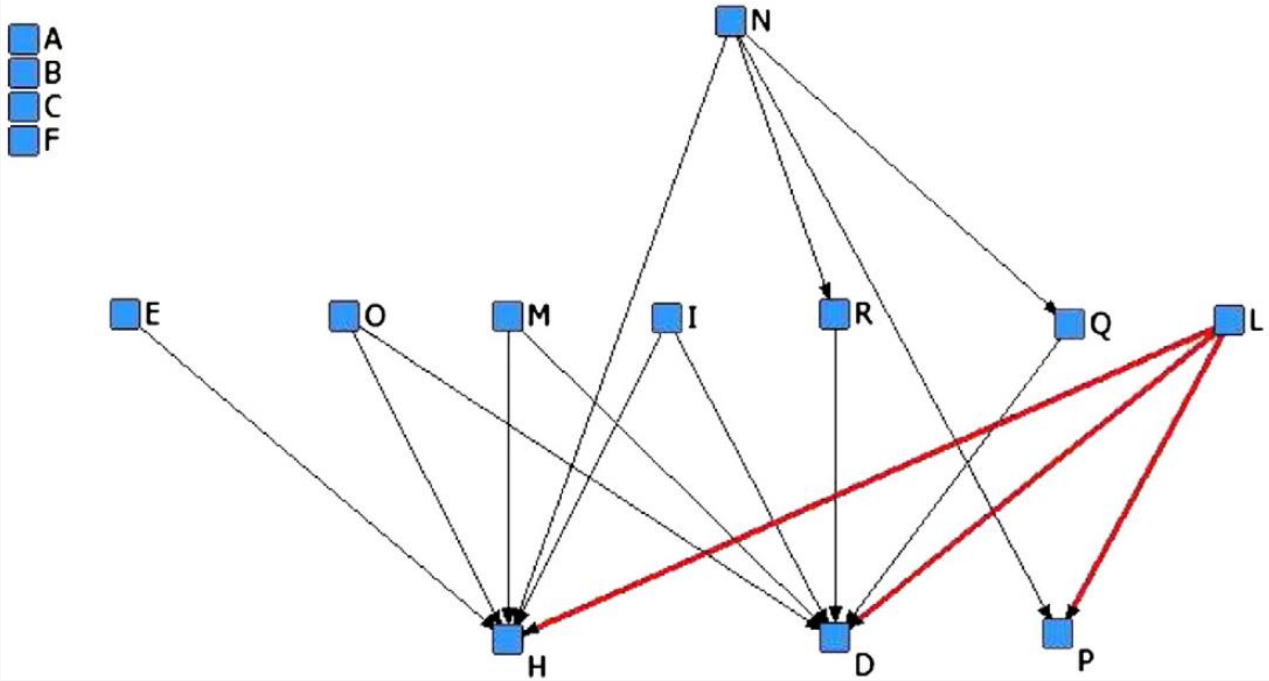
- On Chemistry, student L is preferred to student P

$$U_{(2)}(\mathrm{L}) > U_{(2)}(\mathrm{P}) \Leftrightarrow$$

$$\Leftrightarrow U_{(2,1)}(\mathrm{L}) + U_{(2,2)}(\mathrm{L}) > U_{(2,1)}(\mathrm{P}) + U_{(2,2)}(\mathrm{P}) \Leftrightarrow$$

$$\Leftrightarrow u_{(2,1,1)}(\mathrm{L}) + u_{(2,1,2)}(\mathrm{L}) + u_{(2,2,1)}(\mathrm{L}) + u_{(2,2,2)}(\mathrm{L}) > u_{(2,1,1)}(\mathrm{P}) + u_{(2,1,2)}(\mathrm{P}) + u_{(2,2,1)}(\mathrm{P}) + u_{(2,2,2)}(\mathrm{P})$$

- Necessary preference relation after the 4th piece of preference information



Multiple Criteria Hierarchy Process (MCHP) – value function model

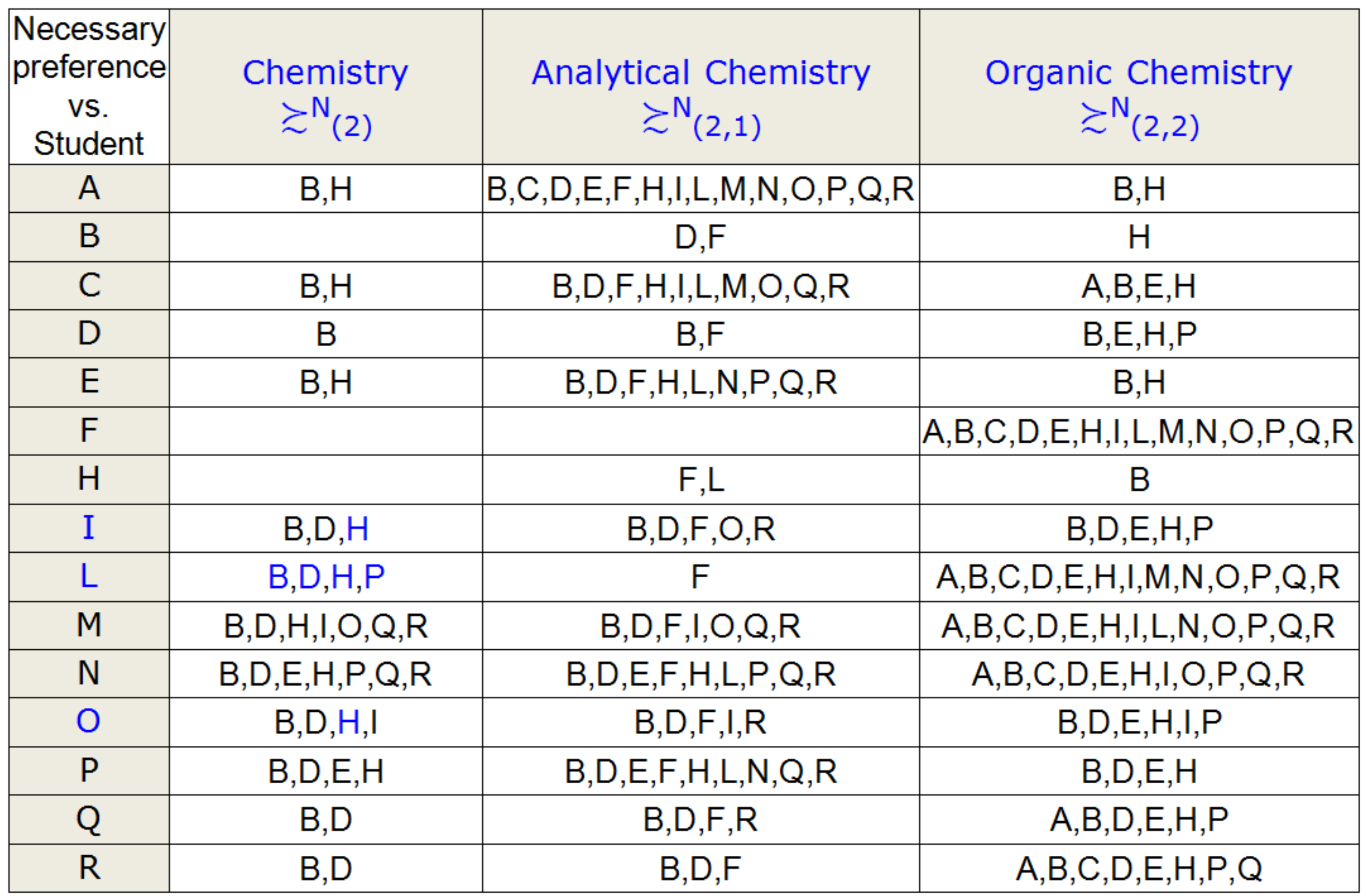
- Necessary preference relation on Mathematics

Necessary preference vs. Student	Mathematics $\succsim^{N}_{(1)}$	Algebra $\succsim^{N}_{(1,1)}$	Analysis $\succsim^{N}_{(1,2)}$
A			
B	A,P	A,P	A,C,D,E,F,H,L,M,O,P,Q,R
C	D	D,F,H,L,M,N,O,P	D,Q,R
D		H,P	R
E	C,D,F,H,M,O,P,Q,R	A,B,C,D,F,H,I,L,M,N,O,P,Q,R	C,D,F,H,M,O,P,Q,R
F	D,H,P	D,H,L,N,P	D,H,O,P,R
H	D	D,P	D,O,R
I	D,F,H,M,O,P,R	D,F,H,L,M,N,O,P,R	C,D,E,F,H,M,O,P,Q,R
L	D,H,P	D,H,P	A,D,F,H,O,P,R
M	D,H	D,H,P	C,D,H,O,Q,R
N	C,D,F,H,P,Q,R	D,F,H,L,P	C,D,E,F,H,I,M,O,P,Q,R
O	D,H	D,F,H,L,M,N,P	D,H,R
P			D,F,H,O,R
Q	C,D,R	A,B,C,D,E,F,H,I,L,M,N,O,P,R	C,D,R
R	D	D,F,H, I,L,M,N,O,P	D

- Remark: the necessary preference relation $N\succsim^{N}_{(1)}C$, $N\succsim^{N}_{(1)}Q$, $N\succsim^{N}_{(1)}R$ is true on Mathematics but it is not true at the level below on Algebra

Multiple Criteria Hierarchy Process (MCHP) – value function model

- Necessary preference relation on Chemistry



Necessary preference vs. Student	Chemistry $\succsim^{N}_{(2)}$	Analytical Chemistry $\succsim^{N}_{(2,1)}$	Organic Chemistry $\succsim^{N}_{(2,2)}$
A	B,H	B,C,D,E,F,H,I,L,M,N,O,P,Q,R	B,H
B		D,F	H
C	B,H	B,D,F,H,I,L,M,O,Q,R	A,B,E,H
D	B	B,F	B,E,H,P
E	B,H	B,D,F,H,L,N,P,Q,R	B,H
F			A,B,C,D,E,H,I,L,M,N,O,P,Q,R
H		F,L	B
I	B,D,H	B,D,F,O,R	B,D,E,H,P
L	B,D,H,P	F	A,B,C,D,E,H,I,M,N,O,P,Q,R
M	B,D,H,I,O,Q,R	B,D,F,I,O,Q,R	A,B,C,D,E,H,I,L,N,O,P,Q,R
N	B,D,E,H,P,Q,R	B,D,E,F,H,L,P,Q,R	A,B,C,D,E,H,I,O,P,Q,R
O	B,D,H,I	B,D,F,I,R	B,D,E,H,I,P
P	B,D,E,H	B,D,E,F,H,L,N,Q,R	B,D,E,H
Q	B,D	B,D,F,R	A,B,D,E,H,P
R	B,D	B,D,F	A,B,C,D,E,H,P,Q

- Remark: the necessary preference relation $I\succsim^{N}_{(2)}H$, $L\succsim^{N}_{(2)}B,D,H,P$, $O\succsim^{N}_{(2)}H$ is true on Chemistry but it is not true at the level below on Analytical Ch.

Multiple Criteria Hierarchy Process (MCHP) – value function model

- Ranking of students by representative value functions

Rank	**Student**
1	M(0.8808)
2	N(0.8622)
3	F(0.6690)
3	L(0.6690)
3	A(0.6690)
6	I(0.5426)
7	C(0.4915)
8	O(0.4893)
9	R(0.4654)
10	Q(0.4617)
11	P(0.4190)
11	E(0.4190)
13	B(0.3808)
14	D(0.2117)
15	H(0.1690)

Multiple Criteria Hierarchy Process (MCHP) – value function model

- Extensions of MCHP applied to value function model
  - Gradual credibility of provided *n* pieces of preference information: for any G_r, nested relations $\succeq^N_{\mathbf{r},1} \subseteq \ldots \subseteq \succeq^N_{\mathbf{r},n}$ and $\succeq^P_{\mathbf{r},1} \supseteq \ldots \supseteq \succeq^P_{\mathbf{r},n}$
  - Extreme ranking analysis: for any G_r, one can see the best and the worst rank of each alternative assigned by compatible value functions
  - Ordinal classification using $UTADIS^{GMS}$: preference information in terms of exemplary assignments wrt any G_r; recommendation in terms of possible & necessary assignments $C^P_r(a)$, $C^N_r(a)$, $\forall a$,r
  - Group decision: for any subset $\mathcal{D}$ of DMs, and for any G_r, one gets 4 types of preference relation - $\gtrsim^{N,N}_r(\mathcal{D})$, $\gtrsim^{N,P}_r(\mathcal{D})$, $\gtrsim^{P,N}_r(\mathcal{D})$, $\gtrsim^{P,P}_r(\mathcal{D})$

S. Corrente, S. Greco, R. Słowiński: Multiple Criteria Hierarchy Process in Robust Ordinal Regression. *Decision Support Systems*, 53 (2012) no.3, 660-674

MCHP and Choquet integral preference model

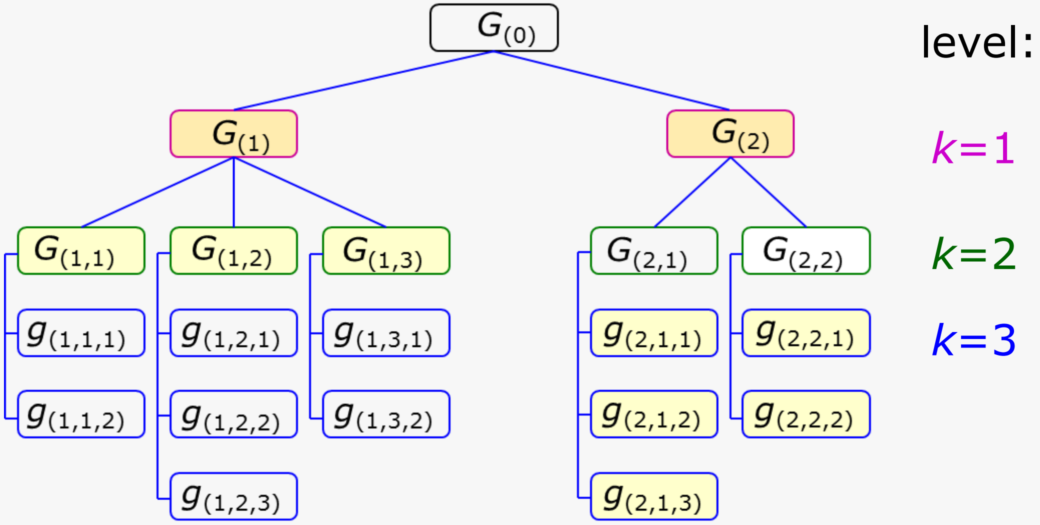
- For each non-elementary criterion $G_{\mathbf{r}}$, located at level h of the tree, the set of descending criteria located at level $k>h$ is denoted by $\mathcal{G}^k_{\mathbf{r}}$:

$$\mathcal{G}^2_{(1)} = \left\{ G_{(1,1)}, G_{(1,2)}, G_{(1,3)} \right\},$$

for $\mathbf{r} = (1),\ h = 1,\ k = 2$

$$\mathcal{G}^3_{(2)} = \left\{ \begin{matrix} g_{(2,1,1)}, g_{(2,1,2)}, g_{(2,1,3)} \\ g_{(2,2,1)}, g_{(2,2,2)} \end{matrix} \right\},$$

for $\mathbf{r} = (2),\ h = 1,\ k = 3$



- To each alternative $a \in A$, there corresponds a performance vector

$$\left[g_{\mathbf{t}_1}(a), \dots, g_{\mathbf{t}_n}(a) \right], \quad \text{where} \quad n = |EL|$$

S. Angilella, S. Corrente, S. Greco, R. Słowiński: Multiple Criteria Hierarchy Process for the Choquet integral. [In]: R.C. Purshouse et al. (eds.): *EMO 2013*, LNCS 7811, Springer, Berlin, 2013, pp. 475–489

MCHP and Choquet integral preference model

- Given capacity μ defined on the power set of *EL* (elementary criteria), a capacity $\mu_{\mathbf{r}}^{k}$ on the power set of $\mathcal{G}^{k}_{\mathbf{r}}$:

$$\mu_{\mathbf{r}}^{k} : 2^{\mathcal{G}_{\mathbf{r}}^{k}} \rightarrow [0,1]$$

such that

$$\text{for all } \mathcal{F} \subseteq \mathcal{G}_{\mathbf{r}}^{k} \qquad \mu_{\mathbf{r}}^{k}(\mathcal{F}) = \frac{\mu(E(\mathcal{F}))}{\mu(E(G_{\mathbf{r}}))}$$

- Considering non-elementary criterion $G_{\mathbf{r}}$ (in node **r**) and alternative $a \in A$, given capacity μ defined on the power set of *EL*, the Choquet integral score of *a*:

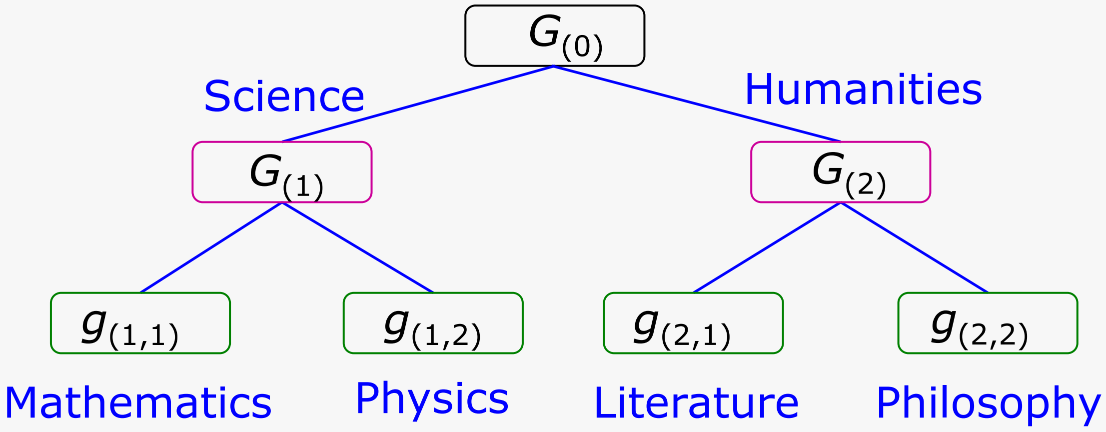
$$Ch_{\mu_{\mathbf{r}}}(a) = \frac{Ch_{\mu}(a_{\mathbf{r}})}{\mu(E(G_{\mathbf{r}}))},$$

$$\text{where } a_{\mathbf{r}} \text{ is such that } g_{\mathbf{t}}(a_{\mathbf{r}}) = \begin{cases} g_{\mathbf{t}}(a) & \text{if } g_{t} \in E(G_{\mathbf{r}}) \\ 0 & \text{otherwise} \end{cases}$$

MCHP and Choquet integral preference model

- Example:

  Ranking of students wrt. hierarchical criteria of Science & Humanities



- 9 students: *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *k*

MCHP and Choquet integral preference model

- Evaluation of students on elementary criteria

Student	Mathematics	Physics	Literature	Philosophy
a	18	18	12	12
b	16	16	16	16
c	14	14	18	18
d	18	12	16	16
e	15	15	18	14
f	18	14	14	18
g	15	17	18	16
h	10	20	10	20
k	14	14	14	14

MCHP and Choquet integral preference model

- Möbius measures on elementary criteria

Mathematics	$m(\{g_{(1,1)}\})$	0.29
Physics	$m(\{g_{(1,2)}\})$	0.19
Literature	$m(\{g_{(2,1)}\})$	0.29
Philosophy	$m(\{g_{(2,2)}\})$	0.19
Mathematics & Physics	$m(\{g_{(1,1)},g_{(1,2)}\})$	-0.1
Mathematics & Literature	$m(\{g_{(1,1)},g_{(2,1)}\})$	0
Mathematics & Philosophy	$m(\{g_{(1,1)},g_{(2,2)}\})$	0
Physics & Literature	$m(\{g_{(1,2)},g_{(2,1)}\})$	0
Physics & Philosophy	$m(\{g_{(1,2)},g_{(2,2)}\})$	0.24
Literature & Philosophy	$m(\{g_{(2,1)},g_{(2,2)}\})$	-0.1

MCHP and Choquet integral preference model

- Choquet integral score

wrt Science ($Ch_{\mu 1}$), Humanities ($Ch_{\mu 2}$), and Overall (Ch_{μ})

$Ch_{\mu 1}(a)$ $Ch_{\mu 2}(a)$ $Ch_{\mu}(a)$	18 12 14.28	$Ch_{\mu 1}(f)$ $Ch_{\mu 2}(f)$ $Ch_{\mu}(f)$	17.05 16 15.92
$Ch_{\mu 1}(b)$ $Ch_{\mu 2}(b)$ $Ch_{\mu}(b)$	16 16 16	$Ch_{\mu 1}(g)$ $Ch_{\mu 2}(g)$ $Ch_{\mu}(g)$	16 17.52 16.58
$Ch_{\mu 1}(c)$ $Ch_{\mu 2}(c)$ $Ch_{\mu}(c)$	14 18 15.52	$Ch_{\mu 1}(h)$ $Ch_{\mu 2}(h)$ $Ch_{\mu}(h)$	13.5 13.5 $\boxed{15.06}$
$Ch_{\mu 1}(d)$ $Ch_{\mu 2}(d)$ $Ch_{\mu}(d)$	16.57 16 15.26	$Ch_{\mu 1}(k)$ $Ch_{\mu 2}(k)$ $Ch_{\mu}(k)$	14 14 $\boxed{14}$
$Ch_{\mu 1}(e)$ $Ch_{\mu 2}(e)$ $Ch_{\mu}(e)$	15 17.05 15.54		

Ranking

Overall:

$g \succ b \succ f \succ e \succ c \succ d \succ h \succ a \succ k$

Science:

$a \succ f \succ d \succ b, g \succ e \succ c, k \succ h$

Humanities:

$c \succ g \succ e \succ b, d, f \succ k \succ h \succ a$

MCHP and Choquet integral preference model

- Shapley values & interaction indices
  - Shapley value of each elementary criterion wrt parent criterion G_r

$G_{\mathbf{r}}$, $\mathbf{r}$=(1),(2)	Science		Humanities	
$G_{(\mathbf{r},w)}$ w=1,2,3,4	Mathematics	Physics	Literature	Philosophy
$\varphi^2_{\mathbf{r}}(G_{(\mathbf{r},w)})$	0.63	0.37	0.63	0.37

  - Shapley value of each elementary criterion wrt root criterion $G_{(0)}$

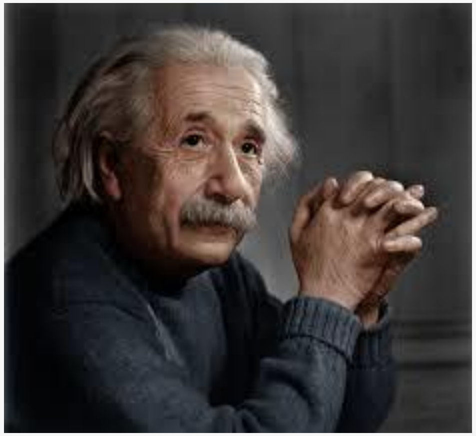
$G_{(\mathbf{r},w)}$, $\mathbf{r}$=(0) w=1,2,3,4	Mathematics	Physics	Literature	Philosophy
$\varphi^2_{\mathbf{r}}(G_{(\mathbf{r},w)})$	0.24	0.26	0.24	0.26

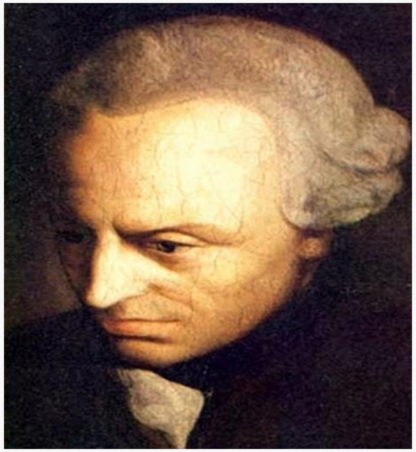
  - Shapley value & interaction index for Science ($G_{(1)}$) & Humanities ($G_{(2)}$)

$G_{(\mathbf{r},w)}$, $\mathbf{r}$=(0), w=1,2	Science	Humanities
$\varphi^1_{\mathbf{r}}(G_{(\mathbf{r},w)})$	0.5	0.5
$\varphi^1_{\mathbf{r}}(\{G_{(\mathbf{r},1)}, G_{(\mathbf{r},2)}\})$	0.24	

Is it reasonable?

A friend of the Einstein family, Max Talmey, tutored Albert and recommended him the “Critique of pure reason” when when he was 12 years old. This became a new Bible for him and he called it his "holy geometry book‟.





MCHP and Choquet integral assessed using ROR

- Types of indirect preference information supplied by the DM:
  - Considering some reference alternatives $a,b,c,d \in A$
    - Pairwise comparison: a is at least as good as b on criterion $G_{\mathbf{r}}$

      $$a \succsim_{\mathbf{r}} b \quad \Leftrightarrow \quad Ch_{\mathbf{r}}(a) \geq Ch_{\mathbf{r}}(b)$$

    - Intensity of preference: considering criterion $G_{\mathbf{r}}$,
      a is preferred to b at least as much, as c is preferred to d

      $$(a,b) \succsim_{\mathbf{r}}^{*} (c,d) \Leftrightarrow Ch_{\mathbf{r}}(a) - Ch_{\mathbf{r}}(b) \geq Ch_{\mathbf{r}}(c) - Ch_{\mathbf{r}}(d) \geq 0$$

  - Considering some criteria $G_{\mathbf{r}1}, G_{\mathbf{r}2}, G_{\mathbf{r}3}, G_{\mathbf{r}4} \in \mathcal{G}^{k}_{\mathbf{r}}$:
    - $G_{\mathbf{r}1}$ is at least as important as $G_{\mathbf{r}2}$

      $$\varphi_{\mathbf{r}}^{k}(G_{\mathbf{r}1}) \geq \varphi_{\mathbf{r}}^{k}(G_{\mathbf{r}2})$$

    - $G_{\mathbf{r}1}$ and $G_{\mathbf{r}2}$ are positively (negatively) interacting

      $$\varphi_{\mathbf{r}}^{k}(G_{\mathbf{r}1}, G_{\mathbf{r}2}) \geq \varepsilon \qquad \left(\varphi_{\mathbf{r}}^{k}(G_{\mathbf{r}1}, G_{\mathbf{r}2}) \leq -\varepsilon\right)$$

    - $G_{\mathbf{r}1}$ is preferred to $G_{\mathbf{r}2}$ at least as much as $G_{\mathbf{r}3}$ is preferred to $G_{\mathbf{r}4}$

      $$\varphi_{\mathbf{r}}^{k}(G_{\mathbf{r}1}) - \varphi_{\mathbf{r}}^{k}(G_{\mathbf{r}2}) \geq \varphi_{\mathbf{r}}^{k}(G_{\mathbf{r}3}) - \varphi_{\mathbf{r}}^{k}(G_{\mathbf{r}4}) \geq 0$$

MCHP and Choquet integral assessed using ROR

- Information provided to the DM:
  - Applying Robust Ordinal Regression to Choquet integral (NAROR), i.e., solving a LP problem for each pair of alternatives $a,b \in A$ in each node r, one gets 2 partial relations (necessary & possible):
  - a is weakly preferred to b wrt G_r for all compatible capacities

$$a \succsim^N_r b$$

  - a is weakly preferred to b wrt G_r for at least one compatible capacity

$$a \succsim^P_r b$$

- Remark: in the LP problem of ROR, the only decision variables are 2-additive Möbius function values representing capacities μ defined on the power set of elementary criteria

S. Angilella, S. Greco, B. Matarazzo (2010). Non-additive robust ordinal regression: A multiple criteria decision model based on the Choquet integral. *European Journal of Operational Research*, *201*(1), 277-288.

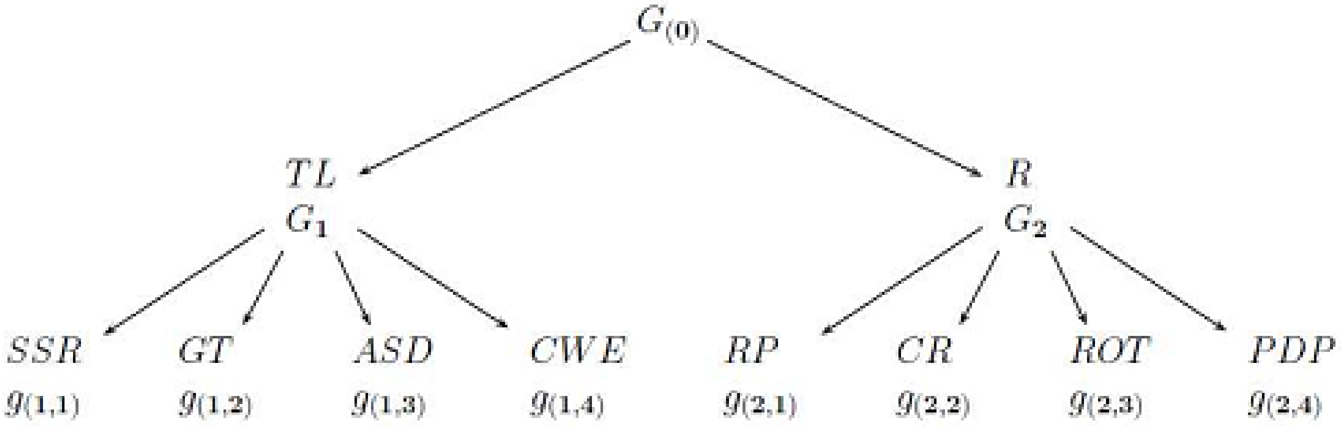
MCHP and Choquet integral assessed using ROR & SMAA

- MCHP & ROR & Choquet integral identify a set of capacities compaptible with preference information
- To explore the set of compatible capacities, we use the Hit-And-Run method (Tervonen et al. 2013) & Stochastic Multiobjctive Acceptability Analysis (SMAA - Lahdelma, Hokkanen, Salminen 1998) that yields:
  - the rank acceptability index $b^i_{k,r}$ – probability that alternative a_k gets position i in the ranking obtained wrt criterion G_r
  - the pairwise winning index $p_r(a,b)$ – probability of preference of a over b on criterion G_r

S. Angilella, S. Corrente, S. Greco: Stochastic multiobjective acceptability analysis for the Choquet integral preference model and the scale construction problem. *European Journal of Operational Research*, (2015), *240*(1), 172-182.

S. Angilella, S. Corrente, S. Greco, R. Słowiński: Robust Ordinal Regression & Stochastic Multiobjective Acceptability Analysis in Multiple Criteria Hierarchy Process for the Choquet integral preference model. *Omega*, 2015

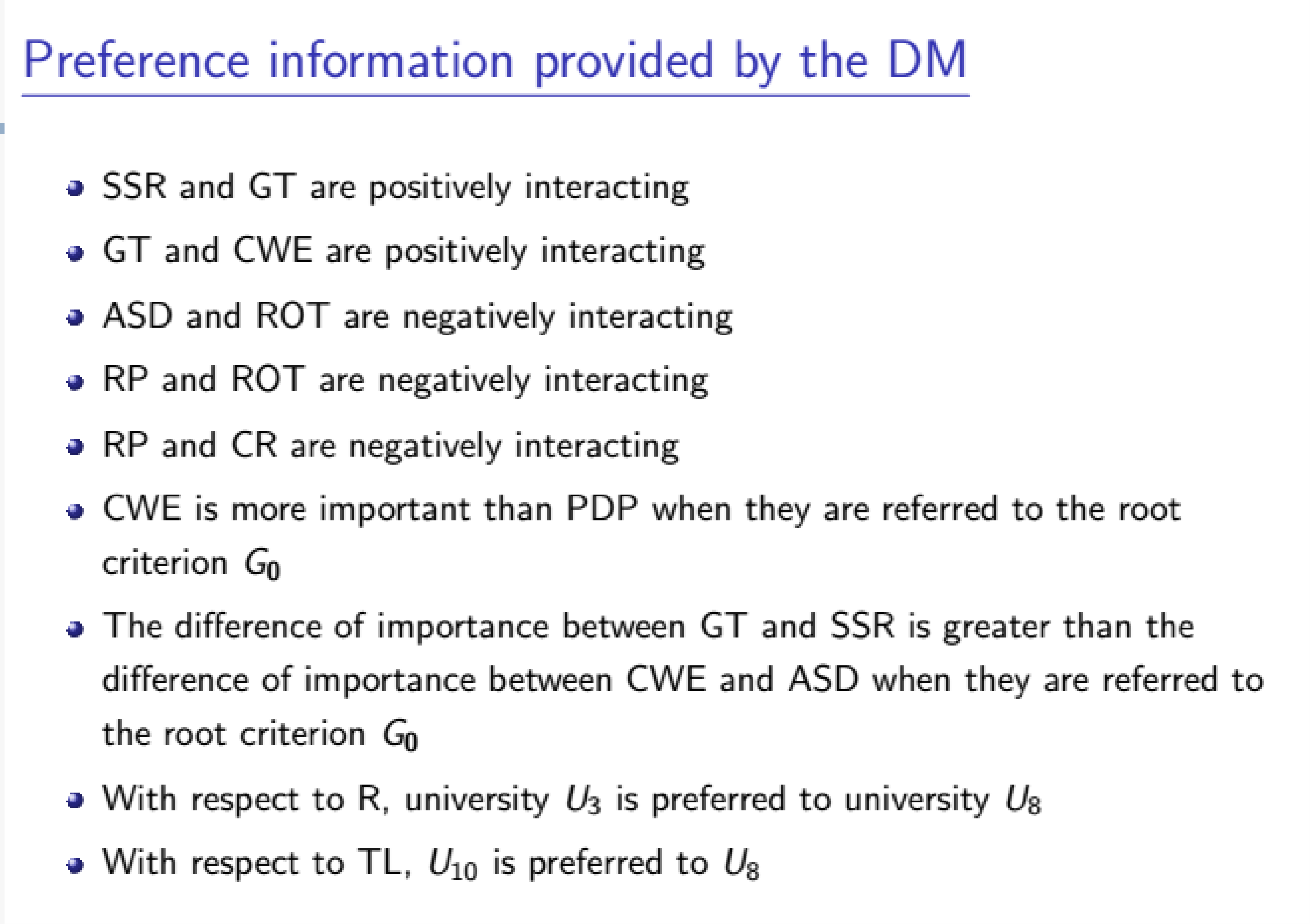
Decision problem



Elementary subriterion	Description
Student-Staff Ratio (SSR)	The number of students per member of the academic staff
Graduating on Time (GT)	The percentage of graduates that graduated within the time expected for their bachelor program
Academic Staff with Doctorates (ASD)	The percentage of academic staff holding a doctorate
Contact with Work Environment (CWE)	A composite measure representing at bachelor level: (1) the inclusion of internships or phases of practical experience in the curriculum; (2) the percentage of students doing an internship (3) teaching by practitioners from outside the university departments
Research Publications (RP)	The number of research publications indexed in the Web of Science database, where at least one author is affiliated to the university
Citation Rate (CR)	The average number of times that the university department's research publications (over the period 2008-2011) get cited in other research, adjusted (normalized) at the global level for the field of science and the year in which a publication appeared
Research Orientation of Teaching (ROT)	The degree to which the education is informed by research in the field (based on a survey of students in the program)
Post-Doc Positions (PDP)	The number of post-doc positions relative to the full-time equivalent number of academic staff

Performances of the universities on the considered criteria

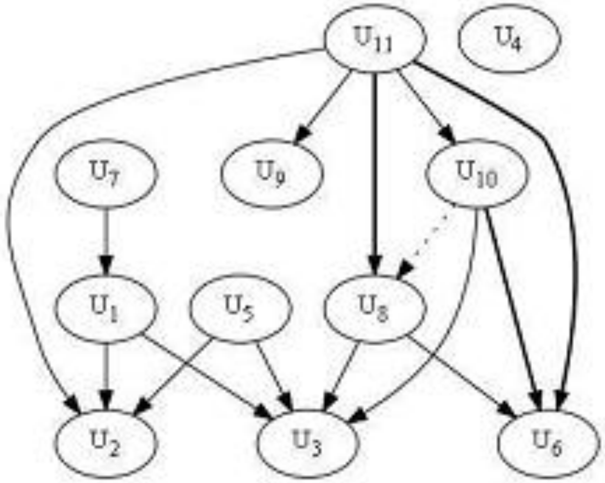
		G_0							
		TL				R			
University	**Country**	**SSR**	**GT**	**ASD**	**CWE**	**RP**	**CR**	**ROT**	**PDP**
WHU School of Management (U_1)	Germany	2	5	5	3	5	5	5	2
Aarhus U (U_2)	Denmark	2	2	5	2	5	3	3	5
U Tampere (U_3)	Finland	2	2	3	3	5	3	3	5
Lille Catholic U (U_4)	France	5	1	4	4	5	4	3	1
U Paris West (U_5)	France	3	4	5	4	5	2	2	1
Polytech, U Milan (U_6)	Italy	5	3	2	4	5	4	3	5
U Trento (U_7)	Italy	2	5	5	5	4	5	3	5
Vilnius Gediminas Technical U (U_8)	Lithuania	5	4	3	4	5	5	4	2
U Porto (U_9)	Portugal	5	2	4	2	5	4	4	2
Bucharest U Economic Studies Marketing (U_{10})	Romania	4	4	4	3	5	2	2	5
Bucharest Faculty of Business Administration (U_{11})	Romania	5	4	5	3	5	2	3	3



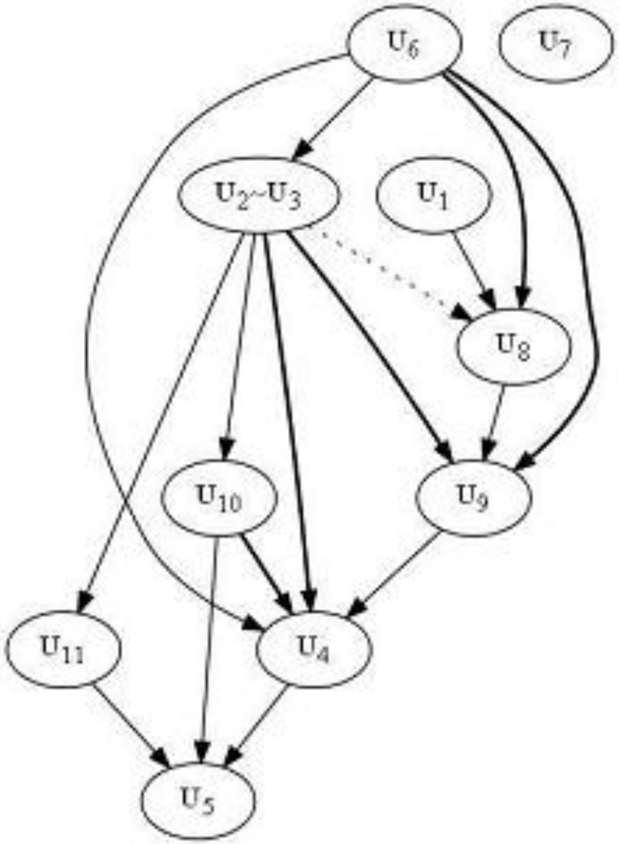
Preference information provided by the DM

- SSR and GT are positively interacting
- GT and CWE are positively interacting
- ASD and ROT are negatively interacting
- RP and ROT are negatively interacting
- RP and CR are negatively interacting
- CWE is more important than PDP when they are referred to the root criterion G_0
- The difference of importance between GT and SSR is greater than the difference of importance between CWE and ASD when they are referred to the root criterion G_0
- With respect to R, university U_3 is preferred to university U_8
- With respect to TL, U_{10} is preferred to U_8

MCHP and NAROR results



(a) Teaching and Learning (TL)



(b) Research (R)

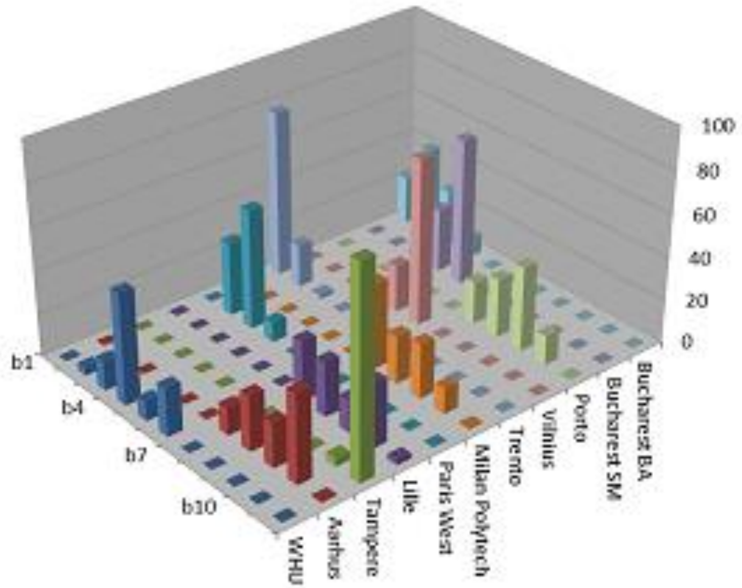
MCHP and SMAA (Rank Acceptability Indices)

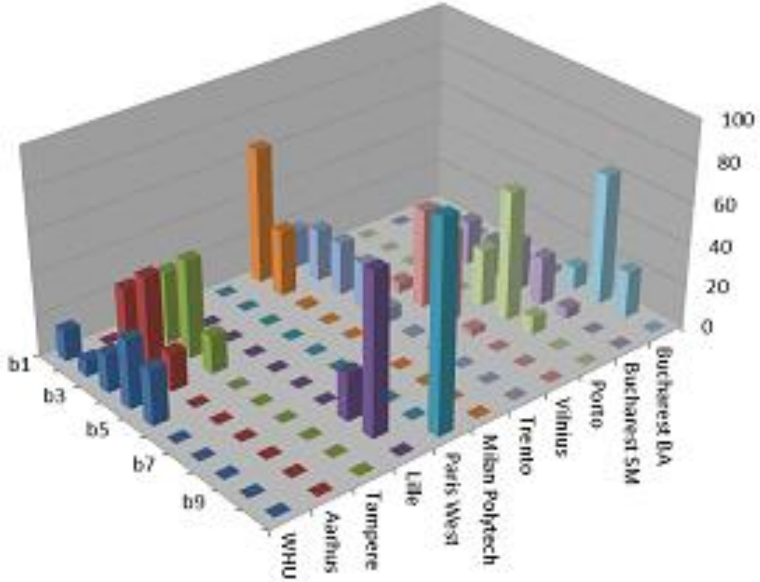
(q) Teaching and Learning

$TL\ (G_1)$	$b^1_{s,1}$	$b^2_{s,1}$	$b^3_{s,1}$	$b^4_{s,1}$	$b^5_{s,1}$	$b^6_{s,1}$	$b^7_{s,1}$	$b^8_{s,1}$	$b^9_{s,1}$	$b^{10}_{s,1}$	$b^{11}_{s,1}$
WHU	0	3.61	13.59	51.48	9.01	22.31	0	0	0	0	0
Aarhus	0	0	0	0	0	0	12.65	25.31	20.34	40.98	0.72
Tampere	0	0	0	0	0	0	0	0	0	3.95	96.05
Lille	0	0	0	0	0	0	27.87	25.13	15.13	29.53	2.34
Paris West	0	34.55	55.8	9.31	0.34	0	0	0	0	0	0
Milan Polytech	0	0	0	0	0	0	39.87	22.19	25.32	11.85	0.77
Trento	77.68	20.05	2.16	0.11	0	0	0	0	0	0	0
Vilnius	0	0	0.11	0.23	21.97	77.69	0	0	0	0	0
Porto	0	0	0	0	0	0	19.61	27.37	39.21	13.69	0.12
Bucharest SM	0	0.17	1.7	29.45	68.68	0	0	0	0	0	0
Bucharest BA	22.32	41.62	26.64	9.42	0	0	0	0	0	0	0

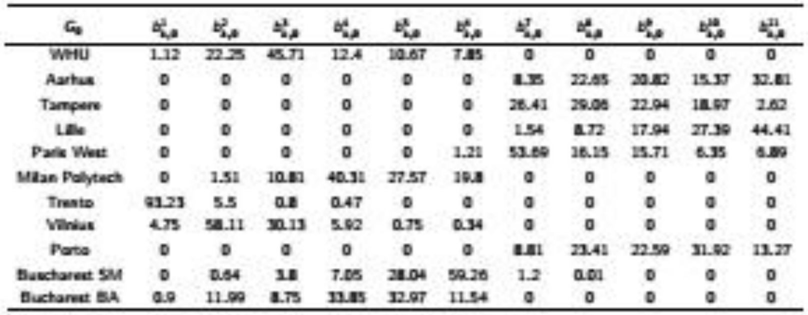
(r) Research

$R\ (G_2)$	$b^1_{s,2}$	$b^2_{s,2}$	$b^3_{s,2}$	$b^4_{s,2}$	$b^5_{s,2}$	$b^6_{s,2}$	$b^7_{s,2}$	$b^8_{s,2}$	$b^9_{s,2}$	$b^{10}_{s,2}$
WHU	14.74	7.56	18.34	32.87	26.48	0.01	0	0	0	0
Aarhus	0	34.46	47.29	18.25	0	0	0	0	0	0
Tampere	0	34.46	47.29	18.25	0	0	0	0	0	0
Lille	0	0	0	0	0	0	0	22.65	77.35	0
Paris West	0	0	0	0	0	0	0	0	0	100
Milan Polytech	66.42	32.46	1.12	0	0	0	0	0	0	0
Trento	18.84	25.52	25.73	22.21	6.84	0.86	0	0	0	0
Vilnius	0	0	0	5.34	48.18	43.66	2.82	0	0	0
Porto	0	0	0	0	0.46	27.96	62.88	8.7	0	0
Bucharest SM	0	0	7.52	21.33	18.03	24.6	23.34	5.18	0	0
Bucharest BA	0	0	0	0	0.01	2.91	10.96	63.47	22.65	0

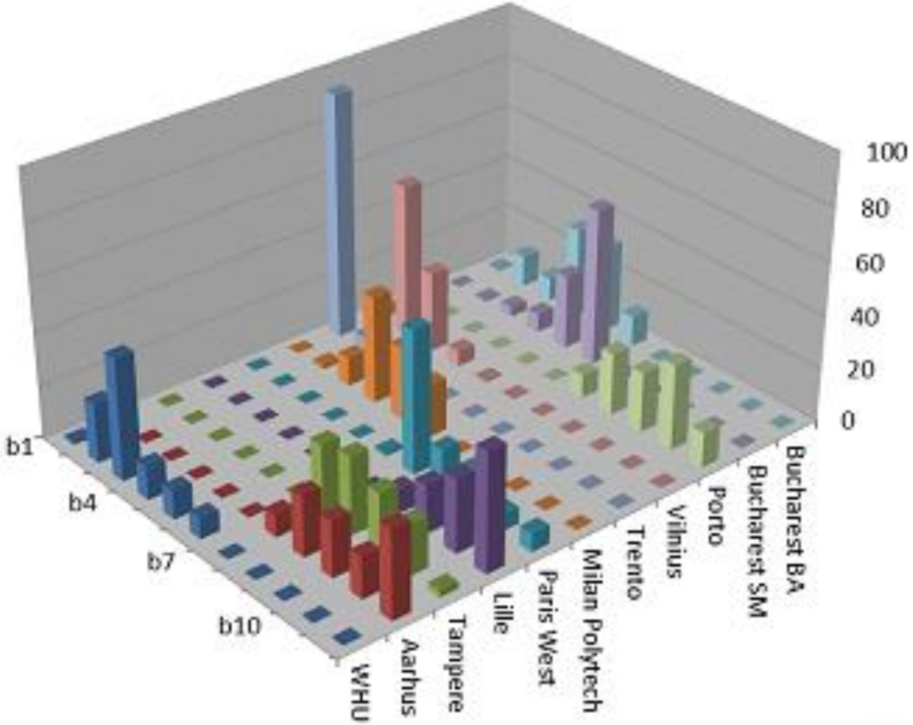




RAI at comprehensive level

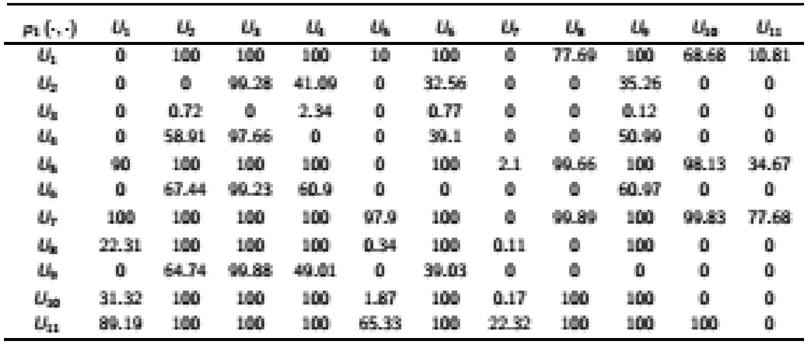


G_s	$b^1_{s,\beta}$	$b^2_{s,\beta}$	$b^3_{s,\beta}$	$b^4_{s,\beta}$	$b^5_{s,\beta}$	$b^6_{s,\beta}$	$b^7_{s,\beta}$	$b^8_{s,\beta}$	$b^9_{s,\beta}$	$b^{10}_{s,\beta}$	$b^{11}_{s,\beta}$
WHU	1.12	22.25	45.71	12.4	10.67	7.85	0	0	0	0	0
Aarhus	0	0	0	0	0	0	8.35	22.65	20.82	15.37	32.81
Tampere	0	0	0	0	0	0	26.41	29.06	22.94	18.97	2.62
Lille	0	0	0	0	0	0	1.54	8.72	17.94	27.39	44.41
Paris West	0	0	0	0	0	1.21	53.69	16.15	15.71	6.35	6.89
Milan Polytech	0	1.51	10.81	40.31	27.57	19.8	0	0	0	0	0
Trento	93.23	5.5	0.8	0.47	0	0	0	0	0	0	0
Vilnius	4.75	58.11	30.13	5.92	0.75	0.34	0	0	0	0	0
Porto	0	0	0	0	0	0	8.81	23.41	22.59	31.92	13.27
Bucharest SM	0	0.64	3.8	7.05	28.04	59.26	1.2	0.01	0	0	0
Bucharest BA	0.9	11.99	8.75	33.85	32.97	11.54	0	0	0	0	0



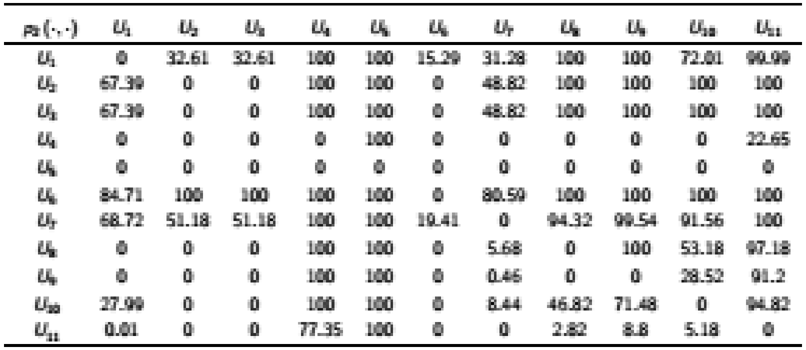
MCHP and SMAA (Pairwise Winning Indices)

(s) Teaching and Learning



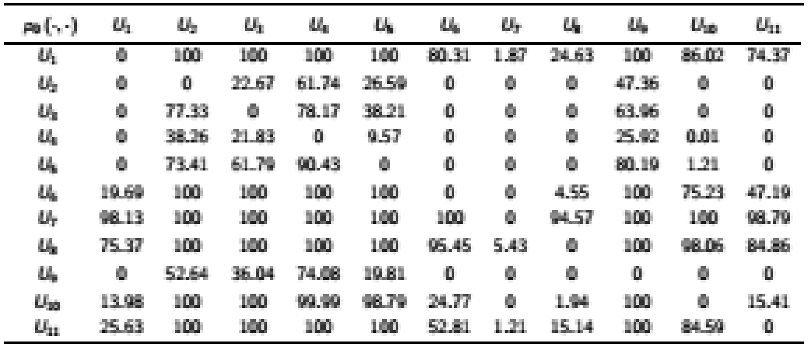
$p_1(\cdot,\cdot)$	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}	U_{11}
U_1	0	100	100	100	10	100	0	77.69	100	68.68	10.81
U_2	0	0	99.28	41.09	0	32.56	0	0	35.26	0	0
U_3	0	0.72	0	2.34	0	0.77	0	0	0.12	0	0
U_4	0	58.91	97.66	0	0	39.1	0	0	50.99	0	0
U_5	90	100	100	100	0	100	2.1	99.66	100	98.13	34.67
U_6	0	67.44	99.23	60.9	0	0	0	0	60.97	0	0
U_7	100	100	100	100	97.9	100	0	99.89	100	99.83	77.68
U_8	22.31	100	100	100	0.34	100	0.11	0	100	0	0
U_9	0	64.74	99.88	49.01	0	39.03	0	0	0	0	0
U_{10}	31.32	100	100	100	1.87	100	0.17	100	100	0	0
U_{11}	89.19	100	100	100	65.33	100	22.32	100	100	100	0

(t) Research



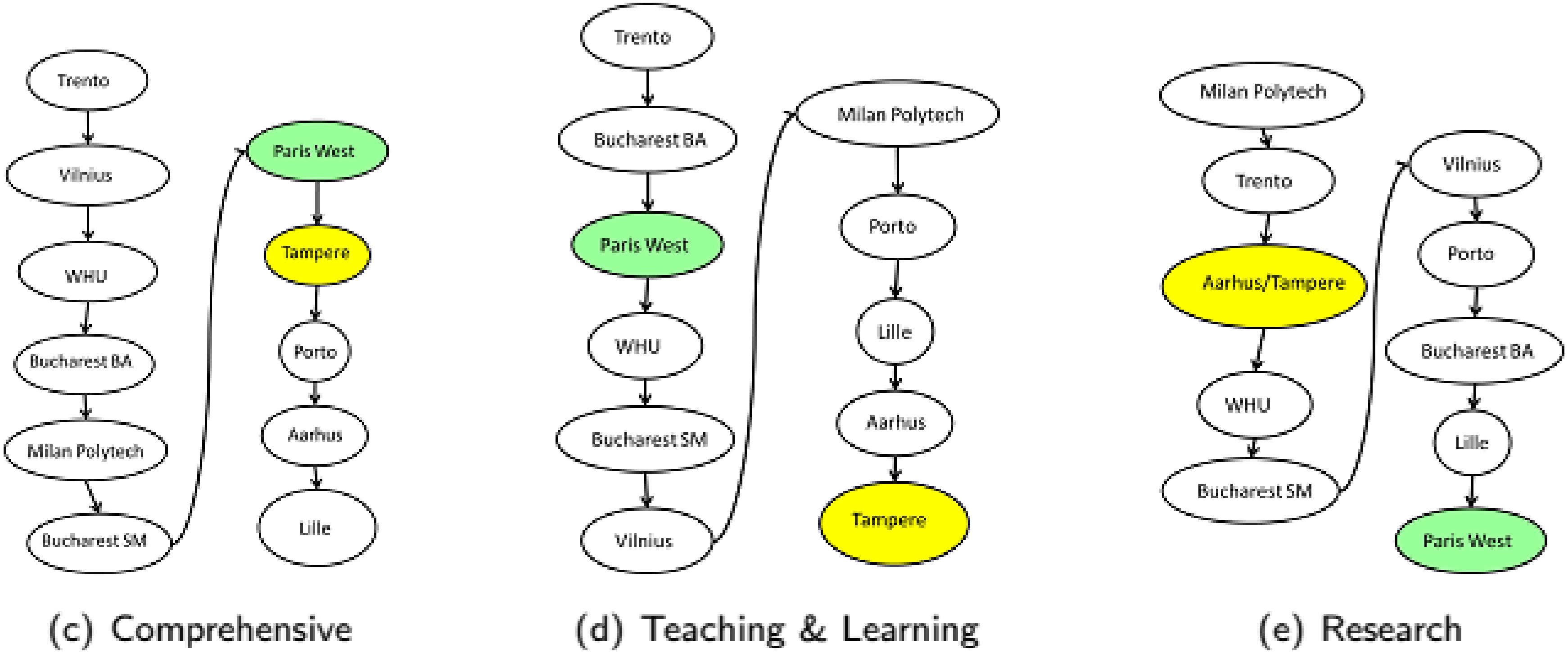
$p_2(\cdot,\cdot)$	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}	U_{11}
U_1	0	32.61	32.61	100	100	15.29	31.28	100	100	72.01	99.99
U_2	67.39	0	0	100	100	0	48.82	100	100	100	100
U_3	67.39	0	0	100	100	0	48.82	100	100	100	100
U_4	0	0	0	0	100	0	0	0	0	0	22.65
U_5	0	0	0	0	0	0	0	0	0	0	0
U_6	84.71	100	100	100	100	0	80.59	100	100	100	100
U_7	68.72	51.18	51.18	100	100	19.41	0	94.32	99.54	91.56	100
U_8	0	0	0	100	100	0	5.68	0	100	53.18	97.18
U_9	0	0	0	100	100	0	0.46	0	0	28.52	91.2
U_{10}	27.99	0	0	100	100	0	8.44	46.82	71.48	0	94.82
U_{11}	0.01	0	0	77.35	100	0	0	2.82	8.8	5.18	0

(u) Comprehensive level



$p_0(\cdot,\cdot)$	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}	U_{11}
U_1	0	100	100	100	100	80.31	1.87	24.63	100	86.02	74.37
U_2	0	0	22.67	61.74	26.59	0	0	0	47.36	0	0
U_3	0	77.33	0	78.17	38.21	0	0	0	63.96	0	0
U_4	0	38.26	21.83	0	9.57	0	0	0	25.92	0.01	0
U_5	0	73.41	61.79	90.43	0	0	0	0	80.19	1.21	0
U_6	19.69	100	100	100	100	0	0	4.55	100	75.23	47.19
U_7	98.13	100	100	100	100	100	0	94.57	100	100	98.79
U_8	75.37	100	100	100	100	95.45	5.43	0	100	98.06	84.86
U_9	0	52.64	36.04	74.08	19.81	0	0	0	0	0	0
U_{10}	13.98	100	100	99.99	98.79	24.77	0	1.94	100	0	15.41
U_{11}	25.63	100	100	100	100	52.81	1.21	15.14	100	84.59	0

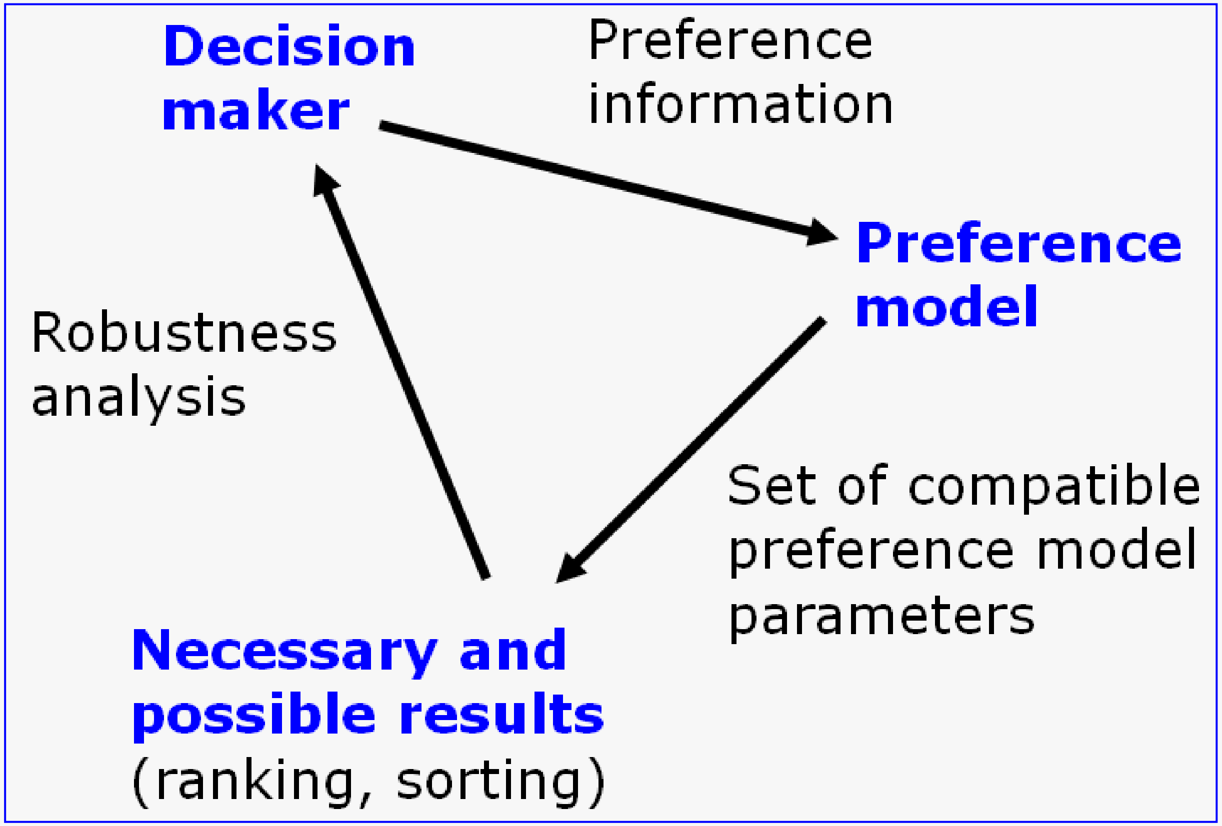
Complete rankings of universities by the most representative capacity



Interactive optimization with Robust Ordinal Regression

Interactive optimization with Robust Ordinal Regression

- Robust Ordinal Regression in a loop:
  preference elicitation with constructive learning
- Results are robust, because they take into account partial preference information



Decision maker

Preference information

Preference model

Set of compatible preference model parameters

Necessary and possible results (ranking, sorting)

Robustness analysis

Interactive optimization with Robust Ordinal Regression

Input (preference information)

- Pairwise comparisons of solutions
- Best (or worst) solution out of a set
- Ranking of several solutions
- Ordinal or cardinal intensity of preference for pairs of solutions
- Sorting of solutions into quality classes
- …

Output (preference model)

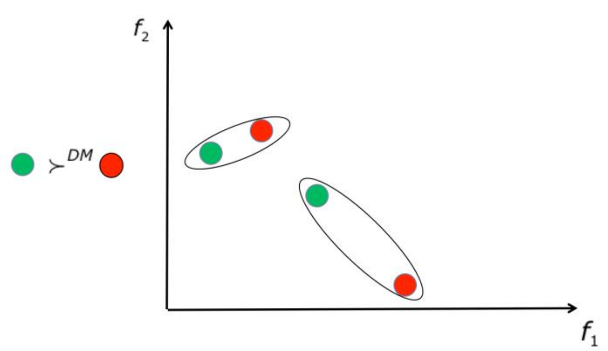
- Value function
- Outranking relation
- Artificial neural network
- Decision rules
- Decision trees
- …

How complex should the preference model be?

- Model too simple

  ➡ not able to represent user’s preferences

- Example: linear model unable to capture preference information



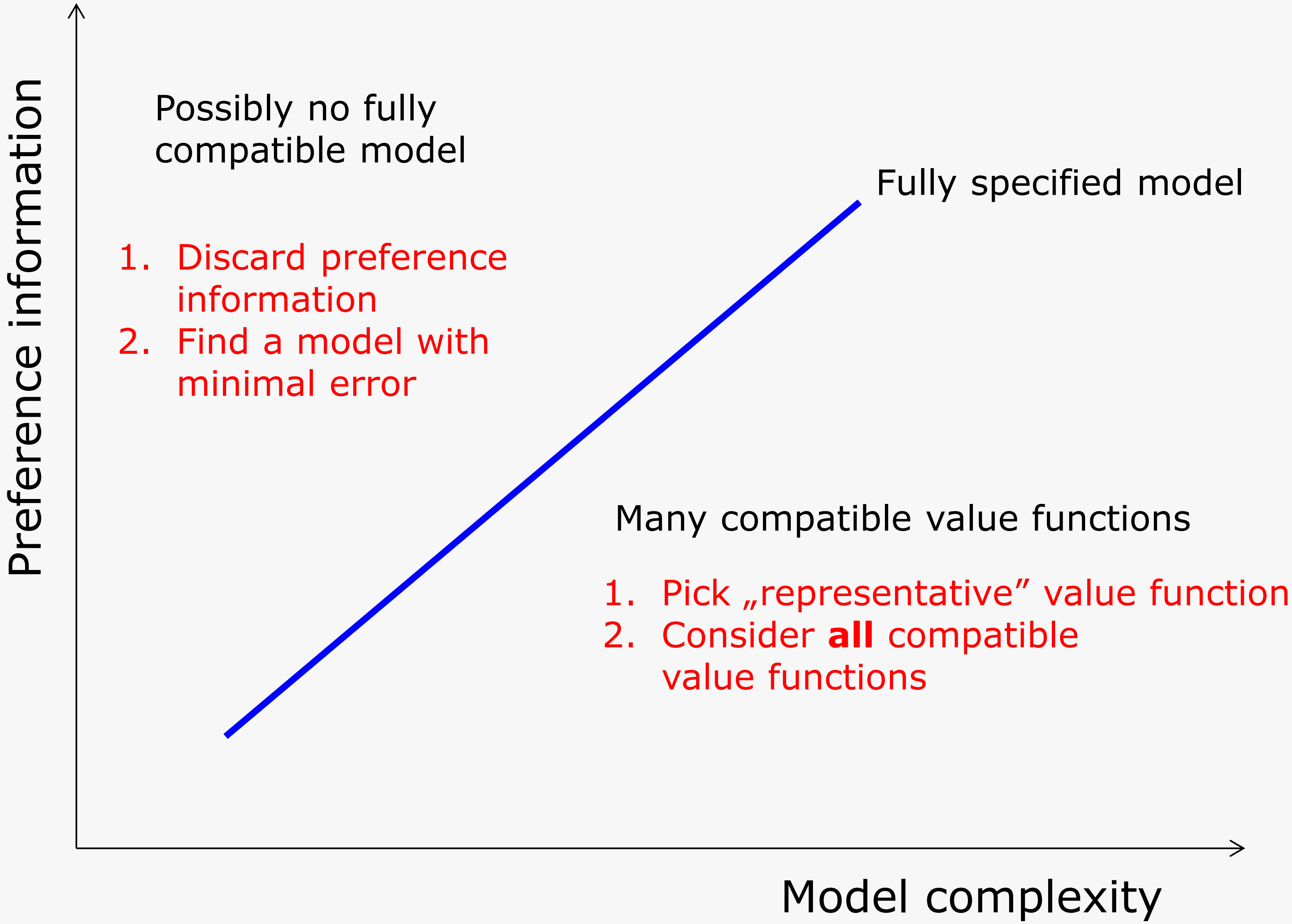
- Model too complex/flexible

  ➡ no generalization power, all solutions enter only one front, takes very long to learn all the parameters

- Example: Dominance relation, general additive model with monotonic marginal value functions

„Everything should be made as simple as possible – but not simpler” [Albert Einstein]

Preference information and model complexity



Preference information

Possibly no fully compatible model

1. Discard preference information
2. Find a model with minimal error

Fully specified model

Many compatible value functions

1. Pick „representative” value function
2. Consider **all** compatible value functions

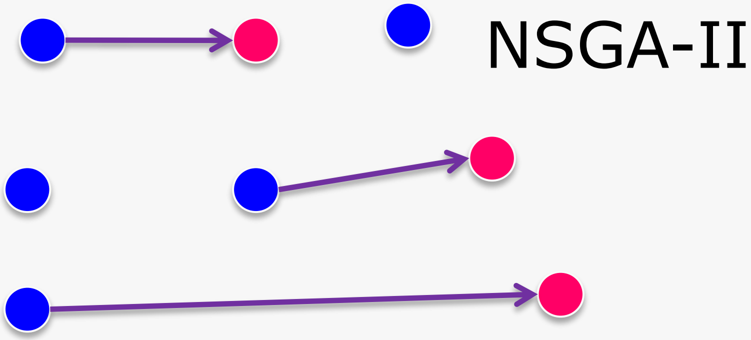
Model complexity

The NEMO framework [Branke, Greco, Słowiński, Zielniewicz 2009, 2010, 2014] [Branke, Corrente, Greco, Słowiński, Zielniewicz 2014]

- NEMO integrates ROR into NSGA-II (Deb et al. 2000)
- Every *q* iterations, the DM is expressing preferences by comparing pairwise some non-dominated solutions in the current population
- Preference model:
  - Linear value function $U(a) = \sum_{i=1}^{n} w_i \times f_i(a)$
  - General additive value function $U(a) = \sum_{i=1}^{n} u_i[f_i(a)]$
  - Choquet integral $U(a) = \sum_{i=1}^{n} \mu(F_i)\left(f_{(i)}(a) - f_{(i-1)}(a)\right)$
  - …
- No scaling of objectives is necessary – NEMO handles heterogeneous objectives

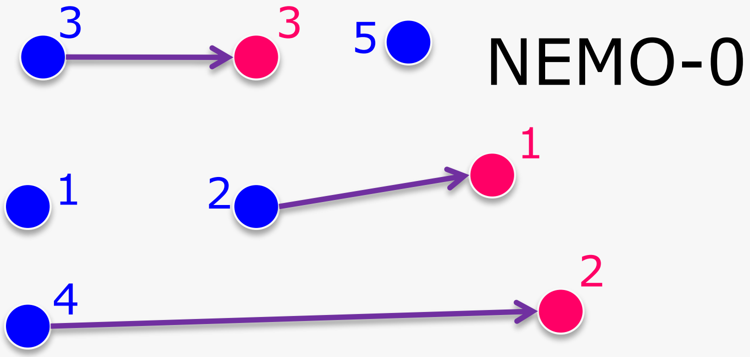
The NEMO framework

- NSGA-II: dominance ranking of solutions from a current population



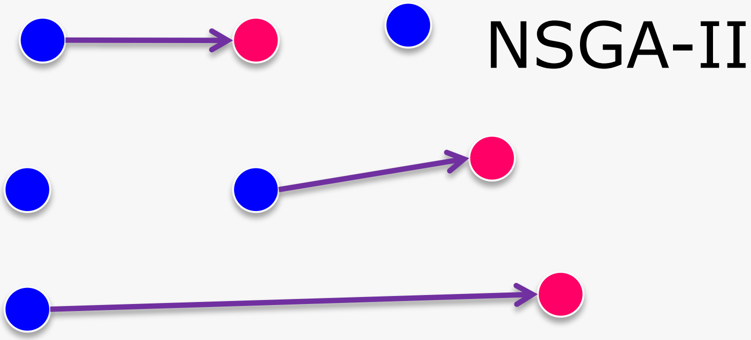
Within the same front, order the individuals with respect to the crowding distance

- NEMO-0: in non-dominated fronts, individuals are ranked by representative value function compatible with preference information



The NEMO framework

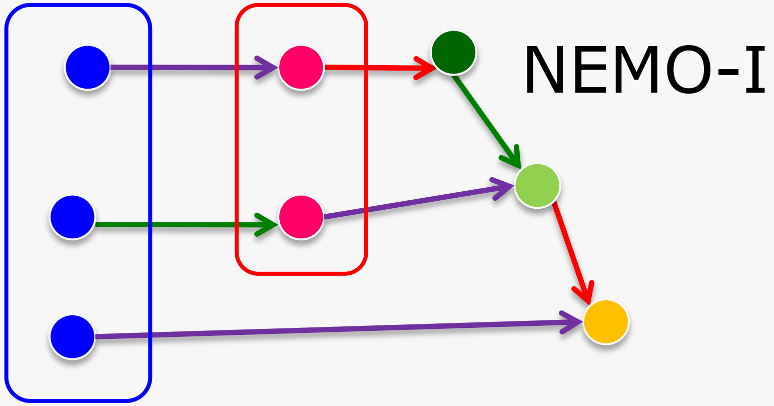
- NSGA-II: dominance ranking of solutions from a current population



Within the same front, order the individuals with respect to the crowding distance

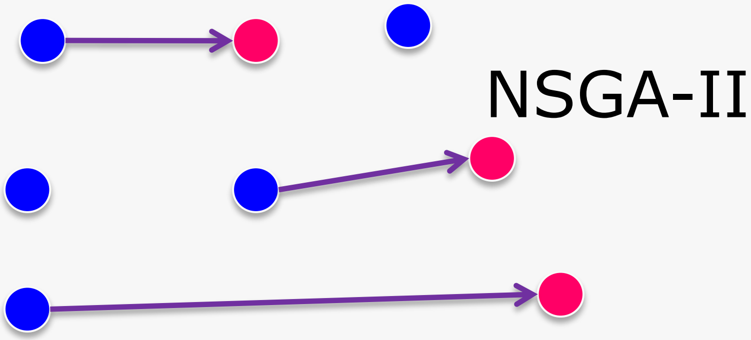
- NEMO-I: replaces dominance relation by pairwise necessary preference relation

$O(p^2)$ LPs to solve
in every iteration



The NEMO framework

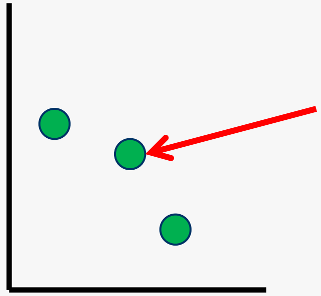
- NSGA-II: dominance ranking of solutions from a current population



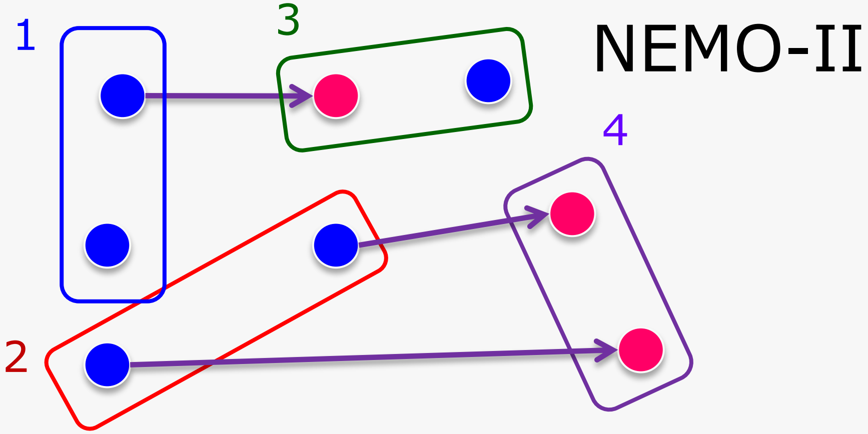
Within the same front, order the individuals with respect to the crowding distance

- NEMO-II: put in the first front solutions that are preferred to all others in the population for at least one compatible value function

only O(p) LPs to solve



Never preferred under NEMO-II-linear



Front of NEMO-II $\subseteq$ Front of NEMO-I $\subseteq$ Front of NSGA-II

Recent work: NEMO-II-Choquet

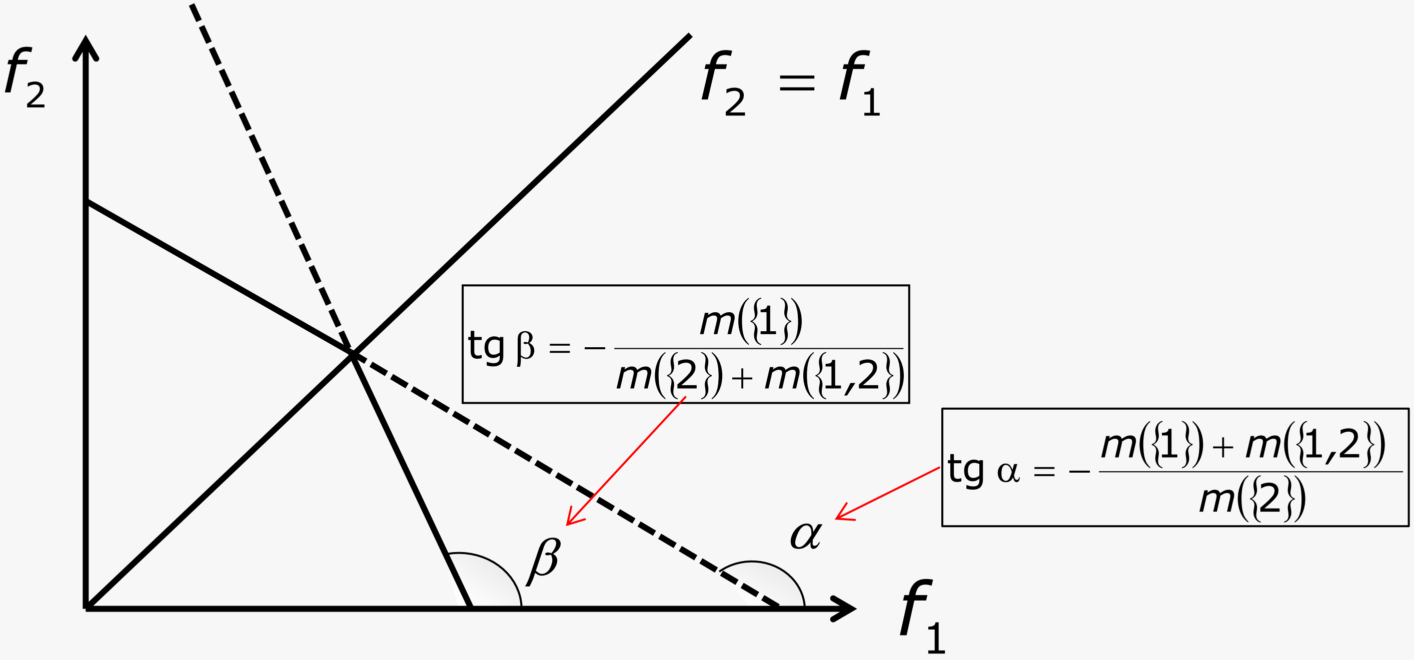
- Use Choquet integral as preference model
  - Well-accepted model in decision theory
  - Allows to model interaction between objectives
- Adapt complexity of preference model to complexity of preferences
  - Start with linear model
  - Switch to 2-additive Choquet once no linear compatible value function can be found
- Every *q* iterations the user is expressing preferences by comparing two non-dominated solutions
- Put in the first front solutions that are preferred to all others in the population for at least one compatible value function
- Within the same front, order the individuals with respect to the crowding distance

A particular case of the Choquet integral: *n*=2

If n=2, then…

$$Ch_\mu(f_1, f_2) = m(\{1\})\, f_1 + m(\{2\})\, f_2 + m(\{1,2\}) \min\{f_1, f_2\} \quad =$$

$$= \begin{cases} (m(\{1\}) + m(\{1,2\}))\, f_1 + m(\{2\})\, f_2 & \text{if } f_1 \le f_2 \\ m(\{1\})\, f_1 + (m(\{2\}) + m(\{1,2\}))\, f_2 & \text{if } f_1 \ge f_2 \end{cases}$$

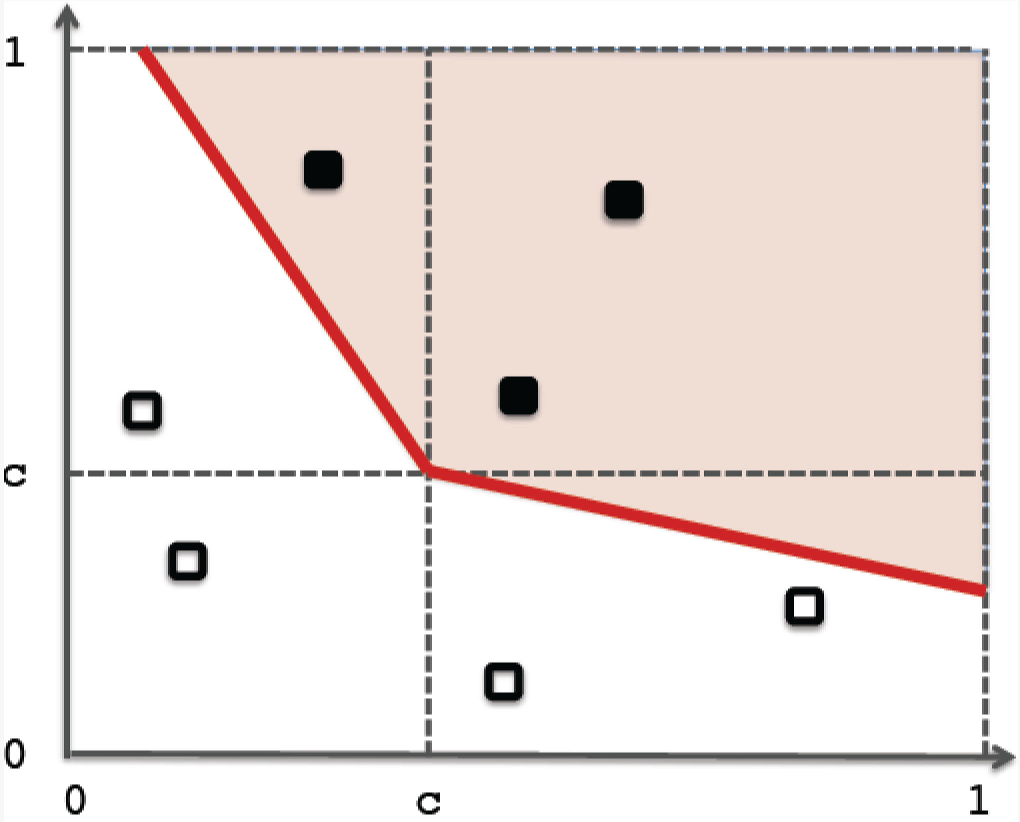


$$\text{tg}\,\beta = -\frac{m(\{1\})}{m(\{2\}) + m(\{1,2\})}$$

$$\text{tg}\,\alpha = -\frac{m(\{1\}) + m(\{1,2\})}{m(\{2\})}$$

Isoquants of the Choquet integral for two criteria – special cases

- 2-additive Choquet – positive interaction (synergy)

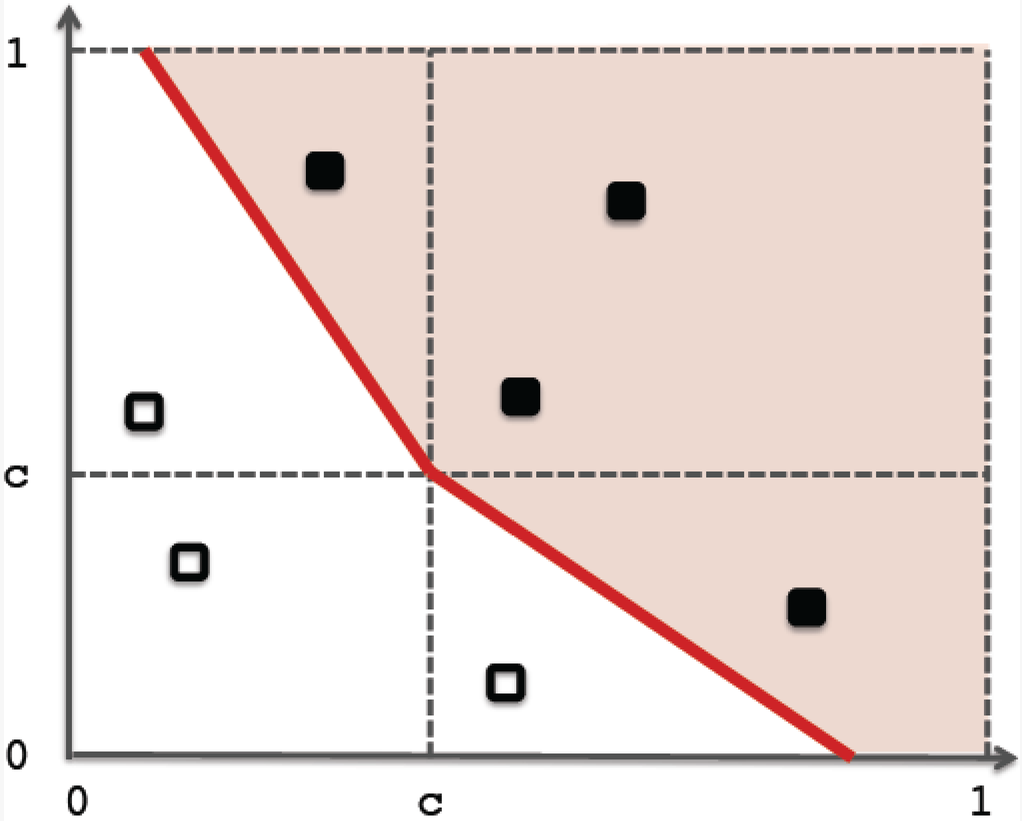


$$U(a) = \mu(\{f_1\})\, f_1(a) + \mu(\{f_2\})\, f_2(a) + [\mu(\{f_1, f_2\}) - \mu(\{f_1\}) - \mu(\{f_2\})]\, min\{f_1(a), f_2(a)\} \geq c$$

positive interaction when $\mu(\{f_1, f_2\}) > \mu(\{f_1\}) + \mu(\{f_2\})$

Isoquants of the Choquet integral for two criteria – special cases

- 2-additive Choquet – positive interaction (synergy)



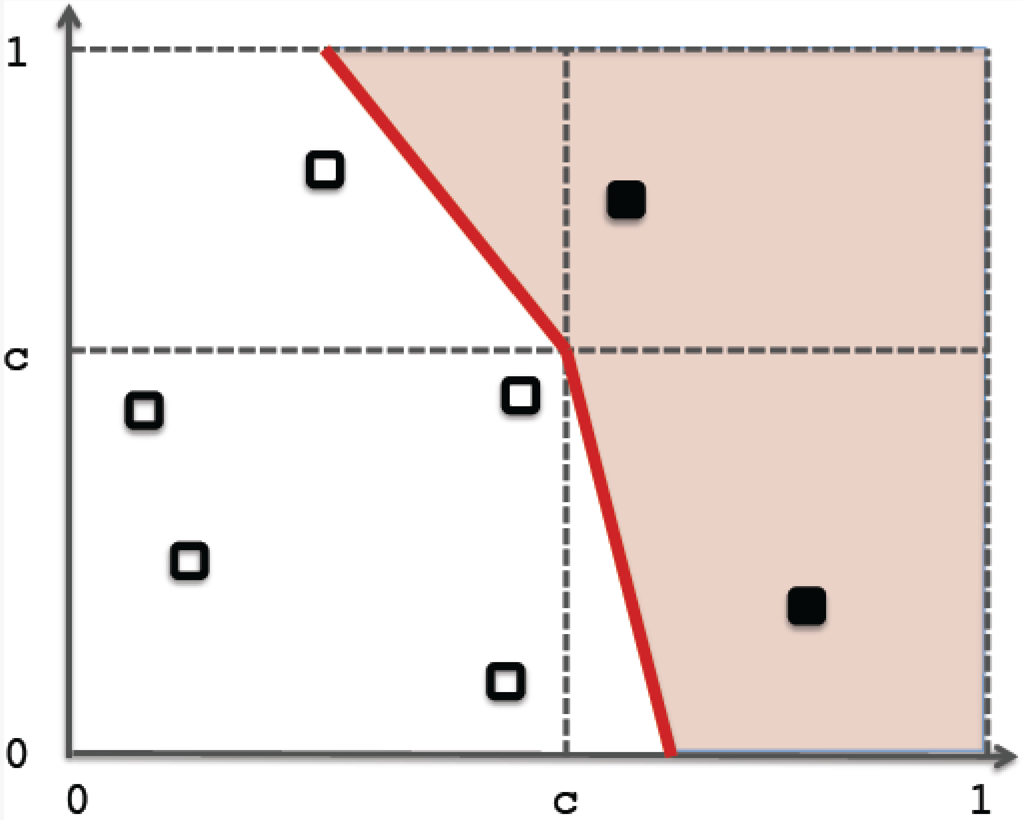
greater capacity=weight of f_1 than before

$$U(a) = \mu(\{f_1\})\,f_1(a) + \mu(\{f_2\})\,f_2(a) + [\mu(\{f_1, f_2\}) - \mu(\{f_1\}) - \mu(\{f_2\})]\,min\{f_1(a), f_2(a)\} \geq c$$

positive interaction when $\mu(\{f_1, f_2\}) > \mu(\{f_1\}) + \mu(\{f_2\})$

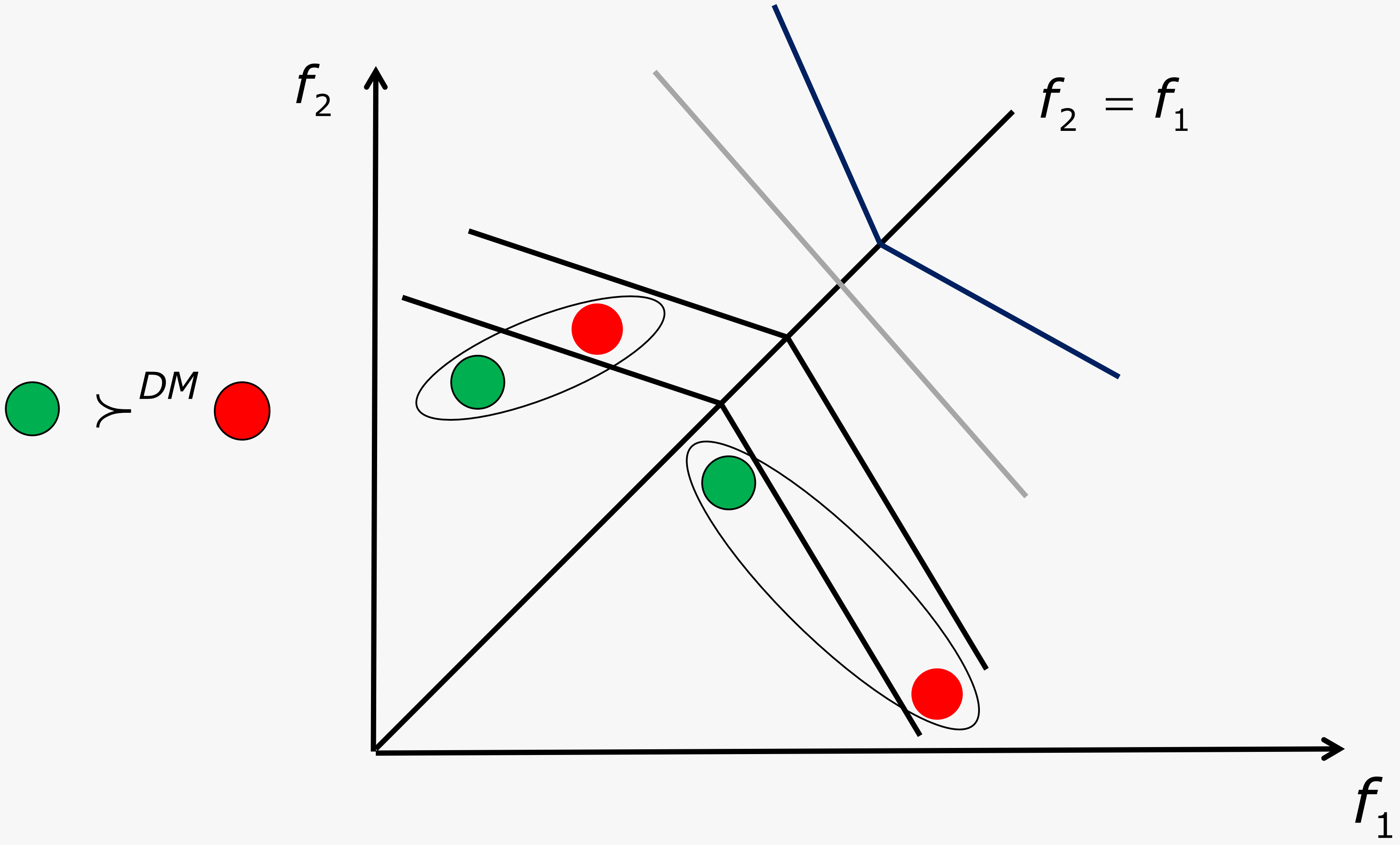
Isoquants of the Choquet integral for two criteria – special cases

- 2-additive Choquet – negative interaction (redundancy)

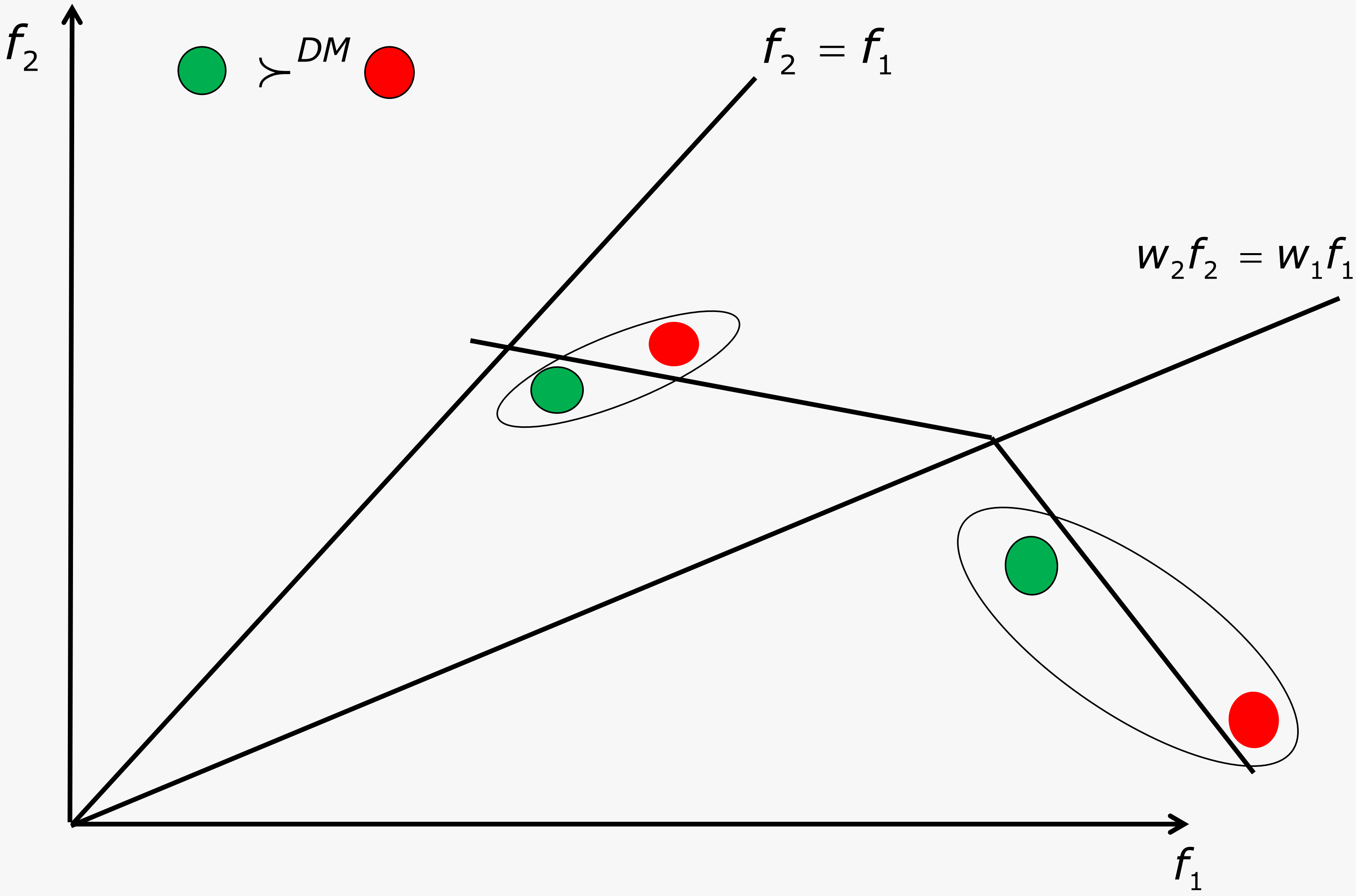


Impossibile visualizzare l'immagine.

Graphical interpretation



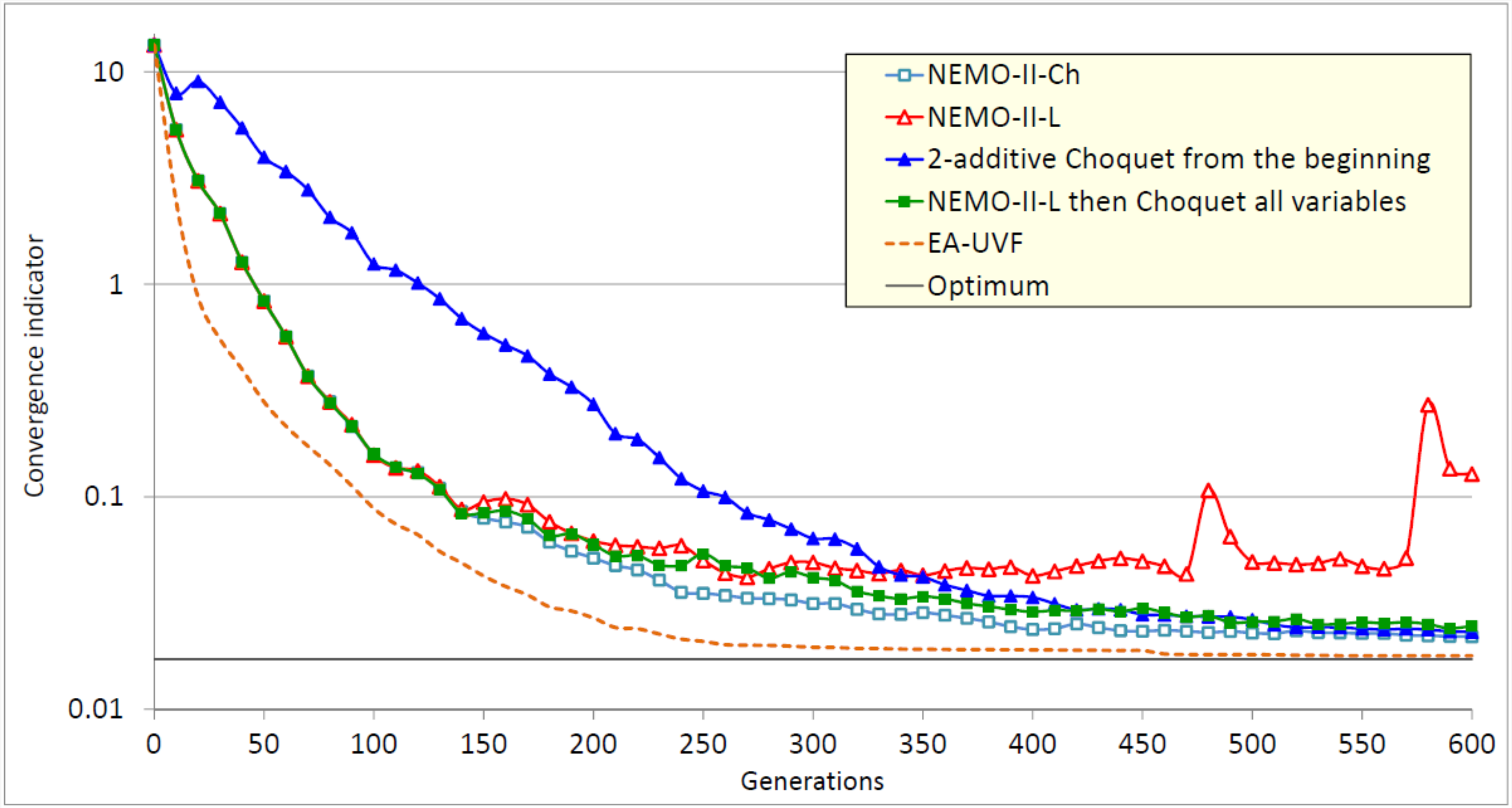
Scaling of objectives



NEMO-II-Ch main points

- Start with the linear value function as preference model
- Ask every *q* iterations DM's preferences by comparing two non-dominated solutions
- Order the solutions by checking if there exists at least one compatible model for which *x* is preferred to all other solutions
- Within the same front order the solutions with respect to the crowding distance
- Switch to the 2-additive Choquet integral preference model as soon as the linear model is not able to represent the preferences of the DM anymore

Why NEMO-II-Ch? (DTLZ1-5D)



- DM compares two n-d solutions in the same front every 10 iterations
- It is better to start with the simplest model (the linear one);
- Passing to the 2-additive Choquet integral preference model produces better results than passing to the complete Choquet integral model;
- In NEMO-II-Ch interactions between pairs of criteria are considered.

Thank you