Multiattribute Utility Theory, Ordinal Regression and Robust Ordinal Regression

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Plan of the talk
Plan

- Multiattribute Utility Theory
- Ordinal regression and inductive learning approaches
- The UTA method
- Robust ordinal regression
- The UTA$^\text{GMS}$ method
- GRIP – *Generalized Regression with Intensities of Preference*
- Representative value functions
- UTADIS$^\text{GMS}$: Robust ordinal regression for sorting
- Robust ordinal regression for group decision
- Robust ordinal regression for outranking methods
- Robust ordinal regression for interacting criteria: NAROR and UTA$^\text{GMS-INT}$
- Stochastic Ordinal Regression and Subjective Stochastic Ordinal Regression
- Multiple Criteria Hierarchy Process
- Interactive optimization with Robust Ordinal Regression
- Conclusions
Ordinal regression and inductive learning approaches
Problem statement – multicriteria choice, ranking and sorting

- Consider a finite set \( A \) of actions (alternatives, solutions, objects) evaluated by \( m \) criteria from a consistent family \( F=\{g_1, \ldots, g_m\} \); \( I=\{1, \ldots, m\} \).
- The only objective information is dominance relation in set \( A \).
Bernard Roy on the constructive approach of MCDA

“MCDA must be based on models that are, at least partially, co-constructed through interaction with the decision maker. The co-constructed model must be a tool for looking deeper into the subject, exploring, interpreting, debating and even arguing.” (Roy 2010)
Bernard Roy on the recommendation in MCDA

“The content of the recommendation may be only the fruit of a conviction constructed in the course of a process necessitating multiple interactions, bringing into play a variety of actors involved in a complex managerial environment.” (Roy 1993)"
Taxonomy of Decision Problems
$P_\alpha$ : Choice problem (optimization)

- Chosen subset $A'$ of best actions
- Rejected subset $A \setminus A'$ of actions
$P_{β_1}$: Classification to preferentially non-ordered classes (classification in the strict sense)
$P_{β2} :$ Classification to preferentially ordered classes (sorting)

Class 1 ⊃ Class 2 ⊃ ... ⊃ Class $p$
### Classification in the strict sense – example of traffic signs

<table>
<thead>
<tr>
<th>Traffic sign</th>
<th>Shape (S)</th>
<th>Primary Color (PC)</th>
<th>Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>triangle</td>
<td>yellow</td>
<td>W</td>
</tr>
<tr>
<td>b)</td>
<td>circle</td>
<td>white</td>
<td>I</td>
</tr>
<tr>
<td>c)</td>
<td>circle</td>
<td>blue</td>
<td>I</td>
</tr>
<tr>
<td>d)</td>
<td>circle</td>
<td>blue</td>
<td>O</td>
</tr>
</tbody>
</table>

**W:** Warning; **I:** Interdiction; **O:** Obligation
## Sorting – example of multiple criteria sorting of students

<table>
<thead>
<tr>
<th>Student</th>
<th>Mathematics (M)</th>
<th>Physics (Ph)</th>
<th>Literature (L)</th>
<th>Overall class</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>good</td>
<td>medium</td>
<td>bad</td>
<td>bad</td>
</tr>
<tr>
<td>S2</td>
<td>medium</td>
<td>medium</td>
<td>bad</td>
<td>medium</td>
</tr>
<tr>
<td>S3</td>
<td>medium</td>
<td>medium</td>
<td>medium</td>
<td>medium</td>
</tr>
<tr>
<td>S4</td>
<td>medium</td>
<td>medium</td>
<td>medium</td>
<td>good</td>
</tr>
<tr>
<td>S5</td>
<td>good</td>
<td>medium</td>
<td>good</td>
<td>good</td>
</tr>
<tr>
<td>S6</td>
<td>good</td>
<td>good</td>
<td>good</td>
<td>good</td>
</tr>
<tr>
<td>S7</td>
<td>bad</td>
<td>bad</td>
<td>bad</td>
<td>bad</td>
</tr>
<tr>
<td>S8</td>
<td>bad</td>
<td>bad</td>
<td>medium</td>
<td>bad</td>
</tr>
</tbody>
</table>
Pγ : Ordering problem (ranking)

Partial or complete ranking
Multiattribute Utility Theory (MAUT)
Setting

- \( N = \{1, 2, \ldots, n\} \) set of attributes
- \( X_i \) : set of possible values of the i-th attribute
- \( X = \prod_{i=1}^{n} X_i = X_1 \times X_2 \times \ldots \times X_n = \{(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\} \): set of all conceivable alternatives
  - \( X \) includes the alternatives under study. . . and many others!
- \( \preceq \): weak preference relation on \( X \) such that for all \( x, y \in X \)
  \[ x \succeq y \]
  means
  «x is at least as good as y»
- \( x \succ y \iff x \succeq y \) and not \( y \succeq x \) (which means «x is preferred to y»)
- \( x \sim y \iff x \succeq y \) and \( y \succeq x \) (which means «x and y are indifferent»)
Marginal preferences

- $J = \{i_1, \ldots, i_k\} \subseteq N$

- $X_J = \prod_{i \in J} X_i = X_{i_1} \times X_{i_2} \times \cdots \times X_{i_k} = \{(x_{i_1}, \ldots, x_{i_k}) : x_{i_1} \in X_{i_1}, \ldots, x_{i_k} \in X_{i_k}\}$: set of all conceivable alternatives with respect to attributes from $J$

- $X_{-J} = \prod_{i \notin J} X_i$ : set of all conceivable alternatives with respect to attributes different from $J$

- $\succeq_J$: weak marginal preference relation on $X_J$ such that for all $x_J, y_J \in X_J$

  $$x_J \succeq_J y_J \iff (x_J, z_{-J}) \succeq (y_J, z_{-J}) \text{ for all } z_{-J} \in X_{-J}$$

  which means

  «$x_J$ is at least as good as $y_J$»

- In case $J = \{i\}$, we write $\succeq_i$ instead of $\succeq_{\{i\}}$. 
Additive value function model

- For all all \( x, y \in X \)

\[
x \succeq y \iff \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)
\]

with \( u_i : X_i \to \mathbb{R} \).

- Sometimes a simplified model is considered: if \( X \subseteq \mathbb{R}^n \) and for all attribute \( i \in N \)

\[
x_i \geq y_i \iff x_i \geq y_i
\]

- For all \( x, y \in X \)

\[
x \succeq y \iff \sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i y_i
\]

with \( w_i \) non negative for all \( i \in N \).
Independence

- $\succeq$ is independent for $J \subseteq N$ if for all $x_j, y_j \in X_J$

  $$[(x_j, z_{-j}) \succeq (y_j, z_{-j}) \text{ for some } z_{-j} \in X_{-j}]$$

  $\Downarrow$

  $$[(x_j, z_{-j}) \succeq (y_j, z_{-j}) \text{ for all } z_{-j} \in X_{-j}]$$

- If $\succeq$ is independent for all $J \subseteq N$, with $J$ non-empty, we say that $\succeq$ is independent.

- If $\succeq$ is independent for all $\{i\}$, $i \in N$, we say that $\succeq$ is weakly independent.

- If $\succeq$ is weakly independent, then dominance arguments apply, i.e. for all $x, y \in X$

  $$[x_i \succeq_i y_i \text{ for all } i \in N] \Rightarrow x \succeq y$$
Independence: illustrative example

If $S_2 \succeq S_1$, then $S_4 \succeq S_3$

If $(x_j, z_{-j}) \succeq (y_j, z_{-j})$, then $(x_j, z'_{-j}) \succeq (y_j, z'_{-j})$
Independence: illustrative example

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<td>Bad</td>
<td>Medium</td>
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$S_2 \succ S_1 \Rightarrow S_4 \succ S_3$
Is independence a reasonable hypothesis?

<table>
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<th>Dinner</th>
<th>Main course</th>
<th>Wine</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>Meat</td>
<td>White</td>
</tr>
<tr>
<td>D2</td>
<td>Meat</td>
<td>Red</td>
</tr>
<tr>
<td>D3</td>
<td>Fish</td>
<td>White</td>
</tr>
<tr>
<td>S4</td>
<td>Fish</td>
<td>Red</td>
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D2 ≻ D1 and D3 ≻ D4
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<td>White</td>
</tr>
<tr>
<td>S4</td>
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<td>Red</td>
</tr>
</tbody>
</table>

\[ D2 \succ D1 \succ D4 \succ D3 \]
Basic results for Multiattribute Utility Theory

If

- restricted solvability holds,
- each attribute is essential,

then the additive value function holds if and only if

is an independent weak order satisfying the Thomsen and the Archimedean conditions.

In case there are more than two attributes, Thomsen condition can be forgotten.
How to assess a multiattribute value function?

Many methods:

- Direct rating
- Bisection techniques
- ... (e.g. Peter C. Fishburn, Methods of Estimating Management Science, 13(7), 1967, 435-453, where 24 methods are presented)
How to assess tradeoff?

- Consider the simplified model
  \[ x \succeq y \iff \sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i y_i \]

- For \( x \in X \) and \( i, j \in N \), consider \( k_{ij} \) such that
  \((x_1, x_2, \ldots, x_i+1, \ldots, x_j, \ldots, x_n) \sim (x_1, x_2, \ldots, x_i, \ldots, x_j + k_{ij}, \ldots, x_n)\)

- We get
  \[ w_1 x_1 + w_2 x_2 + \ldots + w_i (x_i + 1) + \ldots + w_j x_j + \ldots + w_n x_n \]
  \[ = \]
  \[ w_1 x_1 + w_2 x_2 + \ldots + w_i x_i + \ldots + w_j (x_j + k_{ij}) + \ldots + w_n x_n \]

- From which...
How to assess tradeoff?

- From which

\[ w_i = w_j \cdot k_{ij} \]

and therefore

\[ k_{ij} = \frac{w_i}{w_j} \]

- This means that the weights in the MAUT model are related to the concept of tradeoff (I can renounce to \( k_{ij} \) on attribute \( j \), in order to increase one unit on attribute \( i \)).

- Observe that coherence condition is that for all \( i,j,l \in N \)

\[ k_{ij} = k_{il} \times k_{lj} \]

\[ \left( \frac{w_i}{w_j} = \frac{w_i}{w_l} \times \frac{w_l}{w_j} \right) \]
Holistic preference information

- Psychologists confirm that Decision Makers (DMs) are more confident exercising their decisions than explaining them.
- The most natural is a holistic pairwise comparison of some actions relatively well known to the DM, i.e. reference actions.
Psychologists confirm that DMs are more confident exercising their decisions than explaining them.

The most natural is a holistic pairwise comparison of some actions relatively well known to the DM, i.e. reference actions.
Question: what is the consequence of using on the whole set $A$ this information transformed to a compatible preference model?

What ranking will result?

Apply the preference model on $A$.
Principle of the ordinal regression

- The **preference information** is given in the form of partial preorder on a subset of reference actions \( A^R \subset A \)

- Additive **value (or utility) function** on \( A \): for each \( x \in A \)

\[
U(x) = \sum_{i=1}^{n} u_i[g_i(x)]
\]

where \( u_i \) are non-decreasing marginal value functions
The UTA method
The UTA method (Jacquet-Lagreze & Siskos 1982)

- **Marginal value** of action $x_i \in A$ is approximated by linear interpolation.

![Diagram](image)

**Figure 1:** Piecewise linear marginal utility function
The principle of the ordinal regression – the UTA method

(Jacquet-Lagreze & Siskos 1982)

- The marginal value functions (breakpoint variables) are estimated by solving the LP problem

\[
\begin{align*}
\text{Min} & \quad E^{UTA} = \sum_{a \in A^R} (\sigma^+(a) + \sigma^-(a)) \\
\text{subject to} & \\
U(a) + \sigma^+(a) - \sigma^-(a) & \geq U(b) + \sigma^+(b) - \sigma^-(b) + \epsilon \iff a \succ b \\
U(a) + \sigma^+(a) - \sigma^-(a) & = U(b) + \sigma^+(b) - \sigma^-(b) \iff a \sim b \\
u_i(x_{i}^{j+1}) - u_i(x_{i}^{j}) & \geq 0 \quad j = 0, \ldots, \gamma_i - 1; \quad \forall i \in I \\
\sum_{i=1}^{n} u_i(\beta_i) & = 1 \\
u_i(\alpha_i) & = 0 \quad \forall i \in I \\
u_i(x_{i}^{j}) & \geq 0, \quad \sigma^+(a) \geq 0, \quad \sigma^-(a) \geq 0, \quad \forall a \in A^R, \quad \forall i \text{ and } j
\end{align*}
\]

where \( \epsilon \) is a small positive constant, and \( \sigma^+ \) and \( \sigma^- \) are auxiliary variables (errors of approximation)
“The most representative value function” of UTA: UTAMP1 model
After verifying that the set of compatible value function is not empty, the “most representative” value function is estimated by solving the following LP problem,

\[
\begin{align*}
\text{Max } & \quad \varepsilon \\
\text{Min } & \quad F^{'UTA} \quad \sum_{x^k \in A^R} (\sigma^+(x^k) + \sigma^-(x^k)) \\
\text{subject to } & \quad U'(x^k) \geq U'(x^{k+1}) + \varepsilon \Leftrightarrow x^k \succ x^{k+1} \\
& \quad U'(x^k) = U'(x^{k+1}) \Leftrightarrow x^k \sim x^{k+1} \\
& \quad u_i(x_{i,j+1}^k) - u_i(x_{i,j}^k) \geq 0 \quad j = 0, \ldots, \gamma_i ; \quad \forall i \in I \\
& \quad \sum_{i=1}^{n} u_i(\beta_i) = 1 \\
& \quad u_i(\alpha_i) = 0 \quad \forall i \in I \\
& \quad u_i(x_{i,j}^k) \geq 0, \quad \sigma^+(x^k) \geq 0, \quad \sigma^-(x^k) \geq 0, \quad \forall x^k \in A^R, \quad \forall i \text{ and } j \\
& \quad \sigma^+(x^k) = 0, \quad \sigma^-(x^k) = 0
\end{align*}
\]
Intuition behind the Robust Ordinal Regression
Basic question

- Remark 1. If there is one value function representing the preferences of the DM, in general, there are infinitely many others.

- Remark 2. In general, each one of these infinitely many value functions, gives a different ranking of actions from A.

- Why to consider only one of these infinitely many value functions?
One should use all compatible preference models on set $A$

- **Question**: what is the consequence of using all compatible preference models on set $A$?

**Preference information**

$x \succeq y$

$z \succeq w$

$y \succeq v$

$u \succeq t$

$z \succeq u$

$u \succeq z$

**DM**

**All instances of preference model compatible with preference information**

**Apply all compatible instances on $A$**

**What rankings will result?**
Rank related preference information

- Types of indirect preference information in particular nodes of the tree:
  - Desired ranks of alternatives, e.g.,

  - should take place on the podium
  - should (not) be ranked among top/bottom 5 alternatives
  - should be among the 10% of best/worst alternatives
  - is predisposed to secure the place between 4 and 10
  - should be ranked in the second ten of alternatives
  - should be ranked in the upper/lower half of the ranking
  - evaluation profile of predisposes it to have value at least/at most x

The UTA$^\text{GMS}$ method
The **UTA\textsuperscript{GMS}** method (Greco, Mousseau & Słowiński 2004, 2008)

- DM is supposed to provide the following preference information:
  - a partial preorder $\succeq$ on $A^R$, such that $\forall x, y \in A^R$

$$x \succeq y \iff "x \text{ is at least as good as } y"$$
A value function $U$ is called compatible if it satisfies the constraints corresponding to DM’s preference information:

- $U(x) \geq U(y)$ iff $x \succeq y$
- $U(x) > U(y)$ iff $x \succ y$
- $U(x) = U(y)$ iff $x \sim y$
- $u_i(x) \geq u_i(y)$ iff $x \succeq_i y$, $i \in I$

Moreover, the following normalization constraints should also be taken into account:

- $u_i(\alpha_i) = 0$, $i \in I$
- $\sum_{i \in I} u_i(\beta_i) = 1$
The UTA\textsuperscript{GMS} method (Greco, Mousseau & Słowiński 2004, 2008)

- If constraints $a) – f)$ are consistent, then we get the two weak preference relations $\succeq^N$ and $\succeq^P$:
  - the necessary weak preference relation: for all $x,y \in A$,
    $$x \succeq^N y \iff U(x) \geq U(y) \text{ for all compatible value functions}$$
    (i.e. for all compatible value functions $x$ is at least as good as $y$)
  - the possible weak preference relation: for all $x,y \in A$,
    $$x \succeq^P y \iff U(x) \geq U(y) \text{ for at least one compatible value function}$$
    (i.e. for at least one compatible value function $x$ is at least as good as $y$)
Is it necessary the possible preference relation?

- If we do not consider the possible preference relation, we are not able to distinguish these following two cases.

\[
\text{A} \sim^N \text{B} \text{ and } \text{B} \succ^P \text{A}
\]

\[
\text{A} \sim^N \text{B} \text{ and } \text{B} \succ^P \text{A}
\]
The \textbf{UTA}\textsuperscript{GMS} method (Greco, Mousseau & Słowiński 2008)

- Basic properties: for all $x, y, z \in A$
  - $x \succeq^N y \implies x \succeq^P y$
  - $\succeq^N$ is a \textit{partial preorder} (i.e. $\succeq^N$ is reflexive and transitive)
  - $x \succeq^N y$ and $y \succeq^P z \implies x \succeq^P z$
  - $x \succeq^P y$ and $y \succeq^N z \implies x \succeq^P z$
  - $x \succeq^N y$ or $y \succeq^P x$
  - $\succeq^P$ is \textit{strongly complete} (i.e. for all $x, y \in A$, $x \succeq^P y$ or $y \succeq^P x$) and negatively transitive (i.e. for all $x, y, z \in A$, not $x \succeq^P y$ and not $y \succeq^P z \implies$ not $x \succeq^P z$), (in general, $\succeq^P$ is not transitive)

- Giarlotta and Greco (2013) proved that the first 5 properties characterize $\succeq^N$ and $\succeq^P$. 
The $\text{UTA}^{\text{GMS}}$ method (Greco, Mousseau & Słowinski 2004, 2008)

- The **marginal value function** $u_i(x_i)$

Characteristic points of marginal value functions are fixed on actual evaluations of actions from set $A$
GRIP – Generalized Regression with Intensities of Preference
GRIP – Generalized Regression with Intensities of Preference
(Figueira, Greco & Słowiński 2005, 2008)

- GRIP extends the UTA^{GMS} method by adopting all features of UTA^{GMS}
  and by taking into account additional preference information:
  - comprehensive comparisons of intensities of preference between
    some pairs of reference actions,
    e.g. „x is preferred to y at least as much as w is preferred to z”
  - partial comparisons of intensities of preference between some pairs
    of reference actions on particular criteria,
    e.g. „x is preferred to y at least as much as w is preferred to z, on
    criterion g_i \in F”
DM is supposed to provide the following preference information:

- a partial preorder $\succeq$ on $A^R$, such that $\forall x,y \in A^R$
  \[ x \succeq y \iff "x \text{ is at least as good as } y" \]

- a partial preorder $\succeq^*$ on $A^R \times A^R$, such that $\forall x,y,w,z \in A^R$
  \[ (x,y) \succeq^* (w,z) \iff "x \text{ is preferred to } y \text{ at least as much as } w \text{ is preferred to } z" \]

- a partial preorder $\succeq_i^*$ on $A^R \times A^R$, $i=1,...,n$, such that $\forall x,y,w,z \in A^R$
  \[ (x,y) \succeq_i^* (w,z) \iff "x \text{ is preferred to } y \text{ at least as much as } w \text{ is preferred to } z, \text{ on criterion } g_i \in F" \]
A utility function $U$ is called **compatible** if it satisfies the constraints corresponding to DM’s preference information:

a) $U(x) \geq U(y)$ iff $x \succeq y$

b) $U(x) > U(y)$ iff $x \succ y$

c) $U(x) = U(y)$ iff $x \asymp y$

d) $U(x) - U(y) \geq U(w) - U(z)$ iff $(x,y) \succeq^* (w,z)$

e) $U(x) - U(y) > U(w) - U(z)$ iff $(x,y) \succ^* (w,z)$

f) $U(x) - U(y) = U(w) - U(z)$ iff $(x,y) \asymp^* (w,z)$

g) $u_i(x) \geq u_i(y)$ iff $x \succeq_i y, \ i \in I$

h) $u_i(x) - u_i(y) \geq u_i(w) - u_i(z)$ iff $(x,y) \succeq_i^* (w,z), \ i \in I$

i) $u_i(x) - u_i(y) > u_i(w) - u_i(z)$ iff $(x,y) \succ_i^* (w,z), \ i \in I$

j) $u_i(x) - u_i(y) = u_i(w) - u_i(z)$ iff $(x,y) \asymp_i^* (w,z), \ i \in I$
Moreover, the following normalization constraints should also be taken into account:

\( k) \ u_i(\alpha_i) = 0, \ i \in N \)

\( l) \ \sum_{i \in I} u_i(\beta_i) = 1 \)
If constraints $a) - l)$ are consistent, then we get two weak preference relations $\succeq^N$ and $\succeq^P$

- **a necessary weak preference relation**: for all $x, y \in A$,
  \[
  x \succeq^N y \iff U(x) \geq U(y) \text{ for all compatible value functions}
  \]

- **a possible weak preference relation**: for all $x, y \in A$,
  \[
  x \succeq^P y \iff U(x) \geq U(y) \text{ for at least one compatible value function}
  \]
If constraints \(a) – l)\) are consistent, then we get also two overall binary relations comparing intensity of preference \(\succeq^*N\) and \(\succeq^*P\) :

- a necessary relation of preference intensity: for all \(x,y,w,z \in A\),
  \[(x,y) \succeq^*N (w,z) \iff [U(x) - U(y)] \geq [U(w) - U(z)] \text{ for all compatible value functions}\]

- a possible relation of preference intensity: for all \(x,y,w,z \in A\),
  \[(x,y) \succeq^*P (w,z) : [U(x) - U(y)] \geq [U(w) - U(z)] \text{ for at least one compatible value function}\]
If constraints \( a) – l)\) are consistent, then we get two binary relations comparing intensity of preference \( \succeq_i^{*N} \) and \( \succeq_i^{*P} \) for each criterion \( g_i \in F\):

- **a necessary** relation of preference intensity: for all \( x, y, w, z \in A \),
  \[
  (x, y) \succeq_i^{*N} (w, z) \iff [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))]
  \]
  for all compatible value functions

- **a possible** relation of preference intensity: for all \( x, y, w, z \in A \),
  \[
  (x, y) \succeq_i^{*P} (w, z) : [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))]
  \]
  for at least one compatible value function
Some properties:

- $x \succeq^N y \Rightarrow x \succeq^P y$,
- $(x, y) \succeq^* N (w, z) \Rightarrow (x, y) \succeq^* P (w, z)$,
- $(x, y) \succeq^i N (w, z) \Rightarrow (x, y) \succeq^i P (w, z), g_i \in F$
- $\succeq^N, \succeq^* N$ and $\succeq^i N, i \in N$, are partial preorders
- $\succeq^P, \succeq^* P$ and $\succeq^i P$ are strongly complete and negatively transitive, (in general, $\succeq^P \succeq^P, \succeq^* P$ and $\succeq^* P$ are not transitive)
GRIP – the linear programming problem: the result is independent of $\varepsilon$

- **Strict inequalities** such as $b), e), i)$ are rewritten as:
  
  \begin{align*}
  b') & \quad U(x) \geq U(y) + \varepsilon \\
  e') & \quad U(x) - U(y) \geq U(w) - U(z) + \varepsilon \\
  i') & \quad u_i(x) - u_i(y) \geq u_i(w) - u_i(z) + \varepsilon
  \end{align*}

- $x_{\succeq_P} y \iff$ the set of constraints is feasible and $\varepsilon^*>0$, where $\varepsilon^* = \text{Max } \varepsilon$, subject to constraints $a) - l)$, with $b), e), i)$ written as $b'), e'), i')$ and $U(x) \geq U(y)$

- $x_{\succeq_N} y \iff$ the set of constraints is infeasible or $\varepsilon^* \leq 0$, where $\varepsilon^* = \text{Max } \varepsilon$, subject to constraints $a) - l)$, with $b), e), i)$ written as $b'), e'), i')$ and $U(y) \geq U(x) + \varepsilon$
The “most representative” value function
The need for a representative value function

- Recommendations taking into account the whole set of admissible value functions answer to robustness concerns, since they are in general “more robust” than a single ranking obtained by an arbitrarily chosen compatible value function.

- However, in practice, for some decision-making situations, a score is needed to assign to the different actions and despite the interest of the two rankings provided, some users would like to see the “most representative” value function among all the compatible ones.

- This value function should allow assigning a score to each action.

- We propose a way to identify the “most representative” value function in GRIP, without loosing the advantage of taking into account all compatible value functions.
The idea of the „most representative“ value function (Figueira, Greco, Slowinski 2008; see also Kadzinski, Greco, Slowinski 2010, 2011)

- The idea is to select among compatible value functions that value function which better highlights the necessary ranking, maximizing the difference of evaluations between actions for which there is a preference in the necessary ranking.

- As secondary objective, one can consider minimizing the difference of evaluations between actions for which there is not a preference in the necessary ranking.
Procedure to determine the “most representative” value function

1) Determine the necessary and the possible preferences in the considered set of actions.

2) For all pairs of actions \((a,b)\), such that \(a\) is necessarily preferred to \(b\) \((a \succeq^N b\) but not \(b \succeq^N a\)), add the following constraints to the linear programming constraints of GRIP: \(U(a) \geq U(b) + \varepsilon\).

3) Maximize \(\varepsilon\)

4) Add the constraint \(\varepsilon = \varepsilon^*\), with \(\varepsilon^* = \text{Max } \varepsilon\) of point 3), to the linear programming constraints of point 2)

5) For all pairs of actions \((a,b)\), such that neither \(a\) is necessarily preferred to \(b\) nor \(b\) is necessarily preferred to \(a\) \((\text{not } a \succeq^N b\) and \(\text{not } b \succeq^N a\)), add the following constraints to the linear programming constraints of GRIP: \(U(a) - U(b) \leq \delta\) and \(U(b) - U(a) \leq \delta\).

6) Minimize \(\delta\)
Alternative procedure to determine the “most representative” value function

1) Determine the necessary and the possible preferences in the considered set of actions.

2) For all pairs of actions \((a,b)\), such that \(a\) is necessarily preferred to \(b\) \((a \succeq^N b\) but not \(b \succeq^N a\)), add the following constraints to the linear programming constraints of GRIP: 
\[ U(a) \geq U(b) + \varepsilon. \]

3) For all pairs of actions \((a,b)\), such that neither \(a\) is necessarily preferred to \(b\) nor \(b\) is necessarily preferred to \(a\) \((\text{not } a \succeq^N b\) and \(\text{not } b \succeq^N a\))\), add the following constraints to the linear programming constraints of GRIP: 
\[ U(a) - U(b) \leq \delta \quad \text{and} \quad U(b) - U(a) \leq \delta. \]

4') Maximize the following objective function: 
\[ M \varepsilon - \delta, \] where \(M\) is a “big value”. 
First illustrative example:
ROR is easy!
## Illustrative example

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>Medium</td>
<td>Medium</td>
<td>Good</td>
</tr>
<tr>
<td>S2</td>
<td>Good</td>
<td>Good</td>
<td>Medium</td>
</tr>
<tr>
<td>S3</td>
<td>Medium</td>
<td>Good</td>
<td>Medium</td>
</tr>
<tr>
<td>S4</td>
<td>Medium</td>
<td>Medium</td>
<td>Medium</td>
</tr>
<tr>
<td>S5</td>
<td>Good</td>
<td>Good</td>
<td>Bad</td>
</tr>
<tr>
<td>S6</td>
<td>Medium</td>
<td>Bad</td>
<td>Good</td>
</tr>
</tbody>
</table>
Information on preferences given by the DM

- Preferences between students
  - S2 ≻ S1
  - S4 ≻ S5
  - S5 ≻ S6

- Overall intensity of preferences
  - (S5,S6) ≻* (S2,S1)

- Intensity of preference relative to single criteria
  - (Good,Medium) ≻* \text{Mathematics} (Medium, Bad)
Necessary weak preference \( \preceq^N \) from GRIP (Hasse Diagram)
“The most representative value function”

- $\varepsilon = 0.1$, $\delta = 0$

<table>
<thead>
<tr>
<th></th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bad</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Medium</td>
<td>0</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>Good</td>
<td>0.1</td>
<td>0.5</td>
<td>0.4</td>
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</tbody>
</table>
Evaluation of students by means of “the most representative value function”

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>Medium (0)</td>
<td>Medium (0.4)</td>
<td>Good (0.4)</td>
<td>0.8</td>
</tr>
<tr>
<td>S2</td>
<td>Good (0.1)</td>
<td>Good (0.5)</td>
<td>Medium (0.3)</td>
<td>0.9</td>
</tr>
<tr>
<td>S3</td>
<td>Medium (0)</td>
<td>Good (0.5)</td>
<td>Medium (0.3)</td>
<td>0.8</td>
</tr>
<tr>
<td>S4</td>
<td>Medium (0)</td>
<td>Medium (0.4)</td>
<td>Medium (0.3)</td>
<td>0.7</td>
</tr>
<tr>
<td>S5</td>
<td>Good (0.1)</td>
<td>Good (0.5)</td>
<td>Bad (0)</td>
<td>0.6</td>
</tr>
<tr>
<td>S6</td>
<td>Medium (0)</td>
<td>Bad (0)</td>
<td>Good (0.4)</td>
<td>0.4</td>
</tr>
</tbody>
</table>
Value function given by UTAMP1

$\epsilon = 0.167$

<table>
<thead>
<tr>
<th></th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bad</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Medium</td>
<td>0</td>
<td>0.5</td>
<td>0.33</td>
</tr>
<tr>
<td>Good</td>
<td>0.17</td>
<td>0.5</td>
<td>0.33</td>
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</table>
## Evaluation of students by means of UTAMP1

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>Medium (0)</td>
<td>Medium (0.5)</td>
<td>Good (0.33)</td>
<td>0.83</td>
</tr>
<tr>
<td>S2</td>
<td>Good (0.17)</td>
<td>Good (0.5)</td>
<td>Medium (0.33)</td>
<td>1</td>
</tr>
<tr>
<td>S3</td>
<td>Medium (0)</td>
<td>Good (0.5)</td>
<td>Medium (0.33)</td>
<td>0.83</td>
</tr>
<tr>
<td>S4</td>
<td>Medium (0)</td>
<td>Medium (0.5)</td>
<td>Medium (0.33)</td>
<td>0.83</td>
</tr>
<tr>
<td>S5</td>
<td>Good (0.17)</td>
<td>Good (0.5)</td>
<td>Bad (0)</td>
<td>0.67</td>
</tr>
<tr>
<td>S6</td>
<td>Medium (0)</td>
<td>Bad (0)</td>
<td>Good (0.33)</td>
<td>0.33</td>
</tr>
</tbody>
</table>
Comparison of GRIP, “the most representative value function” and UTAMP1

<table>
<thead>
<tr>
<th>Students</th>
<th>“The most representative value function”</th>
<th>UTAMP1</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>0.8</td>
<td>0.83</td>
</tr>
<tr>
<td>S2</td>
<td>0.9</td>
<td>1</td>
</tr>
<tr>
<td>S3</td>
<td>0.8</td>
<td>0.83</td>
</tr>
<tr>
<td>S4</td>
<td>0.7</td>
<td>0.83</td>
</tr>
<tr>
<td>S5</td>
<td>0.6</td>
<td>0.67</td>
</tr>
<tr>
<td>S6</td>
<td>0.4</td>
<td>0.33</td>
</tr>
</tbody>
</table>

UTAMP1 does not represent the necessary weak preference of S3 over S4
A didactic example:
ROR is interactive!
**UTA^{GMS} : an illustrative example**

**Ranking problem:** 20 actions evaluated on 5 criteria

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>$s_5$</th>
<th>$s_6$</th>
<th>$s_7$</th>
<th>$s_8$</th>
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<th>$s_{16}$</th>
<th>$s_{17}$</th>
<th>$s_{18}$</th>
<th>$s_{19}$</th>
<th>$s_{20}$</th>
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<tbody>
<tr>
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<td>1</td>
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<td>4</td>
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<td>1</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>3</td>
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<tr>
<td>$g_2$</td>
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<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>4</td>
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<td>1</td>
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<td>0</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
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<tr>
<td>$g_3$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
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<td>3</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
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<tr>
<td>$g_4$</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>1</td>
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<td>3</td>
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<td>1</td>
<td>2</td>
<td>4</td>
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<td>1</td>
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<tr>
<td>$g_5$</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
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<td>4</td>
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<td>4</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

**Evaluation matrix**

Empty dominance relation !
UTA$^\text{GMS}$ : an illustrative example

First iteration
UTA$^{GMS}$: an illustrative example

Second iteration
UTA$^\text{GMS}$: an illustrative example

Third iteration
GRIP: an illustrative example

Fourth iteration, after addition of intensity condition: \((s_8,s_{10})\sim(s_1,s_2)\)
UTA-DIS\textsuperscript{GMS} for multicriteria sorting problems

- Actions from set $A$ are to be assigned to pre-defined and preference-ordered classes
- Classes have a semantic definition
- Assignment to classes is grounded on absolute evaluation of actions on multiple criteria
- No relative comparison is required because sorting is „context-free”, which is not the case of choice and ranking
UTA-DIS\textsuperscript{GMS} – given data

- \( A = \{a_1, a_2, \ldots, a_i, \ldots, a_m\} \) actions to be assigned to classes,
- \( g_1, g_2, \ldots, g_n, \) \( n \) criteria, \( g_j : A \to \mathbb{R} \) for \( j \in G = \{1, 2, \ldots, n\} \),
- \( C_1, C_2, \ldots, C_p, \) \( p \) ordered classes, \( C_{h+1} \gg C_h, \)
  \( H = \{1, \ldots, p\} \),
- \( X_j = \{x_j \in \mathbb{R} : g_j(a_i) = x_j, a_i \in A\} \) - the set of all different evaluations on \( g_j, j \in G \),
- \( x_j^0, x_j^1, \ldots, x_j^{m_j} \) - the ordered values of \( X_j, (x_j^k < x_j^{k+1}) \).
A* ⊆ A - a set of reference actions,

An assignment example is an action \( a^* \in A^* \) for which the DM defined a desired assignment \( a^* \rightarrow [C_{LDM}(a*), C_{RDM}(a*)] \), where \([C_{LDM}(a*), C_{RDM}(a*)]\) is an interval of contiguous classes \(C_{LDM}(a*), C_{LDM}(a*)+1, \ldots, C_{RDM}(a*)\).

An assignment example is said to be precise if \( L^{DM}(a*) = R^{DM}(a*) \), and imprecise otherwise.

A set of assignment examples is consistent with \( U \) iff

\[
\forall a^*, b^* \in A^*, U(a^*) \geq U(b^*) \Rightarrow R^{DM}(a^*) \geq L^{DM}(b^*) \quad (1)
\]
UTA-DIS\textsuperscript{GMS} – preference model

To represent DM’s preferences, we use a value function $U$:

$$U(a) = \sum_{j=1}^{n} u_j(g_j(a))$$

where the marginal value functions $u_j$ are such that:

$$u_j(x_j^k) \leq u_j(x_j^{k+1}), \quad k = 0, 1, \ldots, m_j - 1, j \in G$$

To normalize $U$ so that $U(a) \in [0, 1], \forall a \in A$, we set:

$$u_j(x_j^0) = 0, \forall j \in G$$

$$\sum_{j=1}^{n} u_j(x_j^{m_j}) = 1$$
Consider the threshold-based sorting procedure

- $a \in A$ is assigned to class $C_h$ ($a \rightarrow C_h$) iff $U(a) \in [b_{h-1}, b_h)$
- $b_{h-1}$ corresponds to the minimum value for an action $a$ to be assigned to class $C_h$.
- $b_h$ is the supremum value for any action $a$ to be assigned to class $C_h$, i.e. if $a$ is assigned to class $C_h$, then $U(a) < b_h$
- we impose $b_{h-1} < b_h$, $\forall h \in H$ and we set $b_0 = 0$ and $b_p > 1$
Threshold-based sorting

\[ U(a) = \sum_{i} u_i(g_i(a)) \]

\[ g_1 \rightarrow u_1(g_1) \]
\[ g_2 \rightarrow u_2(g_2) \]
\[ g_n \rightarrow u_n(g_n) \]

\[ U(a) \]
\[ 0 \rightarrow b_1 \]
\[ \rightarrow b_2 \rightarrow C_1 \]
\[ \rightarrow b_3 \rightarrow C_2 \]
\[ \rightarrow 1 \rightarrow C_3 \]
We consider the example-based sorting procedure

- The example-based sorting procedure is driven by a value function $U$ and its associated assignment examples $A^* \subset A$. It assigns an action $a$ to an interval of classes $[C_{LU(a)}, C_{RU(a)}]$:

$$L^U(a) = \text{Max}\left\{L^{DM}(a^*): U(a^*) \leq U(a), a^* \in A^*\right\} \quad (2)$$

$$R^U(a) = \text{Min}\left\{R^{DM}(a^*): U(a^*) \geq U(a), a^* \in A^*\right\} \quad (3)$$

- Procedure considered in [Köksalan, Ulu 2003] with linear value functions.
Proposition

Consider the case where \( L_{DM}(a^*) \leq R_{DM}(a^*), \forall a^* \in A^* \),

Assuming the use of a single value function \( U \) in the example-based sorting procedure,

if we choose the \( b_h^U, h = 1, ..., p - 1 \) in the interval \( \max_{a^* : R_{DM}(a^*) \leq h} \{ U(a^*) \}, \min_{a^* : L_{DM}(a^*) > h} \{ U(a^*) \} \) \( \),

with \( b_h^U < b_{h+1}^U \),

we obtain a threshold-based sorting procedure that restores the assignment examples and assigns each non-reference action \( a \in A \setminus A^* \) to a single class in the interval \( [C_{LU}(a), C_{RU}(a)] \) stemming from the example-based sorting procedure.
Consider $b \in A$, with $U(b) \in ]U(a_6), U(a_7)[$. 

---

**UTA-DIS**

**GMS**
Consider $b \in A$, with $U(b) \in ]U(a_6), U(a_7)[$

$L^U(b) = C_2$
Consider $b \in A$, with $U(b) \in ]U(a_6), U(a_7)[$

$L^U(b) = C_2$, $R^U(b) = C_4$
Consider \( b \in A \), with \( U(b) \in ]U(a_6), U(a_7)[ \)

\[ L^U(b) = C_2, \quad R^U(b) = C_4 \]
**UTA-DIS$^\text{GMS}$ – possible and necessary assignments**

Application of a set of compatible value functions $U$

**Definition**

Considering a set $A^*$ of assignment examples, the set $U_{A^*}$ of compatible value functions is defined by:

$U_{A^*} = \{ U \in U : R^{DM}(b^*) < L^{DM}(a^*) \Rightarrow U(a^*) > U(b^*) \}$

**Definition**

Given a set $A^*$ of assignment examples and a corresponding set $U_{A^*}$ of compatible value functions, $\forall a \in A$, we define:

$C_P(a) = \{ h \in H : \exists U \in U_{A^*} \text{ for which } h \in [L^U(a), R^U(a)] \}$

$= \bigcup_{U \in U_{A^*}} [[L^U(a), R^U(a)]]$

$C_N(a) = \{ h \in H : \forall U \in U_{A^*} \text{ it holds } h \in [L^U(a), R^U(a)] \}$

$= \bigcap_{U \in U_{A^*}} [[L^U(a), R^U(a)]]$
Consider all increasing value functions (not piece-wise only),

Consider \( a^* \rightarrow [C_{L,DM}(a^*), C_{R,DM}(a^*)] \),
\( a^{*'} \rightarrow [C_{L,DM}(a^{*'}), C_{R,DM}(a^{*'})] \), such that the two intervals of classes have an empty intersection. It holds:

\[
U(a^*) < U(a^{*'}) \text{ iff } R^{DM}(a^*) < L^{DM}(a^{*'})
\] (4)

Assignment of \( a^*, a^{*'} \in A^* \), to intervals of classes having a non-empty intersection, does not induce constraints,

if \(|A^*| > 2\), one should consider constraints (4) for all pairs assigned to series of classes with an empty intersection,

these constraints (with monotonicity and normality) define (if consistent) a non-empty set of value functions \( U_{A^*} \) compatible with the preference information,
Computing possible assignments

Begin

\[ h \leftarrow p \]

While \( \mathcal{U}_{A^*} \cap \mathcal{U}_{\{a' \rightarrow C_h\}} = \emptyset \) do

\[ h \leftarrow h - 1 \]

\[ h' \leftarrow 1 \]

While \( \mathcal{U}_{A^*} \cap \mathcal{U}_{\{a' \rightarrow C_{h'}\}} = \emptyset \) do

\[ h' \leftarrow h' + 1 \]

\[ C_p(a') \leftarrow [C_{h'}, C_h] \]

End
UTA-DIS\textsuperscript{GMS} – posible assignments

\[ u_j(x_j^k) \leq u_j(x_j^{k+1}), \ k = 0, 1, \ldots, m_j - 1, j \in G \]  \hspace{1cm} (5)

\[ u_j(x_j^0) = 0, \ \forall j \in G \]  \hspace{1cm} (6)

\[ \sum_{j=1}^{n} u_j(x_j^{m_j}) = 1 \]  \hspace{1cm} (7)

\[ \sum_{j=1}^{n} u_j(a) \geq b_{c_{\text{min}}(a)-1}, \ \forall a \in A^* \]  \hspace{1cm} (8)

\[ \sum_{j=1}^{n} u_j(a) \leq b_{c_{\text{max}}(a)} - \varepsilon, \ \forall a \in A^* \]  \hspace{1cm} (9)

\[ \sum_{j=1}^{n} u_j(a') \geq b_{h-1} \]  \hspace{1cm} (10)

\[ \sum_{j=1}^{n} u_j(a') \leq b_h - \varepsilon \]  \hspace{1cm} (11)

- Contraints (5)-(9) define \( U_{A^*} \),

- Contraints (10),(11) define \( U_{a' \rightarrow C_h} \),

- The set of constraints (5)-(11) is feasible iff the following linear program has an optimal value \( \varepsilon^* > 0 \):

\[ \text{Max} \rightarrow \varepsilon \]

\text{s.t. constraints (5) – (11)}
UTA-DIS$^{GMS}$ – necessary assignments

- Computing **necessary** assignments

Let us define:

- $S^+(a) = \{ a' \in A^* : \exists U \in \mathcal{U}_{A^*} \text{ such that } U(a') \geq U(a) \}$,
- $S^-(a) = \{ a' \in A^* : \exists U \in \mathcal{U}_{A^*} \text{ such that } U(a') \leq U(a) \}$

$L^N(a)$ and $R^N(a)$ can be computed as:

- $L^N(a) = \text{Max}_{a' \in S^-(a)} \{ L^U(a') \}$,
- $R^N(a) = \text{Min}_{a' \in S^-(a)} \{ R^U(a') \}$

Computing $S^+(a)$ (and $S^-(a)$) can be done via the resolution of $m \times m_{\text{ref}}$ linear programs:

$$\text{Max/Min } \rightarrow U(a') - U(a)$$

s.t. $U \in \mathcal{U}_{A^*}$
Confidence levels and embedded assignment examples

- Consider an ordinal confidence scale $\psi_1 \succ \psi_2 \succ ... \succ \psi_\rho$
  (e.g., sure $\succ$ possible $\succ$ maybe),

- Each assignment example is assigned by the DM to a confidence level
  $\Rightarrow$ embedded sets of assignment examples,

- Embedded sets of assignment examples induce embedded ranges of assignment for non-reference alternatives,
Several DMs cooperate in a decision problem to construct a collective ranking

DMs share the same “description” of the decision problem (set of actions, evaluation criteria, evaluation matrix)

Each DM provides his/her own preference information

The collective ranking should account for the preference expressed by each DM
Set of DMs: $D = \{d_1, \ldots, d_p\}$

Preference information provided by DM $d_h, \ h=1, \ldots, p$: $B^R(d_h)$ a partial preorder on a set of reference actions
We consider the set of value functions for each \( d_h \in D^* \subset D \) stemming from UTA\(^{GMS}\)

For each \( D^* \subset D \), 4 situations are interesting for \((x,y) \in A:\)

\[
\begin{align*}
\text{1. } x \succeq_{N,N}^\subset(D^*) y: & \text{ } x \succeq_N y \text{ for all } d_h \in D^*, \\
\text{2. } x \succeq_{N,P}^\subset(D^*) y: & \text{ } x \succeq_N y \text{ for at least one } d_h \in D^*, \\
\text{3. } x \succeq_{P,N}^\subset(D^*) y: & \text{ } x \succeq_P y \text{ for all } d_h \in D^*, \\
\text{4. } x \succeq_{P,P}^\subset(D^*) y: & \text{ } x \succeq_P y \text{ for at least one } d_h \in D^*
\end{align*}
\]
Ordinal regression for group ranking: UTA-GROUP\textsuperscript{GMS}

Properties

- $\succeq_{N,N}(D^*)$ is a partial preorder
- $\succeq_{N,P}(D^*)$ is not necessarily transitive
- $\succeq_{P,P}(D^*)$ is strongly complete

When $D^* \subset D^{**}$, it holds

- $x \succeq_{N,N}(D^{**}) y \Rightarrow x \succeq_{N,N}(D^*) y$
- $x \succeq_{N,P}(D^{**}) y \Rightarrow x \succeq_{N,P}(D^*) y$
- $x \succeq_{P,N}(D^{**}) y \Rightarrow x \succeq_{P,N}(D^*) y$
- $x \succeq_{P,P}(D^{**}) y \Rightarrow x \succeq_{P,P}(D^*) y$
Ordinal regression for group ranking: UTA-GROUP

- Given a set of DMs $D^* \subseteq D$, a value function $U$ is compatible if it satisfies the following set of constraints:

\[
\begin{align*}
    U(a) > U(b) & \iff a \succ b \\
    U(a) = U(b) & \iff a \sim b \\
    u_i(g_i(a_{\tau_i(j)})) - u_i(g_i(a_{\tau_i(j-1)})) & \geq 0, \quad i = 1, \ldots, n, \quad j = 2, \ldots, m \\
    u_i(g_i(a_{\tau_i(1)})) & \geq 0, \quad u_i(g_i(a_{\tau_i(m)})) \leq u_i(\beta_i), \quad i = 1, \ldots, n, \\
    u_i(\alpha_i) & = 0, \quad i = 1, \ldots, n \\
    \sum_{i=1}^{n} u_i(\beta_i) & = 1,
\end{align*}
\]

where $\tau_i$ is the permutation on the set of indices of alternatives that reorders them according to the increasing evaluation on criterion $g_i$, i.e.

\[
g_i(a_{\tau_i(1)}) \leq g_i(a_{\tau_i(2)}) \leq \cdots \leq g_i(a_{\tau_i(m-1)}) \leq g_i(a_{\tau_i(m)})
\]

- Suppose that set $\mathcal{U}_{D^*}$ of compatible value functions is not empty (DMs statements are not contradictory)…
Ordinal regression for group ranking: UTA-GROUP®

- One obtains two rankings such that for any pair of actions \((x, y) \in A:\)
  - \(x \succeq^N(D^*) y:\) \(x\) is ranked at least as good as \(y\) iff \(U^{D^*}(x) \geq U^{D^*}(y)\) for all value functions compatible with the preference information (necessary weak preference relation \(\succeq^N\) being a partial preorder)
  - \(x \succeq^P(D^*) y:\) \(x\) is ranked at least as good as \(y\) iff \(U^{D^*}(x) \geq U^{D^*}(y)\) for at least one value function compatible with the preference information (possible weak preference relation \(\succeq^P\) being a strongly complete and negatively transitive binary relation)

- However, the set \(U_{D^*}\) of compatible value function can be empty...
Ordinal regression for group ranking: UTA-GROUP$^{\text{GMS}}$

Suppose $\mathcal{U}_{D^*} = \emptyset$

- $\mathcal{U}_{D^*}$ corresponds to the intersection of sets of compatible value functions for all $d_h \in D^*$ (each one being non-empty)
- This means that pairwise comparisons of two (or more) DMs are contradictory
- Identifying which are these contradictory comparisons amounts at solving inconsistency
- This leads to know which comparisons to remove in order to obtain a consistent collective model
- Performing these computations $\forall D^* \subset D$ allows to indentify coalitions of convergent DMs, for which a necessary and possible consensus rankings exist
Ordinal regression for group ranking: UTA-GROUP\textsuperscript{GMS}

- Reasoning in terms of pairwise comparisons decomposes elicitation of preference information into small natural pieces

- UTA-GROUP\textsuperscript{GMS} avoids discussions of DMs on technical parameters (tradeoffs, weights, ...)

- Taking into account all compatible value functions permits to reason in terms of necessary and possible rankings and coalitions
UTA-GROUPGMS: an illustrative example

Ranking problem: 3 DMs ($d_1$, $d_2$ and $d_3$), 20 actions evaluated on 5 criteria

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>$s_5$</th>
<th>$s_6$</th>
<th>$s_7$</th>
<th>$s_8$</th>
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<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Evaluation matrix

Empty dominance relation!
UTA-GROUP\textsuperscript{GMS} : an illustrative example

- Statements of DMs:
  - $d_1: s_1 \succ s_2, s_6 \succ s_7, s_{17} \succ s_{20}$
  - $d_2: s_9 \succ s_{13}, s_4 \succ s_5, s_{14} \succ s_7$
  - $d_3: s_4 \succ s_3, s_{15} \succ s_{11}, s_8 \succ s_{10}$

- $\mathcal{U}_{\{d_1,d_3\}} = \mathcal{U}_{\{d_1,d_2,d_3\}} = \emptyset$, i.e., statements of $d_1$ and $d_3$ are contradictory:
  - $d_1: s_1 \succ s_2 \Rightarrow s_3 \succ s_4$
  - $d_3: s_4 \succ s_3 \Rightarrow s_2 \succ s_1$

- If ($d_1$ removes $s_1 \succ s_2$) or ($d_3$ removes $s_4 \succ s_3$), then $\mathcal{U}_{\{d_1,d_2,d_3\}} \neq \emptyset$
UTA-GROUP$^{GMS}$: an illustrative example

- Although $\mathcal{U}_{\{d_1,d_2,d_3\}} = \emptyset$, the following relations are not empty:
  
  - $\preceq_{N,N}(\{d_1,d_2,d_3\}) = \{(s_6,s_7)\}$, i.e., $s_6 \preceq_{N} s_7$ for all $d_h$
  
  - $\preceq_{N,N}(\{d_1,d_2\}) = \{(s_6,s_7), (s_9,s_{13})\}$
  
  - $\preceq_{N,N}(\{d_1,d_3\}) = \{(s_6,s_7), (s_{17},s_{20})\}$
  
  - $\preceq_{N,N}(\{d_2,d_3\}) = \{(s_6,s_7), (s_{15},s_{11})\}$

- $x \preceq_{N,P}(\{d_1,d_2,d_3\}) y$:
  
  - $x \succ_{N} y$ for at least one $d_h$

- $x \preceq_{P,N}(\{d_1,d_2,d_3\}) y$:
  
  - $x \succ_{P} y$ for all $d_h$

- $x \preceq_{P,P}(\{d_1,d_2,d_3\}) y$:
  
  - $x \succ_{P} y$ for at least one $d_h$
UTA-GROUP$^{GMS}$ : an illustrative example

- Suppose $d_3$ removes $s_4 \succ s_3$ then the collective model leads to the following **collective necessary ranking**:
Multicriteria group sorting with a set of additive value functions: $\text{UTA-DIS-GROUP}^{\text{GMS}}$

- Set of DMs: $D = \{d_1, \ldots, d_p\}$
- Preference information provided by DM $d_h$, $h=1, \ldots, p$:
  \[
  a^* \rightarrow_h \left[ \begin{array}{c}
  C_{\min_h(a^*)} \\
  C_{\max_h(a^*)}
  \end{array} \right]
  \]
  for all reference actions $a^* \in A^R$
- Given a set of DMs $D' \subseteq D$,
  \[
  a^* \rightarrow_D \bigcup_{h \in D'} \left[ \begin{array}{c}
  C_{\min_h(a^*)} \\
  C_{\max_h(a^*)}
  \end{array} \right]
  \]
  \[
  a^* \rightarrow_D \bigcap_{h \in D'} \left[ \begin{array}{c}
  C_{\min_h(a^*)} \\
  C_{\max_h(a^*)}
  \end{array} \right]
  \]
Multicriteria group sorting with a set of additive value functions: 
\textsc{UTA-DIS-GROUP}^{\text{GMS}} (necessary and possible assignment)

\[ a \rightarrow \left[ C_{\min_{D'}}^N(a), C_{\max_{D'}}^N(a) \right] \]

means that all DMs in \( D' \) agree that action \( a \) can be assigned to 
one class from the interval 
\[ \left[ C_{\min_{D'}}^N(a), C_{\max_{D'}}^N(a) \right] \]

\[ a \rightarrow \left[ C_{\min_{D'}}^P(a), C_{\max_{D'}}^P(a) \right] \]

means that that there is at least one DM in \( D' \) who believes that 
action \( a \) can be assigned to one class in the interval 
\[ \left[ C_{\min_{D'}}^P(a), C_{\max_{D'}}^P(a) \right] \]

\[ \left[ C_{\min_{D'}}^N(a), C_{\max_{D'}}^N(a) \right] \subseteq \left[ C_{\min_{D'}}^P(a), C_{\max_{D'}}^P(a) \right] \]
Robust Ordinal Regression approach for outranking methods

- **Preference information** provided by DM:
  
  \[ aSb \text{ or } aScb, \text{ for } a,b \in A^R \]

- **Concordance function**, for \( a,b \in A \):
  
  \[ C(a,b) = \frac{w_1c'_1(a,b)+...+w_nc'_n(a,b)}{(w_1+...+w_n)} \]

  - since \( (w_1+...+w_n)=1 \), we can consider \( C(a,b)=c_1(a,b)+...+c_n(a,b) \),
    where \( c_i(a,b)=w_ic'_i(a,b), \text{ i=1,...,n} \)

  - \( c_i(a,b) \) is a monotone, non-decreasing function w.r.t. \( g_i(a)-g_i(b) \),
    such that \( c_i(a,b) \geq 0 \) for all \( a,b \in A \) (alt. for \( g_i(a)-g_i(b) \geq q_i \geq 0 \), \text{ i=1,...,n} \),
    and

    \[ c_1(a,b)+...+c_n(a,b)=1 \text{ in case } g_i(a)-g_i(b) = \beta_i-\alpha_i \text{ for all } i=1,...,n \]
Robust Ordinal Regression approach for outranking methods

- **Ordinal regression constraints**, for \( a, b \in A^R \):

\[
\begin{align*}
&c_1(a,b)+\ldots+c_n(a,b) \geq \lambda \quad \text{and} \quad g_i(b)-g_i(a) \leq v_i, \quad i=1,\ldots,n, \quad \text{if} \quad aSb \\
&c_1(a,b)+\ldots+c_n(a,b) \leq \lambda+\varepsilon+M_0(a,b) \quad \text{and} \quad g_i(b)-g_i(a) \geq v_i+\varepsilon-\delta M_i(a,b), \\
&M_i(a,b) \in \{0,1\}, \quad i=1,\ldots,n, \quad M_0(a,b)+M_1(a,b) \ldots+M_n(a,b) \leq n, \quad \text{if} \quad aS^c b \\
&\lambda \geq 0.5, \quad v_i \geq 0 \quad (\text{alt.} \quad v_i \geq p_i \geq q_i \geq 0), \quad i=1,\ldots,n, \\
&c_i(a,b) \geq 0 \quad \text{for all} \quad a, b \in A^R \quad \text{and} \quad i=1,\ldots,n, \\
&c_1(a,b)+\ldots+c_n(a,b)=1 \quad \text{for} \quad g_i(a)-g_i(b)=\beta_i-\alpha_i, \quad i=1,\ldots,n \\
&c_i(a,b) \geq c_i(c,d) \quad \text{if} \quad g_i(a)-g_i(b) \geq g_i(c)-g_i(d), \quad \text{for all} \quad a, b, c, d \in A^R, \quad i=1,\ldots,n
\end{align*}
\]

where \( \varepsilon \) is a small positive value and \( \delta \) is a big positive value

(if specified, preference and indifference thresholds \( p_i, q_i \) are given)
Robust Ordinal Regression approach for outranking methods

- Given a pair of actions $x, y \in A$, $x$ necessarily outranks $y$:
  \[ xS^N y \iff d(x, y) \geq 0 \]
  
  \[
  d(x, y) = \min \{ c_1(x, y) + \ldots + c_n(x, y) - \lambda \}, \quad \text{s.t. } E(A^R),
  \]
  where

  \[
  c_i(a, b) \geq c_i(c, d) \quad \text{if} \quad g_i(a) - g_i(b) \geq g_i(c) - g_i(d), \quad \text{for all } a, b, c, d \in A^R \cup \{ x, y \},
  \]

  \[
  i = 1, \ldots, n, \text{ and} \]

  \[
  g_i(y) - g_i(x) \leq v_i, \quad i = 1, \ldots, n
  \]

- $d(x, y) \geq 0$ means that for all compatible outranking models $x$ outranks $y$.

- For $x, y \in A^R$:
  \[ xS y \Rightarrow xS^N y \]
Robust Ordinal Regression approach for outranking methods

Given a pair of actions $x, y \in A$, $x$ possibly outranks $y$:

$$x S^P y \iff D(x, y) \geq 0$$

$$D(x, y) = \text{Max}\{c_1(x, y) + \ldots + c_n(x, y) - \lambda\}, \text{ s.t. } E(A^R), \text{ where}$$

$$c_i(a, b) \geq c_i(c, d) \text{ if } g_i(a) - g_i(b) \geq g_i(c) - g_i(d), \text{ for all } a, b, c, d \in A^R \cup \{x, y\}, \text{ } i = 1, \ldots, n, \text{ and}$$

$$g_i(y) - g_i(x) \leq v_i, \text{ } i = 1, \ldots, n$$

$D(x, y) \geq 0$ means that **for at least one** compatible outranking model $x$ outranks $y$

For $x, y \in A^R$:

$$x S y \Rightarrow \text{not } y S^P x$$
For any pair of actions $x, y \in A$:

$$x S^N y \iff \neg x S^{CP} y$$

$$x S^P y \iff \neg x S^{CN} y$$

so, only $x S^N y$ and $x S^P y$ are to be checked
Robust Ordinal Regression approach for outranking methods: the case of group decision

- Generalization for **group decision** is analogical to UTA-GROUP$^{GMS}$ and UTA-DIS-GROUP$^{GMS}$
- For each DM $d_h \in D' \subseteq D$ we consider all compatible outranking models
- Four situations are interesting for $x, y \in A$:
  - $x \ S^{N,N}(D') \ y: x \ S^N y$ for all $d_h \in D'$
  - $x \ S^{N,P}(D') \ y: x \ S^N y$ for at least one $d_h \in D'$
  - $x \ S^{P,N}(D') \ y: x \ S^P y$ for all $d_h \in D'$
  - $x \ S^{P,P}(D') \ y: x \ S^P y$ for at least one $d_h \in D'$
ROR and Interaction among criteria
Basic concepts
Setting

- $N = \{1, 2, \ldots, n\}$ set of criteria
- $X_i$ : set of possible values of the $i$-th criterion
- $X = \prod_{i=1}^{n} X_i = X_1 \times X_2 \times \ldots \times X_n = \{(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}$: set of all conceivable alternatives
  - $X$ include the alternatives under study. . . and many others!
- In this case we suppose that $X_1 = X_2 = \ldots = X_n = X \subseteq \mathbb{R}_+$ such that $X = X^n$
- $\succeq$: weak preference relation on $X$ such that for all $x, y \in X$
  - $x \succeq y$
  - means «$x$ is at least as good as $y$»
Marginal preferences

- \( \succeq_i \): weak **marginal preference** relation on \( X_i, \ i \in \mathbb{N} \), such that for all \( x_i, y_i \in X_i \)

\[
x_i \succeq_i y_i \text{ means } \langle x_i \text{ is at least as good as } y_i \rangle
\]

- We suppose also that

\[
x_i \geq y_i \iff x_i \succeq y_i
\]

- \( \succeq^\circ \): weak **marginal preference** relation on \( \bigcup_{i=1}^{n} X_i \), such that for all \( x_i \in X_i, y_j \in X_j, \ i, j \in \mathbb{N} \)

\[
x_i \succeq^\circ y_j \text{ means } \langle x_i \text{ is at least as good as } y_i \rangle
\]

- We suppose also that

\[
x_i \geq y_j \iff x_i \succeq^\circ y_j
\]
Weighted sum model

For all all $x, y \in X$

$$x \succeq y \iff \sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i y_i$$

with $w_i$ non negative for all $i \in \mathbb{N}$.

In this case $w_i$ can be interpreted as the importance of criterion $i \in \mathbb{N}$.

The importance of couple of criteria $\{i,j\} \subseteq \mathbb{N}$ is given by $w_i + w_j$.

The importance of set of criteria $A \subseteq \mathbb{N}$, denoted by $\mu(A)$, is given by

$$\mu(A) = \sum_{i \in A} w_i$$

Observe that for $A, B \subseteq \mathbb{N}$ such that $A \cap B = \emptyset$

$$\mu(A \cup B) = \sum_{i \in A} w_i + \sum_{i \in B} w_i = \sum_{i \in A \cup B} w_i = \mu(A) + \mu(B)$$
Introductory example
### Illustrative example (Grabisch 1996)

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
</tr>
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<td>18</td>
<td>16</td>
<td>10</td>
</tr>
<tr>
<td>S2</td>
<td>10</td>
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</tr>
<tr>
<td>S3</td>
<td>14</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>
Suppose that the school is more scientifically than literary oriented, so that weights could be for example 3, 3 and 2 respectively.

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
<th>Global evaluation (weighted sum)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>18</td>
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<td>15.25</td>
</tr>
<tr>
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<tr>
<td>S3</td>
<td>14</td>
<td>15</td>
<td>15</td>
<td>14.62</td>
</tr>
</tbody>
</table>
Illustrative example (Grabisch 1996)

“If the school wants to favor well equilibrated students without weak points, the above ranking is not fully satisfactory, since student S1 has a severe weakness in literature, but has been considered better than student S3, who has no weak point. The reason is that too much importance is given to mathematics and physics, which are in a sense redundant since, usually, students good at mathematics are also good at physics (and vice versa), so that the evaluation is overestimated (resp. underestimated) for students good (resp. bad) at mathematics and/or physics.”

How to solve the problem?
Representing importance of criteria by means of a fuzzy measure (or capacity)

\[ \mu : 2^N \rightarrow [0, 1] \text{ is a fuzzy measure satisfying the following axioms:} \]

\[ \bullet \mu(\emptyset) = 0, \mu(N) = 1 \text{ (boundary conditions);} \]
\[ \bullet A \subseteq B \subseteq N \text{ implies } \mu(A) \leq \mu(B) \text{ (monotonicity conditions);} \]

For any \( A \subseteq N \), \( \mu(A) \) represent the importance of the set of criteria \( A \).

It is no more true that for any \( A, B \subseteq N \) such that \( A \cap B = \emptyset \) we have

\[ \mu(A \cup B) = \mu(A) + \mu(B) \]
Representing importance of criteria by means of a fuzzy measure in the illustrative example (Grabisch 1996)

<table>
<thead>
<tr>
<th>Set of subjects $A$</th>
<th>$\mu(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>0</td>
</tr>
<tr>
<td>{Mathematics}</td>
<td>0.45</td>
</tr>
<tr>
<td>{Physics}</td>
<td>0.45</td>
</tr>
<tr>
<td>{Literature}</td>
<td>0.3</td>
</tr>
<tr>
<td>{Mathematics, Physics}</td>
<td>0.5</td>
</tr>
<tr>
<td>{Mathematics, Literature}</td>
<td>0.9</td>
</tr>
<tr>
<td>{Physics, Literature}</td>
<td>0.9</td>
</tr>
<tr>
<td>{Mathematics, Physics, Literature}</td>
<td>1</td>
</tr>
</tbody>
</table>

$\mu(\{\text{Mathematics, Physics}\}) \leq \mu(\{\text{Mathematics}\}) + \mu(\{\text{Physics}\})$

(*redundancy* between Mathematics and Physics)

$\mu(\{\text{Mathematics, Literature}\}) \leq \mu(\{\text{Mathematics}\}) + \mu(\{\text{Literature}\})$

(*synergy* between Mathematics and Literature)

$\mu(\{\text{Physics, Literature}\}) \leq \mu(\{\text{Physics}\}) + \mu(\{\text{Literature}\})$

(*synergy* between Physics and Literature)
The **Choquet integral** (1952): computing a “weighted sum” using the non additive weights given by the fuzzy measure

The Choquet integral of \( x \in X \subseteq \mathbb{R}^n_+ \) is defined as follows:

\[
C_\mu(x) = \sum_{i=1}^{n} [x(i) - x(i-1)] \mu(A_i)
\]

with \((\cdot)\) stands for a permutation of the indices evaluations of criteria such that:

\[
x(1) \leq x(2) \leq x(3) \leq \ldots \leq x(n)
\]

with \(A_i = \{(i), \ldots, (n)\}\) where \(A_{n+1} = \{\emptyset\}\) \((i = 1, \ldots, n)\) and \(x(0) = 0\).

Observe that the Choquet integral can be written also as follows:

\[
C_\mu(x) = \int_{0}^{\max_{i \in N} x_i} \mu(\{j \in N : x_j \geq x_i\})\,dt
\]
Illustrative example (Grabisch 1996): computing the Choquet integral for students S1

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>18</td>
<td>16</td>
<td>10</td>
</tr>
</tbody>
</table>

\[ x_{\text{Lit}} \leq x_{\text{Phys}} \leq x_{\text{Math}} \]

\[ A_1 = \{ \text{Mathematics, Physics, Literature} \} \]

\[ A_2 = \{ \text{Mathematics, Physics} \} \]

\[ A_3 = \{ \text{Mathematics} \} \]

\[ C_\mu(18,16,10) = (10-0) \times \mu(A_1) + (16-10) \times \mu(A_2) + (18-16) \times \mu(A_2) = (10-0) \times 1 + (16-10) \times 0.5 + (18-16) \times 0.45 = 13.9 \]
Illustrative example (Grabisch 1996): computing the Choquet integral for students S2

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>S2</td>
<td>10</td>
<td>12</td>
<td>18</td>
</tr>
</tbody>
</table>

\[
x_{\text{Math}} \leq x_{\text{Phys}} \leq x_{\text{Lit}}
\]

\[
A_1 = \{\text{Mathematics, Physics, Literature}\}
\]

\[
A_2 = \{\text{Physics, Literature}\}
\]

\[
A_3 = \{\text{Literature}\}
\]

\[
C_\mu(10,12,18) = (10-0)\mu(A_1) + (12-10)\mu(A_2) + (18-12)\mu(A_2) = (10-0)\times1 + (12-10)\times0.9 + (18-12)\times0.3 = 13.6
\]
Illustrative example (Grabisch 1996): computing the Choquet integral for students S3

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>S3</td>
<td>14</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

\[ x_{\text{Math}} \leq x_{\text{Phys}} \leq x_{\text{Lit}} \]

\[ A_1 = \{ \text{Mathematics, Physics, Literature} \} \]

\[ A_2 = \{ \text{Physics, Literature} \} \]

\[ A_3 = \{ \text{Physics} \} \]

\[ C_\mu(18,16,10) = (14-0) \times \mu(A_1) + (15-14) \times \mu(A_2) + (15-15) \times \mu(A_2) = (14-0) \times 1 + (15-14) \times 0.9 + (15-15) \times 0.45 = 14.9 \]
Illustrative example (Grabisch 1996)

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
<th>Global evaluation (Choquet integral)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>18</td>
<td>16</td>
<td>10</td>
<td>13.9</td>
</tr>
<tr>
<td>S2</td>
<td>10</td>
<td>12</td>
<td>18</td>
<td>13.6</td>
</tr>
<tr>
<td>S3</td>
<td>14</td>
<td>15</td>
<td>15</td>
<td>14.9</td>
</tr>
</tbody>
</table>

Choquet integral ranks student S1, that has a severe weakness in literature, worse than student S3, that has no weak point.
Specific cases of Choquet integral

- $C_{\mu}(x_1,\ldots,x_n) = \text{Max}(x_1,\ldots,x_n)$ if $\mu(A)=1$ for all $\emptyset \subset A \subseteq N$ (and, of course, $\mu(\emptyset)=0$)

- $C_{\mu}(x_1,\ldots,x_n) = \text{min}(x_1,\ldots,x_n)$ if $\mu(A)=0$ for all $\emptyset \subset A \subseteq N$ (and, of course, $\mu(N)=1$)

- $C_{\mu}(x_1,\ldots,x_n) = \text{OWA}(w_1,\ldots,w_n; x_1,\ldots,x_n)$ if $\mu(A)=\mu(B)$ when $|A|=|B|$, for all $A,B \subseteq N$ with

  $$\text{OWA}(w_1,\ldots,w_n; x_1,\ldots,x_n) = w_1x_{(1)} + \ldots + w_nx_{(n)}$$

  (Yager 1988)

  and $w_i = \mu(A) - \mu(B)$ with $A,B \subseteq N$ such that $|A|=i$ and $|B|=i-1$;

- $C_{\mu}(x_1,\ldots,x_n) = w_{(k)}$ (k-th order statistic, $0<k\leq n$) if $\mu(A)=0$ for $|A|<k$ and $\mu(A)=1$ for $|A|\geq k$ for all $A \subseteq N$. 
The Möbius transformation of a fuzzy measure (or capacity)

$$a(R) = \sum_{T \subseteq R} (-1)^{|R-T|} \mu(T), \text{ for each } R \subseteq G$$

is a Möbius transformation with $a : 2^G \to \mathbb{R}$ such that:

- $a(\emptyset) = 0$, $\sum_{T \subseteq G} a(T) = 1$ (boundary);
- $\sum_{T \subseteq R} a(T) \geq 0 \ \forall \ g_i \in R$ and $\forall R \subseteq G$ (monotonicity).

$$C(a) = \sum_{T \subseteq G} a(T) \min_{g_i \in T} g_i(a)$$

is the Choquet integral in terms of the Möbius representation.
The Shapley value

- The global importance of a criterion \( i \in N \) is not solely determined by the value \( \mu(\{i\}) \), but also by all \( \mu(A) \) with \( A \subseteq N \) such that \( i \in A \).

- But how to extract from these values the contribution of \( i \) alone?

- By the Shapley value

\[
\varphi(\{i\}) = \sum_{A \subseteq N: i \in A} \frac{a(A)}{|A|} = \sum_{A \subseteq N: i \in A} \frac{(|A|-1)!(n-|A|)!}{n!} \left( \mu(A) - \mu(A - \{i\}) \right)
\]
The interaction index

- The interaction between $i$ and $j$ is not only determined by the difference $\mu(\{i, j\}) - \mu(\{i\}) - \mu(\{j\})$ but also by all the coefficients $\mu(A)$ such that $\{i, j\} \subseteq A$. Then, how to compute a degree of interaction which is meaningful?

- By the interaction index (Murofushi 1993)

$$\varphi(\{i, j\}) = \sum_{A \subseteq N: \{i, j\} \subseteq A} \frac{a(A)}{|A| - 1}$$
2-additive fuzzy measures

\[ a(\emptyset) = 0, \quad \sum_{g_i \in G} a(\{g_i\}) + \sum_{\{g_i, g_j\} \subseteq G} a(\{g_i, g_j\}) = 1 \text{ (boundary)}; \]

\[ a(\{g_i\}) \geq 0, \quad \forall g_i \in G, \quad a(\{g_i\}) + \sum_{g_j \in T} a(\{g_i, g_j\}) \geq 0, \]

\[ \forall g_i \in G \text{ and } \forall T \subseteq G \setminus \{g_i\} \text{ (monotonicity.)} \]

In this case, the **Choquet integral** is given by:

\[ C_\mu(a) = \sum_{g_i \in G} a(\{g_i\}) g_i(a) + \sum_{g_i, g_j \subseteq G} a(\{g_i, g_j\}) \min \{g_i(a), g_j(a)\}. \]
The *importance index* (Shapley index) of \( i \in G \) is:

\[
\varphi (\{i\}) = a (\{i\}) + \sum_{j \in G \setminus \{i\}} \frac{a (\{i, j\})}{2},
\]

The *interaction index* for a couple of criteria \( i, j \in G \) is:

\[
\varphi (\{i, j\}) = a (\{i, j\}).
\]
Non Additive Robust Ordinal Regression
(Angilella, Greco, Matarazzo 2010)

• Non Additive Robust Ordinal Regression (NAROR) multicriteria aggregation-disaggregation method for ranking alternatives including preference information on interaction and importance of criteria
• A fuzzy measure $\mu$ is called compatible if it restores the DM’s preference information on $A' \subseteq A$
• Two preference relations: the necessary weak preference relation
  \[ x \succeq^N y \iff C_\mu(x) \geq C_\mu(y) \]
  for all fuzzy measures $\mu$ and $x, y \in A$
• and the possible weak preference relation
  \[ x \succeq^P y \iff C_\mu(x) \geq C_\mu(y) \]
  for at least one fuzzy measure $\mu$ with $x, y \in A$. 
The Decision Maker’s preference information

- Partial preorder $\succeq$ on $A'$ for $a, b \in A'$:
  \[ a \succeq b \iff a \text{ is at least as good as } b \]
- Partial preorder $\succeq^*$ on $A' \times A'$ for $a, b, c, d \in A'$:
  \[ (a, b) \succeq^* (c, d) \iff \]
  \[ a \text{ is preferred to } b \text{ at least as much as } c \text{ is preferred to } d \]
- Partial preorder $\succeq$ on $G$ for $i, j \in G$:
  \[ i \succeq j \iff i \text{ is at least as important as criterion } j \]
- Partial preorder $\succeq^*$ on $G \times G$ for $i, j, l, k \in G$:
  \[ (i, j) \succeq^* (l, k) \iff \]
  \[ \text{Difference of importance of criteria } i \text{ and } j \text{ is at least as much as difference of importance of criteria } l \text{ and } k \]
The Decision Maker’s preference information

- Positive or negative interaction of pairs of criteria
- Partial preorder $\succeq_{\text{Int}}$ on $G \times G$ for $i, j, l, k \in G$:
  \[(i, j) \succeq_{\text{Int}} (l, k)\]

- Partial preorder $\succeq^{*}_{\text{Int}}$ on $G \times G \times G \times G$ for $i, j, l, k, r, s, t, w \in G$:
  \[[(i, j), (l, k)] \succeq^{*}_{\text{Int}} [(r, s), (t, w)]\]

Difference of interaction between criteria $i$ and $j$ and interaction between criteria $l$ and $k$ is at least as much as difference of interaction between criteria $r$ and $s$ and interaction between criteria $t$ and $w$. 
The Decision Maker’s preference information

- Set of DMs: \( D = \{DM_h : h = 1, \ldots, p\} \)

From a mathematical point of view, we have the following system of linear constraints:

\[
\begin{align*}
C_\mu(a) &> C_\mu(b) \text{ if } a \succ b \text{ with } a, b \in A', \\
C_\mu(a) &= C_\mu(b) \text{ if } a \sim b \text{ with } a, b \in A', \\
C_\mu(a) - C_\mu(b) &> C_\mu(c) - C_\mu(d) \text{ if } (a, b) \succ^* (c, d) \text{ with } a, b, c, d \in A', \\
\varphi(\{i\}) &> \varphi(\{j\}) \text{ if } i \succ j \text{ with } i, j \in G, \\
\varphi(\{i\}) - \varphi(\{j\}) &> \varphi(\{l\}) - \varphi(\{k\}) \text{ if } (i, j) \succ^* (l, k), \\
\varphi(\{i, j\}) &\leq 0 \text{ (redundancy) or } \varphi(\{i, j\}) \geq 0 \text{ (synergy)}, \text{ with } i, j \in G, \\
|\varphi(\{i, j\})| &> |\varphi(\{l, k\})| \text{ if } (i, j) \succ_{\text{int}}^* (l, k) \text{ with } i, j, l, k \in G, \\
|\varphi(\{i, j\})| - |\varphi(\{l, k\})| &> |\varphi(\{r, s\})| - |\varphi(\{t, w\})| \text{ if } [(i, j), (l, k)] \succ_{\text{int}}^* [(r, s), (t, w)], \\
\end{align*}
\]

with \( i, j, l, k, r, s, t, w \in G, \)

\[
\begin{align*}
a(\emptyset) &= 0, \quad \sum_{i \in G} a(\{i\}) + \sum_{\{i, j\} \subseteq G} a(\{i, j\}) = 1, \\
a(\{i\}) &\geq 0, \quad \forall i \in G, \quad a(\{i\}) + \sum_{j \in T} a(\{i, j\}) \geq 0, \quad \forall i \in G \text{ and } \forall T \subseteq G \setminus \{i\}
\end{align*}
\]
Since linear programming is not able to handle strict inequalities in $\mathcal{I}_{DM_h}$, we put the constraints in the form of weak inequalities, by adding $\varepsilon > 0$

\[\begin{align*}
C_\mu(a) &> C_\mu(b) + \varepsilon \text{ if } a > b \text{ with } a, b \in A', \\
C_\mu(a) &= C_\mu(b) + \varepsilon \text{ if } a \sim b \text{ with } a, b \in A', \\
C_\mu(a) - C_\mu(b) &> C_\mu(c) - C_\mu(d) + \varepsilon \text{ if } (a, b) \succ^* (c, d) \text{ with } a, b, c, d \in A', \\
\varphi(\{i\}) &> \varphi(\{j\}) + \varepsilon \text{ if } i \succ j \text{ with } i, j \in G, \\
\varphi(\{i\}) - \varphi(\{j\}) &> \varphi(\{l\}) - \varphi(\{k\}) + \varepsilon \text{ if } (i, j) \succ^* (l, k), \\
\varphi(\{i, j\}) &\leq 0 \text{ (redundancy) or } \varphi(\{i, j\}) \geq 0 \text{ (synergy), with } i, j \in G, \\
|\varphi(\{i, j\})| &> |\varphi(\{l, k\})| + \varepsilon \text{ if } (i, j) \triangleright_{int} (l, k) \text{ with } i, j, l, k \in G, \\
|\varphi(\{i, j\})| - |\varphi(\{l, k\})| &> |\varphi(\{r, s\})| - |\varphi(\{t, w\})| + \varepsilon \text{ if } [(i, j), (l, k)] \triangleright_{int} [(r, s), (t, w)], \\
\text{with } i, j, l, k, r, s, t, w &\in G, \\
a(\emptyset) &= 0, \sum_{i \in G} a(\{i\}) + \sum_{\{i, j\} \subseteq G} a(\{i, j\}) = 1, \\
a(\{i\}) &\geq 0, \forall i \in G, a(\{i\}) + \sum_{j \in T} a(\{i, j\}) \geq 0, \forall i \in G \text{ and } \forall T \subseteq G \setminus \{i\}.
\end{align*}\]
The polyhedron defined by the linear constraints can be empty due to some **inconsistencies** in the DM’s preference information:

Decision Analyst’s interactive arrangements can help the DM to solve such inconsistencies
NAROR methodology

\[
\max \varepsilon \quad \text{s.t.} \quad \begin{cases} 
\mathcal{I}_{DM_h}^\varepsilon \\
C_\mu(y) \geq C_\mu(x) + \varepsilon.
\end{cases}
\]

If \( \varepsilon \leq 0 \), then \( C_\mu(x) \geq C_\mu(y) \) for all compatible \( \mu \) that implies \( x \gtrsim^N y \) with \( x, y \in A \).

\[
\max \varepsilon \quad \text{s.t.} \quad \begin{cases} 
\mathcal{I}_{DM_h}^\varepsilon \\
C_\mu(x) \geq C_\mu(y).
\end{cases}
\]

If \( \varepsilon > 0 \), then there exists at least one compatible fuzzy measures \( \mu \) such that \( C_\mu(x) \geq C_\mu(y) \), that implies \( x \gtrsim^P y \) with \( x, y \in A \).
A recruitment problem

<table>
<thead>
<tr>
<th></th>
<th>Education</th>
<th>Experience</th>
<th>Age</th>
<th>Interview</th>
</tr>
</thead>
<tbody>
<tr>
<td>Odile</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>Slobodan</td>
<td>3</td>
<td>1</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Charles</td>
<td>10</td>
<td>9</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>Irene</td>
<td>5</td>
<td>9</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>Katherin</td>
<td>8</td>
<td>0</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>Felix</td>
<td>5</td>
<td>9</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>Germaine</td>
<td>8</td>
<td>10</td>
<td>5</td>
<td>7</td>
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<tr>
<td>Benedicte</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>Arthur</td>
<td>0</td>
<td>10</td>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

Example inspired from Pomerol and Barbera-Romero, 1993; criteria on a [0, 10] scale.
The Decision Maker’s preference information

\[ A' = \{ \text{Odile, Slobodan, Benedicte, Charles, Katherine} \} \]

- Charles \( \succeq \) Slobodan
- Germaine \( \succeq \) Slobodan
- (Odile, Charles) \( \succ^* \) (Benedicte, Slobodan)
- (Benedicte, Katherine) \( \succ^* \) (Charles, Slobodan)
- Ed \( \triangleright \) Ex, Ex \( \triangleright \) Ag and Ag \( \triangleright \) In
- (Ed, Ex) \( \triangleright^* \) (Ag, In)
- \( \varphi(Ed, Ex) > 0 \), \( \varphi(Ex, Ag) > 0 \) and \( \varphi(Ex, In) < 0 \)
- (Ed, Ex) \( \triangleright_{\text{Int}} \) (Ex, Ag)
Necessary and possible preference relations
ROR and the enriched additive value functions: the UTA$^{GMS}$-INT method
Plan

- Interaction among criteria explained on an example
- Incapacity of additive value function and Choquet integral
- Robust Ordinal Regression dealing with interactions – $UTA^{GMS-INT}$
  - Input preference information
  - Discovering the need of handling interactions
  - Identifying the pairs of interacting criteria
  - Calculating the necessary and possible preference relations
  - Didactic example
- Interaction on bipolar scales - $UTA^{GSS}$
- Conclusions
Interaction among criteria explained on an example
Interactions between two criteria

- **Positive interactions** (e.g., maximum speed & price of a car):
  \[ u_{i_1, i_2}(x_{i_1}, x_{i_2}) > u_{i_1}(x_{i_1}) + u_{i_2}(x_{i_2}) \]

- **Negative interactions** (e.g., maximum speed & acceleration of a car):
  \[ u_{i_1, i_2}(x_{i_1}, x_{i_2}) < u_{i_1}(x_{i_1}) + u_{i_2}(x_{i_2}) \]
What preference model would be able to represent the preference:

\[ S_2 \succ S_1 \text{ and } S_3 \succ S_4 \]
Additive value function

- Consider an additive value function

\[ U(x) = \sum_{i=1}^{n} u_i[g_i(x)] \]

- Does there exist an additive value function representing the preferences $S2 \succ S1$ and $S3 \succ S4$?
Additive value function

\[ S_2 \succ S_1 \]

\[ U(S_2) = U_{\text{Math}}(\text{GOOD}) + U_{\text{Phys}}(\text{BAD}) + U_{\text{Lit}}(\text{MEDIUM}) > \]

\[ U_{\text{Math}}(\text{GOOD}) + U_{\text{Phys}}(\text{MEDIUM}) + U_{\text{Lit}}(\text{BAD}) = U(S_1) \]

\[ S_3 \succ S_4 \]

\[ U(S_3) = U_{\text{Math}}(\text{MEDIUM}) + U_{\text{Phys}}(\text{MEDIUM}) + U_{\text{Lit}}(\text{BAD}) > \]

\[ U_{\text{Math}}(\text{MEDIUM}) + U_{\text{Phys}}(\text{BAD}) + U_{\text{Lit}}(\text{MEDIUM}) = U(S_4) \]
Additive value function

\[ U(S2) = U_{\text{Math}}(\text{GOOD}) + U_{\text{Phys}}(\text{BAD}) + U_{\text{Lit}}(\text{MEDIUM}) > U_{\text{Math}}(\text{GOOD}) + U_{\text{Phys}}(\text{MEDIUM}) + U_{\text{Lit}}(\text{BAD}) = U(S1) \]

\[ U(S3) = U_{\text{Math}}(\text{MEDIUM}) + U_{\text{Phys}}(\text{MEDIUM}) + U_{\text{Lit}}(\text{BAD}) > U_{\text{Math}}(\text{MEDIUM}) + U_{\text{Phys}}(\text{BAD}) + U_{\text{Lit}}(\text{MEDIUM}) = U(S4) \]

contradiction!
Additive value function: violation of preferential independence

<table>
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<tbody>
<tr>
<td>S1</td>
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<td>Medium</td>
<td>Bad</td>
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<td>Good</td>
<td>Bad</td>
<td>Medium</td>
</tr>
<tr>
<td>S3</td>
<td>Medium</td>
<td>Medium</td>
<td>Bad</td>
</tr>
<tr>
<td>S4</td>
<td>Medium</td>
<td>Bad</td>
<td>Medium</td>
</tr>
</tbody>
</table>

$S2 \succ S1$ and $S3 \succ S4$
**Choquet integral**

- Numerical encoding on a unique scale of the evaluations on each criterion:

<table>
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<th>Physics</th>
<th>Literature</th>
</tr>
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</tr>
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<td>S3</td>
<td>Medium</td>
<td>Medium</td>
<td>Bad</td>
</tr>
<tr>
<td>S4</td>
<td>Medium</td>
<td>Bad</td>
<td>Medium</td>
</tr>
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</table>

<table>
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<th>Physics</th>
<th>Literature</th>
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<tbody>
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<tr>
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<td>0.5</td>
</tr>
<tr>
<td>S3</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>S4</td>
<td>0.5</td>
<td>0</td>
<td>0.5</td>
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Choquet integral

**Definition**

\[ C_\mu(a) = \sum_{i=1}^{n} [u(g_i(a)) - u(g_{i-1}(a))] \times \mu(R_i) \]

where \( u(g_i) \) is encoding \( g_i \) on a common numerical scale, \((·)\) stands for the permutation of the indices of criteria:

\[ g_{(n)}(a) \succeq^I g_{(n-1)}(a) \succeq^I \ldots \succeq^I g_{(1)}(a) \]

\( \mu(R_i) \) is called capacity of \( R_i \), i.e. weight for subset of criteria \( R_i \)

\( R_i = \{(i),\ldots,(n)\}, \; i=1,\ldots,n, \; u(g_0) = 0 \)
The Choquet integral is **not able** to represent the dean’s preferences $S_2 \succ S_3$ and $S_3 \succ S_4$ for any order-preserving numerical encoding $u$ of the evaluations and for any values of capacity $\mu$.

- **$S_2 \succ S_1$** means $C_\mu(S_2) > C_\mu(S_1)$, which implies

  \[
  [u(\text{Good}) - u(\text{Medium})] \times \mu(\{\text{Math}\}) + [u(\text{Medium}) - u(\text{Bad})] \times \mu(\{\text{Math,Lit}\}) > [u(\text{Good}) - u(\text{Medium})] \times \mu(\{\text{Math}\}) + [u(\text{Medium}) - u(\text{Bad})] \times \mu(\{\text{Math,Phys}\})
  \]

- **$S_3 \succ S_4$** means $C_\mu(S_3) > C_\mu(S_4)$, which implies

  \[
  [u(\text{Medium}) - u(\text{Bad})] \times \mu(\{\text{Math,Phys}\}) > [u(\text{Medium}) - u(\text{Bad})] \times \mu(\{\text{Math,Lit}\})
  \]

contradiction!
Bipolar Choquet integral could handle this interaction but it is yet less intuitive.

Problems with respect to numerical encoding. How to transform performances on criteria into numerical values of a common scale? Questions, like: „is maximum speed of 180km/h worth a fuel consumption of 12 l/100km“?

Problems with respect to non-additive weights (capacity). How to translate the possible interaction among criteria into the capacities? Is there an intelligible relation between the preference information provided by the DM and the obtained value of the capacity? (see Mayag, Grabisch, Labreuche, 2008; Gonzales, Perny, 2005)

Problems with respect to interpretation of the Choquet integral. Is it possible to clearly justify preference of alternative a over alternative b in terms of values of the integral’s components? (see Roy, 2009)
We propose to enrich the additive value function...

We consider a value function of the type

\[
U^{int}(a) = \sum_{i=1}^{n} u_i(g_i(a)) + \\
+ \sum_{(i_1, i_2) \in \text{Syn}^+} \text{syn}^+_{i_1,i_2}(g_{i_1}(a), g_{i_2}(a)) - \sum_{(i_1, i_2) \in \text{Syn}^-} \text{syn}^-_{i_1,i_2}(g_{i_1}(a), g_{i_2}(a))
\]

„bonus”

„malus”

\(\text{Syn}^+\) is the set of pairs of criteria in a positive interaction

\(\text{Syn}^-\) is the set of pairs of criteria in a negative interaction

\(\text{Syn}^+ \cap \text{Syn}^- = \emptyset\)

\(\text{syn}^+_{i_1,i_2}(\cdot,\cdot), \text{syn}^-_{i_1,i_2}(\cdot,\cdot)\) are non-decreasing functions in the two arguments
**Illustrative example: value function** $U^{int}$

Preferences of the dean:  $S2 \succ S1$ and $S3 \succ S4$

violate the principle of preferential independence
Illustrative example: value function $U^{int}$

<table>
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<tr>
<th></th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
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<td>0.32</td>
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</tr>
<tr>
<td>Bad</td>
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<td>0.01</td>
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</tbody>
</table>

Non-interacting part of $U^{int}$
Assuming Mathematics & Literature in positive interaction

„Bonus” component of $U^{int}$

$\text{syn}^+_{\text{math, lit}}(\cdot, \cdot)$
Illustrative example

<table>
<thead>
<tr>
<th>Students</th>
<th>math</th>
<th>phys</th>
<th>lit</th>
<th>( \text{syn}_{\text{math,lit}}^+(\cdot) )</th>
<th>Total score</th>
</tr>
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<td>Bad</td>
<td>Good,Bad</td>
<td>0.31</td>
</tr>
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<td>0.03</td>
<td>0.28</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S2</td>
<td>Good</td>
<td>Bad</td>
<td>Medium</td>
<td>Good,Medium</td>
<td>0.32</td>
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<td>0.03</td>
<td>0</td>
<td>0.26</td>
<td>0.03</td>
<td></td>
</tr>
<tr>
<td>S3</td>
<td>Medium</td>
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<td>Bad</td>
<td>Medium,Bad</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>0.02</td>
<td>0.28</td>
<td>0</td>
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</tr>
<tr>
<td>S4</td>
<td>Medium</td>
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<td></td>
<td>0.02</td>
<td>0</td>
<td>0.26</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

\[ S2 \succ S1 \text{ and } S3 \succ S4 \]

\[ (0.32 > 0.31 \text{ and } 0.30 > 0.28) \]
Advantages of the new value function when used within ROR

- **With respect to numerical encoding:** we do not need an a priori expression of all the evaluations on a common numerical scale; i.e. the marginal value functions are not supposed to be known.

- **Problems with respect to non-additive weights** (capacity): we do not need non-additive weights; the value function is computed using ROR and even does not need to be shown to the Decision Maker.

- **Problems with interpretation of the interaction components:** interpretation of „bonus” and „malus” with respect to the sum of marginal values is cognitively simple.
Interactions between two criteria

- Positive interactions (e.g., maximum speed & price of a car):
  \[ u_{i_1,i_2}(x_{i_1}, x_{i_2}) > u_{i_1}(x_{i_1}) + u_{i_2}(x_{i_2}) \]

- Negative interactions (e.g., maximum speed & acceleration of a car):
  \[ u_{i_1,i_2}(x_{i_1}, x_{i_2}) < u_{i_1}(x_{i_1}) + u_{i_2}(x_{i_2}) \]

- \( I^{(2)} = \{\{i_1, i_2\}: i_1,i_2 \in I\}, \ x_{i_1} \in X_{i_1}, x_{i_2} \in X_{i_2} \)

- \( Syn^+ \subseteq I^{(2)} \), set of pairs of criteria for which there is a positive synergy

- \( Syn^- \subseteq I^{(2)} \), set of pairs of criteria for which there is a negative synergy

- Synergy strength is measured by functions \( syn^+_{i_1,i_2} : X_{i_1} \times X_{i_2} \rightarrow [0,1], \ syn^-_{i_1,i_2} : X_{i_1} \times X_{i_2} \rightarrow [0,1] \) not decreasing in both arguments, called “bonus” and “malus”
Interactions between two criteria

- **Positive interactions** (e.g., maximum speed & price of a car):
  \[
  u_{i_1,i_2}(x_{i_1}, x_{i_2}) = u_{i_1}(x_{i_1}) + u_{i_2}(x_{i_2}) + \text{syn}^+_{i_1,i_2}(x_{i_1}, x_{i_2})
  \]

- **Negative interactions** (e.g., maximum speed & acceleration of a car):
  \[
  u_{i_1,i_2}(x_{i_1}, x_{i_2}) = u_{i_1}(x_{i_1}) + u_{i_2}(x_{i_2}) - \text{syn}^-_{i_1,i_2}(x_{i_1}, x_{i_2})
  \]

- \( I^{(2)} = \{ \{i_1, i_2\} : i_1,i_2 \in I \}, \ x_{i_1} \in X_{i_1}, x_{i_2} \in X_{i_2} \)

- \( \text{Syn}^+ \subseteq I^2 \), set of pairs of criteria for which there is a positive synergy

- \( \text{Syn}^- \subseteq I^2 \), set of pairs of criteria for which there is a negative synergy

- Synergy strength is measured by functions \( \text{syn}^+_{i_1,i_2} : X_{i_1} \times X_{i_2} \rightarrow [0,1] \), \( \text{syn}^-_{i_1,i_2} : X_{i_1} \times X_{i_2} \rightarrow [0,1] \) not decreasing in both arguments, called „bonus” and „malus”

\[
\text{syn}^+_{i_1,i_2}(x_{i_1}, x_{i_2}) \times \text{syn}^-_{i_1,i_2}(x_{i_1}, x_{i_2}) = 0, \ \forall(i_1,i_2) \in I^2, \ \forall(x_{i_1},x_{i_2}) \in X_{i_1} \times X_{i_2}
\]
UTAGMS–INT

- We consider a value function of the type

\[
U^{int}(a) = \sum_{i=1}^{n} u_i(g_i(a)) + \\
+ \sum_{(i_1,i_2) \in \text{Syn}^+} \text{syn}_{i_1,i_2}^+ (g_{i_1}(a), g_{i_2}(a)) - \sum_{(i_1,i_2) \in \text{Syn}^-} \text{syn}_{i_1,i_2}^- (g_{i_1}(a), g_{i_2}(a))
\]

- Preference information elicited by the DM is the same as in the GRIP method:
  - pairwise comparisons of some reference alternatives \( a',b' \in A' \) (partial preorder \( \succeq \) on \( A' \) – set of reference alternatives)
  - ordinal intensity of preference for quadruples of reference alternatives \( a',b',c',d' \in A' \), comprehensively or on specific criteria (partial preorder \( \succeq^* \) or \( \succeq_i^* \) on \( A' \times A' \))
Three options to consider interaction

- Considering a value function of the type

\[
U^{\text{int}}(a) = \sum_{i=1}^{n} u_i(g_i(a)) + \\
+ \sum_{(i_1,i_2) \in \text{Syn}^+} \text{syn}_{i_1,i_2}(g_{i_1}(a), g_{i_2}(a)) - \sum_{(i_1,i_2) \in \text{Syn}^-} \text{syn}_{i_1,i_2}(g_{i_1}(a), g_{i_2}(a))
\]

- a) bonus and malus are not mutually exclusive, so that positive and negative synergies interplay,

- β) bonus and malus are mutually exclusive,

- γ) only one of the two synergies is considered, either the positive, or the negative
Compatible value functions

\( U^{\text{int}}(a') \geq U^{\text{int}}(b') + \varepsilon \) if \( a' \succ b' \)

\( U^{\text{int}}(a') = U^{\text{int}}(b') \) if \( a' \sim b' \)

\( U^{\text{int}}(a') - U^{\text{int}}(b') \geq U^{\text{int}}(c') - U^{\text{int}}(d') \) if \( (a', b') \succsim^* (c', d') \)

\( u_i(g_i(a')) - u_i(g_i(b')) \geq u_i(g_i(c')) - u_i(g_i(d')) \),

if \( (a', b') \succsim^*_i (c', d'), i \in I \)

\( U^{\text{int}}(a') \geq U^{\text{int}}(b') \) if \( g_i(a') \geq g_i(b') \) for \( i = 1, \ldots, n \), and for all \( a', b' \in A' \)

\( u_i(g_i(a'_{\tau_i(j)})) - u_i(g_i(a'_{\tau_i(j-1)})) \geq 0, i = 1, \ldots, n, j = 2, \ldots, m' \) (29)

\( u_i(g_i(a'_{\tau_i(j)})) - u_i(g_i(a'_{\tau_i(j-1)})) = 0, \) if \( g_i(a'_{\tau_i(j)})) = g_i(a'_{\tau_i(j-1)}), i = 1, \ldots, n, j = 2, \ldots, m' \) (30)

\( u_i(g_i(a'_{\tau_i(1)})) \geq 0, i = 1, \ldots, n \) monotonicity of the non-interacting part of \( U^{\text{int}} \) (31)

\( u_i(x_i^*) = 0, i = 1, \ldots, n \) (32)

\( U^{\text{int}}(x_1^*, \ldots, x_n^*) = 1 \) (33)

\( U^{\text{int}}(a') \geq 0, \) for all \( a' \in A' \) normalization and non-negativity of \( U^{\text{int}} \) (34)
Compatible value functions (cont.)

\[
\text{syn}_{i_1,i_2}^+(g_{i_1}(a'), g_{i_2}(a')) \geq \text{syn}_{i_1,i_2}^+(g_{i_1}(b'), g_{i_2}(b')),
\]
\[
(i_1, i_2) \in I^2, \ i_1 > i_2, \text{ if } g_{i_1}(a') \geq g_{i_1}(b') \text{ and } g_{i_2}(a') \geq g_{i_2}(b'), \text{ for all } a', b' \in A'
\]

monotonicity of the bonus and malus functions

\[
\text{syn}_{i_1,i_2}^+(g_{i_1}(a'), g_{i_2}(a')) = \text{syn}_{i_1,i_2}^+(g_{i_1}(b'), g_{i_2}(b')),
\]
\[
(i_1, i_2) \in I^2, \ i_1 > i_2, \text{ if } g_{i_1}(a') = g_{i_1}(b') \text{ and } g_{i_2}(a') = g_{i_2}(b'), \text{ for all } a', b' \in A'
\]

\[
\text{syn}_{i_1,i_2}^+(g_{i_1}(a'), g_{i_2}(a')) \geq \text{syn}_{i_1,i_2}^+(g_{i_1}(b'), g_{i_2}(b')),
\]
\[
(i_1, i_2) \in I^2, \ i_1 > i_2, \text{ if } g_{i_1}(a') \geq g_{i_1}(b') \text{ and } g_{i_2}(a') \geq g_{i_2}(b'), \text{ for all } a', b' \in A'
\]

\[
\text{syn}_{i_1,i_2}^+(g_{i_1}(a'), g_{i_2}(a')) = \text{syn}_{i_1,i_2}^+(g_{i_1}(b'), g_{i_2}(b')),
\]
\[
(i_1, i_2) \in I^2, \ i_1 > i_2, \text{ if } g_{i_1}(a') = g_{i_1}(b') \text{ and } g_{i_2}(a') = g_{i_2}(b'), \text{ for all } a', b' \in A'
\]

\[
\text{syn}_{i_1,i_2}^+(x_{i_1*}, x_{i_2*}) = 0, \ \text{syn}_{i_1,i_2}^-(x_{i_1*}, x_{i_2*}) = 0, \ (i_1, i_2) \in I \times I, \ i_1 > i_2
\]

\[
u_{i_1}(g_{i_1}(a')) + u_{i_2}(g_{i_2}(a')) - \text{syn}_{i_1,i_2}^-(g_{i_1}(a'), g_{i_2}(a')) \geq u_{i_1}(g_{i_1}(b')) + u_{i_2}(g_{i_2}(b'))
\]
\[-\text{syn}_{i_1,i_2}^-(g_{i_1}(b'), g_{i_2}(b')), (i_1, i_2) \in I^2, \ i_1 > i_2, \forall a', b' \in A', \text{ if } g_{i_1}(a') \geq g_{i_1}(b'), g_{i_2}(a') \geq g_{i_2}(b')
\]

\[
x_{i*} \leq g_i(a_{i}(1)) \leq g_i(a_{i}(2)) \leq \cdots \leq g_i(a_{i}(m'-1)) \leq g_i(a_{i}(m')) \leq x_{i*}
\]
Eliciting a minimal set of pairs of interacting criteria

Let us introduce two binary variables:

\[
\begin{align*}
\delta_{i_1i_2}^+ &= 1 \text{ iff } (i_1, i_2) \text{ are positively interacting,} \\
\delta_{i_1i_2}^- &= 0 \text{ otherwise,} \\
\delta_{i_1i_2}^- &= 1 \text{ iff } (i_1, i_2) \text{ are negatively interacting,} \\
\delta_{i_1i_2}^- &= 0 \text{ otherwise.}
\end{align*}
\]

Any pair of criteria can be either in positive or in negative interaction:

\[
\delta_{i_1i_2}^+ + \delta_{i_1i_2}^- \leq 1, \text{ for all } (i_1, i_2) \in I \times I, \ i_1 > i_2
\]
In order to find a **minimal set of pairs of criteria** with either **positive** or **negative** interactions, one has to solve the following program \((P)\)

\[
\text{Minimize } z = \sum_{i_1 < i_2, i_1, i_2 \in I} \left( \delta_{i_1i_2}^+ + \delta_{i_1i_2}^- \right)
\]

subject to

\[
\delta_{i_1i_2}^+ + \delta_{i_1i_2}^- \leq 1, \text{ for all } (i_1, i_2) \in I \times I, i_1 > i_2 \\
\text{syn}_{i_1,i_2}^+(x_{i_1}^*, x_{i_2}^*) \leq \delta_{i_1i_2}^+, \text{ for all } (i_1, i_2) \in I \times I, i_1 > i_2 \\
\text{syn}_{i_1,i_2}^-(x_{i_1}^*, x_{i_2}^*) \leq \delta_{i_1i_2}^-, \text{ for all } (i_1, i_2) \in I \times I, i_1 > i_2 \\
\delta_{i_1i_2}^+, \delta_{i_1i_2}^- \in \{0, 1\}, \text{ for all } (i_1, i_2) \in I \times I, i_1 > i_2,
\]

\((P)\) yields a value function \(U^{int}\) involving a minimal set of pairs of interacting criteria.

\[
\text{Syn}^+ = \{(i_1, i_2) \in I \times I, i_1 > i_2 : \delta_{i_1i_2}^+ = 1\} \text{ and } \text{Syn}^- = \{(i_1, i_2) \in I \times I, i_1 > i_2 : \delta_{i_1i_2}^- = 1\}
\]
Eliciting a minimal set of pairs of interacting criteria

These two sets are then presented to the DM for validation. The DM can react in one of the following ways:

\( \alpha \) accept \( Syn^+ \) and \( Syn^- \) as relevant sets of interacting pairs of criteria,

\( \beta \) refuse a pair \( (i_1, i_2) \) in \( Syn^+ \) and \( Syn^- \), i.e. deny the interaction between criteria \( i_1 \) and \( i_2 \),

\( \gamma \) add a pair \( (i_1, i_2) \) to \( Syn^+ \) and \( Syn^- \), i.e. impose interaction between criteria \( i_1 \) and \( i_2 \).

In case \( \beta \) or \( \gamma \), one has to solve again program \( (P) \) with the following additional constraints:

- in case of \( \beta \): \( \delta_{i_1 i_2}^+ = \delta_{i_1 i_2}^- = 0 \),

- in case of \( \gamma \): \( \delta_{i_1 i_2}^+ = 1 \), in case of positive interaction, and \( \delta_{i_1 i_2}^- = 1 \), in case of negative interaction.
Computing necessary and possible preference relations

- To confirm relation \( a \succeq^N b \) it is enough to check if \( \varepsilon^* \leq 0 \) or the set of constraints is infeasible, where \( \varepsilon^* \):

Maximize: \( \varepsilon \)

subject to

\[
U^{\text{int}}(b) \geq U^{\text{int}}(a) + \varepsilon, \quad \text{can } b \text{ be preferred to } a \text{ for some compatible } U^{\text{int}}? \quad (48)
\]

constraints (28), (33), (34)

\[
U^{\text{int}}(a') \geq U^{\text{int}}(b') \text{ if } g_i(a') \geq g_i(b'), \text{ for } i = 1, \ldots, n, \text{ and for all } a', b' \in A' \cup \{a, b\} \quad (49)
\]

\[
u_i(g_i(a'_{\pi_i(j)})) - u_i(g_i(a'_{\pi_i(j-1)})) \geq 0, \text{ for } i = 1, \ldots, n, \text{ and } j = 2, \ldots, m' + \omega \quad (50)
\]

\[
u_i(g_i(a'_{\pi_i(j)})) - u_i(g_i(a'_{\pi_i(j-1)})) = 0, \text{ if } g_i(a'_{\pi_i(j)})) = g_i(a'_{\pi_i(j-1)}), \text{ for } i = 1, \ldots, n \quad (51)
\]

\[
u_i(g_i(a'_{\pi_i(1)})) \geq 0, \text{ for all } a' \in A' \cup \{a, b\} \quad (52)
\]

monotonicity of \( U^{\text{int}} \), of the non-interacting part of \( U^{\text{int}} \) and non-negativity of \( U^{\text{int}} \) for \( A' \) augmented by \( \{a, b\} \)
Computing necessary and possible preference relations (cont.)

\[ \text{syn}_{i_1, i_2}^+(g_{i_1}(a'), g_{i_2}(a')) \geq \text{syn}_{i_1, i_2}^+(g_{i_1}(b'), g_{i_2}(b')) \], for all \((i_1, i_2) \in \text{Syn}^+\),

if \(g_{i_1}(a') \geq g_{i_1}(b')\) and \(g_{i_2}(a') \geq g_{i_2}(b')\), for all \(a', b' \in A' \cup \{a, b\}\)

\[ \text{syn}_{i_1, i_2}^+(g_{i_1}(a'), g_{i_2}(a')) = \text{syn}_{i_1, i_2}^+(g_{i_1}(b'), g_{i_2}(b')) \], for all \((i_1, i_2) \in \text{Syn}^+\),

if \(g_{i_1}(a') = g_{i_1}(b')\) and \(g_{i_2}(a') = g_{i_2}(b')\), for all \(a', b' \in A' \cup \{a, b\}\)

\[ \text{syn}_{i_1, i_2}^-(g_{i_1}(a'), g_{i_2}(a')) \geq \text{syn}_{i_1, i_2}^-(g_{i_1}(b'), g_{i_2}(b')) \], for all \((i_1, i_2) \in \text{Syn}^-\),

if \(g_{i_1}(a') \geq g_{i_1}(b')\) and \(g_{i_2}(a') \geq g_{i_2}(b')\), for all \(a', b' \in A' \cup \{a, b\}\)

\[ \text{syn}_{i_1, i_2}^-(g_{i_1}(a'), g_{i_2}(a')) = \text{syn}_{i_1, i_2}^-(g_{i_1}(b'), g_{i_2}(b')) \], for all \((i_1, i_2) \in \text{Syn}^-\),

if \(g_{i_1}(a') = g_{i_1}(b')\) and \(g_{i_2}(a') = g_{i_2}(b')\), for all \(a', b' \in A' \cup \{a, b\}\)

\[ \text{syn}_{i_1, i_2}^+(x_{i_1*}, x_{i_2*}) = 0, \ \text{syn}_{i_1, i_2}^-(x_{i_1*}, x_{i_2*}) = 0 \], for all \((i_1, i_2) \in \text{Syn}^+ \cup \text{Syn}^-\)

\[ u_{i_1}(g_{i_1}(a')) + u_{i_2}(g_{i_2}(a')) - \text{syn}_{i_1, i_2}^-(g_{i_1}(a'), g_{i_2}(a')) \geq 0 \], for all \((i_1, i_2) \in \text{Syn}^-\), \(\forall a', b' \in A' \cup \{a, b\}\),

if \(g_{i_1}(a') \geq g_{i_1}(b')\), \(g_{i_2}(a') \geq g_{i_2}(b')\)

\[ U^{\text{int}}(a) = \sum_{i=1}^n u_i(a) + \sum_{(i_1, i_2) \in \text{Syn}^+} \text{syn}_{i_1, i_2}^+(g_{i_1}(a), g_{i_2}(a)) - \sum_{(i_1, i_2) \in \text{Syn}^-} \text{syn}_{i_1, i_2}^-(g_{i_1}(a), g_{i_2}(a)) \]
Computing necessary and possible preference relations

- To confirm relation $a \succeq^P b$ one has to check if the set of constraints is feasible and $\varepsilon^* > 0$ in the previous problem, where constraint (48) is replaced by $U_{int}(a) \geq U_{int}(b) + \varepsilon$

- Using $\succ^N$ and $\succeq^P$ one can compute:
  - necessary ranking (partial preorder in $A$)
  - possible ranking (strongly complete and negatively transitive relation in $A$)

- If a score is needed to assign to the different alternatives, one can calculate a „representative” value function among all the compatible ones.
Example of application of $UTA^{GMS-INT}$
### Illustrative example

#### Performance matrix

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics <em>(math)</em></th>
<th>Physics <em>(phys)</em></th>
<th>Literature <em>(lit)</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>Good</td>
<td>Medium</td>
<td>Bad</td>
</tr>
<tr>
<td>S2</td>
<td>Good</td>
<td>Bad</td>
<td>Medium</td>
</tr>
<tr>
<td>S3</td>
<td>Medium</td>
<td>Medium</td>
<td>Bad</td>
</tr>
<tr>
<td>S4</td>
<td>Medium</td>
<td>Bad</td>
<td>Medium</td>
</tr>
<tr>
<td>S5</td>
<td>Medium</td>
<td>Medium</td>
<td>Medium</td>
</tr>
<tr>
<td>S6</td>
<td>Medium</td>
<td>Good</td>
<td>Medium</td>
</tr>
<tr>
<td>S7</td>
<td>Good</td>
<td>Good</td>
<td>Bad</td>
</tr>
</tbody>
</table>
Preference information elicited by the DM

- Pairwise comparisons of (reference) students
  - $S_2 \succ S_1$
  - $S_3 \succ S_4$
  - $S_5 \succ S_1$

- Overall intensity of preferences
  - $(S_3, S_4) \succ^* (S_2, S_1)$

- Intensity of preference relative to single criteria
  - $(\text{Medium}, \text{Bad}) \succ^*_i (\text{Good}, \text{Medium}), \quad i=\text{math, lit}$
  - $(\text{Good}, \text{Medium}) \succ^*_i (\text{Medium}, \text{Bad}), \quad i=\text{phys}$

- $A' = \{S_1, S_2, S_3, S_4, S_5\}$
Preference information given by the DM

- Dominance relation and pairwise comparisons of reference students
First, we solve program \((P)\), and obtain at the optimum
\[
\delta^+_{i_1,i_2} = \delta^-_{i_1,i_2} = 0, \text{ for all } (i_1,i_2) \in I \times I, \text{ except } \delta^-_{\text{math,lit}} = 1
\]

This means there exists \(U^{\text{int}}\) compatible with the dean’s preferences, involving a negative interaction between Mathematics and Literature.

Suppose that the dean is not willing to consider a negative interaction between these two criteria \(\rightarrow \delta^-_{\text{math,lit}} = 0\) enters \((P) \rightarrow (P')\)

Solving \((P')\), we get
\[
\delta^+_{i_1,i_2} = \delta^-_{i_1,i_2} = 0, \text{ for all } (i_1,i_2) \in I \times I, \text{ except } \delta^-_{\text{math,phys}} = 1
\]

Suppose that the dean is not willing to consider a negative interaction between these two criteria \(\rightarrow \delta^-_{\text{math,phys}} = 0\) enters \((P') \rightarrow (P'')\)

Solving \((P'')\), we get
\[
\delta^+_{i_1,i_2} = \delta^-_{i_1,i_2} = 0, \text{ for all } (i_1,i_2) \in I \times I, \text{ except } \delta^+_{\text{math,lit}} = 1
\]

Suppose the dean accepts to consider this pos. interaction \((\text{math, lit})\)
### Necessary preference relation $\succeq^N$

<table>
<thead>
<tr>
<th>Students $(S_i, S_j)$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
<th>$S_5$</th>
<th>$S_6$</th>
<th>$S_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
</tr>
<tr>
<td>$S_3$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
</tr>
<tr>
<td>$S_4$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
</tr>
<tr>
<td>$S_5$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
</tr>
<tr>
<td>$S_6$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
</tr>
<tr>
<td>$S_7$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
<td>$\succeq^N$</td>
</tr>
</tbody>
</table>
Using this information, the dean is able to identify the best student, even if the value function $U^{int}$ is not unique.
Representative value function $U^{int}$

- In order to give a score to each student, we compute a representative value function
- Non-interacting part of the representative $U^{int}$

<table>
<thead>
<tr>
<th></th>
<th>$u_{math}$</th>
<th>$u_{phys}$</th>
<th>$u_{lit}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good</td>
<td>0.14</td>
<td>0.52</td>
<td>0.14</td>
</tr>
<tr>
<td>Medium</td>
<td>0.10</td>
<td>0.24</td>
<td>0.10</td>
</tr>
<tr>
<td>Bad</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Representative value function $U_{int}$

- „Bonus” component $syn_{math,lit}^{+}$ of the representative $U_{int}$

<table>
<thead>
<tr>
<th>Mathematics / Literature</th>
<th>Good</th>
<th>Medium</th>
<th>Bad</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good</td>
<td>0.19</td>
<td>0.19</td>
<td>0</td>
</tr>
<tr>
<td>Medium</td>
<td></td>
<td>0.05</td>
<td>0</td>
</tr>
<tr>
<td>Bad</td>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>
Representative value function $U^{int}$ – scores and ranking

- „Bonus” component $syn^+_{math, lit}$ of the representative $U^{int}$ and scores of students

<table>
<thead>
<tr>
<th>Students</th>
<th>$u_{math}$</th>
<th>$u_{phys}$</th>
<th>$u_{lit}$</th>
<th>$syn^+_{math, lit}$</th>
<th>$U^{int}(S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>0.14 (Good)</td>
<td>0.24 (Medium)</td>
<td>0 (Bad)</td>
<td>0</td>
<td>0.38</td>
</tr>
<tr>
<td>$S_2$</td>
<td>0.14 (Good)</td>
<td>0 (Bad)</td>
<td>0.10 (Medium)</td>
<td>0.19</td>
<td>0.43</td>
</tr>
<tr>
<td>$S_3$</td>
<td>0.10 (Medium)</td>
<td>0.24 (Medium)</td>
<td>0 (Bad)</td>
<td>0</td>
<td>0.34</td>
</tr>
<tr>
<td>$S_4$</td>
<td>0.10 (Medium)</td>
<td>0 (Bad)</td>
<td>0.10 (Medium)</td>
<td>0.05</td>
<td>0.25</td>
</tr>
<tr>
<td>$S_5$</td>
<td>0.10 (Medium)</td>
<td>0.24 (Medium)</td>
<td>0.10 (Medium)</td>
<td>0.05</td>
<td>0.49</td>
</tr>
<tr>
<td>$S_6$</td>
<td>0.10 (Medium)</td>
<td>0.52 (Good)</td>
<td>0.10 (Medium)</td>
<td>0.05</td>
<td>0.77</td>
</tr>
<tr>
<td>$S_7$</td>
<td>0.14 (Good)</td>
<td>0.52 (Good)</td>
<td>0 (Bad)</td>
<td>0</td>
<td>0.67</td>
</tr>
</tbody>
</table>

- $S_6 > S_7 > S_5 > S_2 > S_1 > S_3 > S_4$
SMAA and Robust Ordinal Regression
SMAA methods (Lahdelma et al. 1998)

Basic assumptions:
- Imprecision or lack of data (weights of criteria and evaluations of alternatives over criteria)
- Density function $f_W(w)$ over the weight space $W$,
- Density function $f_\chi(\xi)$ over evaluation space $\chi \subseteq \mathbb{R}^{m \times n}$,

Computations for each alternative of:
- Rank acceptability index

$$b_j^r = \int_{\xi \in \chi} f_\chi(\xi) \int_{w \in W_j^r(\xi)} f_W(w) \, dw \, d\xi$$

- Pairwise winning index:

$$p(a_h, a_k) = \int_{w \in W} f_W(w) \int_{\xi \in \chi : u(\xi_h, w) \geq u(\xi_k, w)} f_\chi(\xi) \, d\xi \, dw$$

where $W_j^r(\xi) = \{w \in W : \text{rank}(j, \xi, w) = r\}$.
ROR and SMAA under a unified framework

Very often $a \succsim^P b$ and $b \succsim^P a$. Therefore it is interesting to know which is the frequency of the preference of $a$ over $b$ and vice versa the frequency of the preference of $b$ over $a$ or the frequency with which an alternative fills the $k$-th position in a ranking and so on.

The link between ROR and SMAA has been already investigated in M. Kadziński, and T. Tervonen (2013a, 2013b).
Indirect preference information

- indirect elicitation based on Ordinal Regression, (E. Jacquet-Lagreze and Y. Siskos 1982, 2001; Y. Siskos and E. Grigoroudis 2010);

- robust indirect elicitation based on Robust Ordinal Regression (ROR) (S. Corrente, S. Greco, M. Kadziński, R. Słowiński 2013, 2014; S. Greco, V. Mousseau and R. Słowiński 2008);

- stochastic indirect elicitation based on Stochastic Ordinal Regression (SOR) (M. Kadziński and T. Tervonen 2013a, 2013b).
SOR: ROR and SMAA under a unified framework

Very often $a \succeq_P b$ and $b \succeq_P a$. Therefore it is interesting to know which is the frequency of the preference of $a$ over $b$ and viceversa the frequency of the preference of $b$ over $a$ or the frequency with which an alternative fills the $k$-th position in a ranking and so on.

The link between ROR and SMAA has been already investigated in M. Kadziński and T. Tervonen 2013a,2013b.
SMAA-Choquet
(Angilella, Corrente, Greco 2012, 2014)
On the DM’s preference constraints, a sampling of compatible preference parameters (Möbius measures) is obtained by a Hit-and-Run algorithm (Smith, 1984) that is outperformed for a maximum number of iterations.

The Hit and Run algorithm

At each iteration generates a candidate point (the preference parameters) that is uniformly distributed along a randomly chosen direction within the feasible region defined by the DM’s preference information.
Indices of SMAA-Choquet

Computation of:

- the rank acceptability index of every alternative by considering the different compatible preference parameters (the Möbius measures obtained after each iteration) giving to alternative $a_j \in A$ the rank $r$ on the basis of a utility function expressed in terms of a Choquet integral.

- the Möbius measures corresponding to the capacities for which the Choquet integral ranks every alternative $a_j$ as the best.
# Evaluation matrix

<table>
<thead>
<tr>
<th>Criteria</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
<th>$a_7$</th>
<th>$a_8$</th>
<th>$a_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_1$</td>
<td>8</td>
<td>3</td>
<td>10</td>
<td>5</td>
<td>8</td>
<td>5</td>
<td>8</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>$g_2$</td>
<td>6</td>
<td>1</td>
<td>9</td>
<td>9</td>
<td>0</td>
<td>9</td>
<td>10</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>$g_3$</td>
<td>7</td>
<td>10</td>
<td>0</td>
<td>2</td>
<td>8</td>
<td>4</td>
<td>5</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>$g_4$</td>
<td>5</td>
<td>10</td>
<td>5</td>
<td>9</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>
DM’s preference constraints

\[
\begin{align*}
\varphi(\{g_1\}) &> \varphi(\{g_2\}) \\
\varphi(\{g_2\}) &> \varphi(\{g_3\}) \\
\varphi(\{g_3\}) &> \varphi(\{g_4\}) \\
\varphi(\{g_1, g_2\}) &> 0 \\
\varphi(\{g_2, g_3\}) &> 0 \\
\varphi(\{g_2, g_4\}) &< 0 \\
\end{align*}
\]

\[
a(\emptyset) = 0, \sum_{g_i \in G} a(\{g_i\}) + \sum_{\{g_i, g_j\} \subseteq G} a(\{g_i, g_j\}) = 1
\]

\[
a(\{g_i\}) \geq 0, \forall g_i \in G \\
a(\{g_i\}) + \sum_{g_j \in T} a(\{g_i, g_j\}) \geq 0, \forall g_i \in G \text{ and } \forall T \subseteq G \setminus \{g_i\}
\]

The *Hit and Run Sampling* has been outperformed with MaxIter = 100,000
## Rank acceptabilities ($b_i$) in percentages

<table>
<thead>
<tr>
<th>Alt</th>
<th>$b_1^j$</th>
<th>$b_2^j$</th>
<th>$b_3^j$</th>
<th>$b_4^j$</th>
<th>$b_5^j$</th>
<th>$b_6^j$</th>
<th>$b_7^j$</th>
<th>$b_8^j$</th>
<th>$b_9^j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0.03</td>
<td>51.32</td>
<td>43.64</td>
<td>4.35</td>
<td>0.66</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0.00</td>
<td>0.29</td>
<td>0.60</td>
<td>1.37</td>
<td>5.00</td>
<td>5.78</td>
<td>34.51</td>
<td>50.04</td>
<td>2.41</td>
</tr>
<tr>
<td>$a_3$</td>
<td>13.28</td>
<td>34.08</td>
<td>20.51</td>
<td>12.25</td>
<td>9.62</td>
<td>5.49</td>
<td>3.94</td>
<td>0.83</td>
<td>0.01</td>
</tr>
<tr>
<td>$a_4$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.16</td>
<td>2.58</td>
<td>19.16</td>
<td>56.33</td>
<td>19.92</td>
<td>1.85</td>
<td>0.00</td>
</tr>
<tr>
<td>$a_5$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.48</td>
<td>4.60</td>
<td>5.29</td>
<td>7.94</td>
<td>36.55</td>
<td>44.50</td>
<td>0.65</td>
</tr>
<tr>
<td>$a_6$</td>
<td>0.00</td>
<td>0.00</td>
<td>5.10</td>
<td>20.90</td>
<td>48.51</td>
<td>21.10</td>
<td>4.37</td>
<td>0.03</td>
<td>0.00</td>
</tr>
<tr>
<td>$a_7$</td>
<td>86.70</td>
<td>13.15</td>
<td>0.15</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$a_8$</td>
<td>0.00</td>
<td>1.15</td>
<td>29.37</td>
<td>53.96</td>
<td>11.76</td>
<td>3.37</td>
<td>0.39</td>
<td>0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>$a_9$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.32</td>
<td>2.75</td>
<td>96.94</td>
</tr>
</tbody>
</table>
First rank acceptability \((b_1)\) and central weights

<table>
<thead>
<tr>
<th>Alt</th>
<th>(a_1)</th>
<th>(a_3)</th>
<th>(a_7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b_1)</td>
<td>0.03</td>
<td>13.28</td>
<td>86.70</td>
</tr>
<tr>
<td>(a({1}))</td>
<td>0.32799</td>
<td>0.4833</td>
<td>0.23032</td>
</tr>
<tr>
<td>(a({2}))</td>
<td>0.1308</td>
<td>0.12818</td>
<td>0.15372</td>
</tr>
<tr>
<td>(a({3}))</td>
<td>0.055755</td>
<td>0.19372</td>
<td>0.14022</td>
</tr>
<tr>
<td>(a({4}))</td>
<td>0.1808</td>
<td>0.16042</td>
<td>0.1788</td>
</tr>
<tr>
<td>(a({1, 2}))</td>
<td>0.045792</td>
<td>0.21504</td>
<td>0.2164</td>
</tr>
<tr>
<td>(a({1, 3}))</td>
<td>0.1751</td>
<td>-0.12405</td>
<td>0.053909</td>
</tr>
<tr>
<td>(a({1, 4}))</td>
<td>-0.10253</td>
<td>-0.044048</td>
<td>0.0067834</td>
</tr>
<tr>
<td>(a({2, 3}))</td>
<td>0.25613</td>
<td>0.053126</td>
<td>0.092624</td>
</tr>
<tr>
<td>(a({2, 4}))</td>
<td>-0.040672</td>
<td>-0.049708</td>
<td>-0.063965</td>
</tr>
<tr>
<td>(a({3, 4}))</td>
<td>-0.029203</td>
<td>-0.015979</td>
<td>-0.0088056</td>
</tr>
</tbody>
</table>
Subjective Stochastic Ordinal Regression (Corrente, Greco, Kadzinski and Slowinski 2015)
Which probability to use in the space of compatible value functions?

We induce a probability distribution that permits to represent uncertain DM’s preferences.

Differently from SOR, the probability distribution is not given in a exogeneous way.
Uncertain preference

In addition to the certain preference information, the DM could also give some uncertain preference information as the following:

"The preference of a over b" is at least as credible as "the preference of b over a".

All value functions have to be compatible with the certain preference information provided by the DM. At the same time, we propose to induce a probability distribution on the set of compatible value functions reflecting the uncertain preference information.
The procedure

- Sampling a number $s_V$ of value functions satisfying monotonicity and normalization constraints and being compatible with the certain preference information provided by the DM,

- Induce a probability distribution $\{ w_t(U_t) \in [0, 1] : \sum_{t=1}^{s_V} w_t(U_t) = 1 \}$ on the set composed by these value functions on the base of the uncertain preference relation.

Denoting by $a \succ_L b$ and $(a, b) \succ_{L,*} (c, d)$ the uncertain preference relations between pairs and quadruples of alternatives, respectively, we shall translate these preferences as follow:

$$a \succ_L b \iff \sum_{t: U_t(a) > U_t(b)} w(U_t) \geq \sum_{t: U_t(b) > U_t(a)} w(U_t)$$

$$ (a, b) \succ_{L,*} (c, d) \iff \sum_{t: U_t(a) > U_t(b)} w(U_t) \geq \sum_{t: U_t(c) > U_t(d)} w(U_t) $$
Inducing a probability distribution

Solving the following LP problem . . .

\[ \varepsilon_L = \max \varepsilon, \quad \text{subject to} \]

\[ \sum_{t: U_t(a) > U_t(b)} w(U_t) \geq \sum_{t: U_t(b) > U_t(a)} w(U_t) + \varepsilon, \quad \text{if } a \succeq_L b, \]

\[ \sum_{t: U_t(a) > U_t(b)} w(U_t) \geq \sum_{t: U_t(c) > U_t(d)} w(U_t) + \varepsilon, \quad \text{if } (a, b) \succeq_{L,*} (c, d), \]

\[ \sum_{t=1}^{s_V} w(U_t) = 1 \]

\[ w(U_t) \geq 0, \quad t = 1, \ldots, s_V \]

if \( E_L \) is feasible and \( \varepsilon_L > 0 \), then there exist at least one probability distribution on the set of value functions compatible with the uncertain preference provided by the DM.
Three possibilities

1. Considering the most discriminant probability distribution obtained by solving the previous LP problem;

2. Sampling a certain number of probability distributions compatible with the uncertain preference information provided by the DM by using the Hit-And-Run (HAR) method (R.L. Smith 1984) and computing their barycenter \( w^* \) called *representative probability distribution*;

3. Applying ROR on the space of probability distributions compatible with the uncertain preference information provided by the DM.
In the first two cases . . .

We can define the indices typical of SMAA:

\[ b_k^r = \sum_{t: \text{rank}(k, U_t) = r} w(U_t), \]

\[ p(a_h, a_k) = \sum_{t: U_t(a_h) > U_t(a_k)} w(U_t). \]

Based on the pairwise winning index, the following preference relation can be defined:

\[ a_h \succ^R_L a_k \iff p(a_h, a_k) \geq p(a_k, a_h). \]
In the third case...

Since more than one probability distribution can be induced on the space of compatible value functions, the probabilistic necessary and probabilistic possible preference relations could be defined:

- \( a_h \succeq^N_L a_k \) iff the preference of \( a_h \) over \( a_k \) is more credible than the preference of \( a_k \) over \( a_h \) for all induced probability distributions,

- \( a_h \succeq^P_L a_k \) iff the preference of \( a_h \) over \( a_k \) is more credible than the preference of \( a_k \) over \( a_h \) for at least one induced probability distribution.
Proposition

- \( \succsim_L^R \) is strongly complete, that is for all \( a, b \in A \), \( a \succsim_L^R b \) or \( b \succsim_L^R a \),
- \( \succsim^N \subseteq \succsim_L^N \subseteq \succsim_L^R \subseteq \succsim_L^P \subseteq \succsim^P \),
- For all \( a, b \in A \), \( a \succsim_L^N b \) or \( b \succsim_L^P a \),
- For all \( a, b, c \in A \), if \( a \succsim_L^N b \) and \( b \succsim_L^N c \), then \( a \succsim_L^N c \),
- For all \( a, b, c \in A \), if \( a \succsim_L^N b \) and \( b \succsim_L^N c \), then \( a \succsim_L^N c \),
- For all \( a, b, c \in A \), if \( a \succsim_L^N b \) and \( b \succsim_L^P c \), then \( a \succsim_L^P c \),
- For all \( a, b, c \in A \), if \( a \succsim_L^P b \) and \( b \succsim_L^N c \), then \( a \succsim_L^P c \).

Here by \( \succsim^R \) we denoted the preference relation obtained by considering whichever probability distribution compatible with the uncertain preferences provided by the DM.
Illustrative example
Problem definition

Let us suppose that the owner of a firm has to employ one among six sales manager evaluated on the following criteria: sales management experience ($g_1$), international experience ($g_2$) and human qualities ($g_3$).

Table: Evaluations of the sales managers

<table>
<thead>
<tr>
<th></th>
<th>$g_1$</th>
<th>$g_2$</th>
<th>$g_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bassama</td>
<td>28</td>
<td>18</td>
<td>28</td>
</tr>
<tr>
<td>Calvet</td>
<td>26</td>
<td>40</td>
<td>44</td>
</tr>
<tr>
<td>Ferret</td>
<td>35</td>
<td>62</td>
<td>25</td>
</tr>
<tr>
<td>Frechet</td>
<td>9</td>
<td>62</td>
<td>88</td>
</tr>
<tr>
<td>Petron</td>
<td>6</td>
<td>15</td>
<td>100</td>
</tr>
<tr>
<td>Varlot</td>
<td>62</td>
<td>43</td>
<td>0</td>
</tr>
</tbody>
</table>
The owner of the firm provided the following certain preference information:

- Varlot is preferred to Petron \((\text{Varlot} \succ \text{Petron})\),
- Varlot is preferred to Petron more than Ferret is preferred to Calvet \(((\text{Varlot}, \text{Petron}) \succ^* (\text{Ferret}, \text{Calvet}))\).

We sampled \(s_V = 10,000\) value functions satisfying the monotonicity and normalization constraints as well as the constraints translating the certain preference information provided by the DM.
Besides, the owner of the firm wishes to provide also this further uncertain preference information:

- The preference of Calvet over Frechet is more credible than the vice versa (Calvet $\succ^L$ Frechet),

- The preference of Varlot over Frechet is more credible than the preference of Calvet over Frechet ($(Varlot,Frechet) \succ^L{*} (Calvet,Frechet)$).
Rank acceptability indices

Table: Rank acceptability indices in percentage by using the representative probability distribution. Among brackets the rank acceptability indices got by applying the classical SMAA methodology.

<table>
<thead>
<tr>
<th></th>
<th>$b_k^1$</th>
<th>$b_k^2$</th>
<th>$b_k^3$</th>
<th>$b_k^4$</th>
<th>$b_k^5$</th>
<th>$b_k^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bassama</td>
<td>1.02 (0.73)</td>
<td>1.89 (1.97)</td>
<td>18.62 (10.79)</td>
<td>21.37 (19.15)</td>
<td>47.5 (54.03)</td>
<td>9.6 (13.33)</td>
</tr>
<tr>
<td>Calvet</td>
<td>0 (0)</td>
<td>1.52 (2.5)</td>
<td>37.75 (19.54)</td>
<td>47.17 (55.8)</td>
<td>12.89 (21.09)</td>
<td>0.66 (1.07)</td>
</tr>
<tr>
<td>Ferret</td>
<td>26.95 (45.13)</td>
<td>64.36 (42.77)</td>
<td>8.29 (11.39)</td>
<td>0.38 (0.66)</td>
<td>0.02 (0.05)</td>
<td>0 (0)</td>
</tr>
<tr>
<td>Frechet</td>
<td>15.87 (26.75)</td>
<td>11.55 (19.07)</td>
<td>19.74 (33.22)</td>
<td>23.06 (11.74)</td>
<td>28.27 (8.94)</td>
<td>1.50 (0.28)</td>
</tr>
<tr>
<td>Petron</td>
<td>0 (0)</td>
<td>0.27 (0.09)</td>
<td>0.78 (0.41)</td>
<td>1.56 (1.89)</td>
<td>9.15 (12.29)</td>
<td>88.23 (85.32)</td>
</tr>
<tr>
<td>Varlot</td>
<td>56.16 (27.39)</td>
<td>20.41 (33.6)</td>
<td>14.82 (24.65)</td>
<td>6.45 (10.76)</td>
<td>2.15 (3.6)</td>
<td>0 (0)</td>
</tr>
</tbody>
</table>
## Pairwise winning indices

**Table**: Pairwise winning indices in percentage by using the representative probability distribution. Among brackets the pairwise winning indices got by applying the classical SMAA methodology.

<table>
<thead>
<tr>
<th>p(a, b)</th>
<th>Bassama</th>
<th>Calvet</th>
<th>Ferret</th>
<th>Frechet</th>
<th>Petron</th>
<th>Varlot</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bassama</td>
<td>0 (0)</td>
<td>29.49 (26.59)</td>
<td>2.17 (1.48)</td>
<td>33 (15.58)</td>
<td>89.74 (86.3)</td>
<td>4.34 (6.28)</td>
</tr>
<tr>
<td>Calvet</td>
<td>70.51 (73.41)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>50.02 (16.29)</td>
<td>97.25 (96.94)</td>
<td>8.79 (14.67)</td>
</tr>
<tr>
<td>Ferret</td>
<td>97.83 (98.52)</td>
<td>100 (100)</td>
<td>0 (0)</td>
<td>80.85 (67.89)</td>
<td>99.57 (99.6)</td>
<td>39.58 (66.26)</td>
</tr>
<tr>
<td>Frechet</td>
<td>67 (84.42)</td>
<td>49.98 (83.71)</td>
<td>19.15 (32.11)</td>
<td>0 (0)</td>
<td>97.73 (99.5)</td>
<td>25.3 (42.37)</td>
</tr>
<tr>
<td>Petron</td>
<td>10.26 (13.7)</td>
<td>2.75 (3.06)</td>
<td>0.43 (0.4)</td>
<td>2.27 (0.5)</td>
<td>0 (0)</td>
<td>0 (0)</td>
</tr>
<tr>
<td>Varlot</td>
<td>95.66 (93.72)</td>
<td>91.21 (85.33)</td>
<td>60.42 (33.74)</td>
<td>74.7 (57.63)</td>
<td>100 (100)</td>
<td>0 (0)</td>
</tr>
</tbody>
</table>
(a) Probabilistic necessary preference relation

<table>
<thead>
<tr>
<th></th>
<th>Bassama</th>
<th>Calvet</th>
<th>Ferret</th>
<th>Frechet</th>
<th>Petron</th>
<th>Varlot</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bassama</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Calvet</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Ferret</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Frechet</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Petron</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Varlot</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
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</table>

(b) Probabilistic possible preference relation

<table>
<thead>
<tr>
<th></th>
<th>Bassama</th>
<th>Calvet</th>
<th>Ferret</th>
<th>Frechet</th>
<th>Petron</th>
<th>Varlot</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bassama</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Calvet</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Ferret</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Frechet</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Petron</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Varlot</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Multiple Criteria Hierarchy Process (Corrente, Greco, Slowinski 2012, 2013)
Hierarchical decomposition of complex decision problems

- „Almost everyone who has seriously thought about the objectives in a complex problem has come up with some sort of hierarchy of objectives.”
  (R.L. Keeney & H. Raiffa, 1976, p. 41)

- „A hierarchy is an abstraction of the structure of a system to study the functional interactions of its components and their impacts on the entire system.”
  (T.L. Saaty, 1980, p.5)

- „In the process of structuring the problem, it is possible (even likely) that the criteria may have been constructed hierarchically in terms of a value tree.”
  (V. Belton & T.J. Stewart, 2002, p. 80)
Tree structure of objectives-criteria – an example

Quality of life

Social benefits
- Household income
- # Jobs
- Water supply
- Agricultural output
- Forestry output
- Secondary industry
- Area conserved
- # Ecotypes conserved

Economic benefits
- Household income
- # Jobs
- Water supply
- Agricultural output
- Forestry output
- Secondary industry
- Area conserved
- # Ecotypes conserved

Environmental benefits
- Dissolved solids
- Dry season flow
- Flood level

Land use problem
(V. Belton & T.J. Stewart, 2002, p. 81)
Multiple Criteria Hierarchy Process (MCHP)

Multiple Criteria Hierarchy Process (MCHP) - notation

$G_0$ – root criterion

Level criterion $G_r$

$G_{(1)}, G_{(2)} \{G_r, r=(1), \ldots, (m)\}$

$G_{(1,1)}, G_{(1,2)}, \ldots, G_{(2,2)}$

$\{G_{(r,j)}, j=1, \ldots, n(r)\}$

Elementary criterion $g_t$

$EL$– set of elementary criteria

$EL=\{g_{(1,1,1)}, g_{(1,1,2)}, \ldots, g_{(2,2,2)}\}$

alternatives
Multiple Criteria Hierarchy Process (MCHP) - notation

\[ E(G_r) – \text{set of elementary criteria descending from } G_r \]

\[ E(G_{(1)}) = \{ g_{(1,1,1)}, g_{(1,1,2)}, g_{(1,2,1)}, g_{(1,2,2)}, g_{(1,3,1)}, g_{(1,3,2)} \} \]

\[ E(G_{(2,2)}) = \{ g_{(2,1,1)}, g_{(2,1,2)}, g_{(2,1,3)}, g_{(2,2,1)}, g_{(2,2,2)} \} \]

\[ G_k^r – \text{set of criteria descending from } G_r \text{ and located at level } k \]

\[ G^2_{(0)} = \{ G_{(1,1)}, G_{(1,2)}, G_{(1,3)} \} \]

\[ G_{(2,1)}, G_{(2,2)} \}

\[ G^3_{(2)} = \{ g_{(2,1,1)}, g_{(2,1,2)}, g_{(2,1,3)}, g_{(2,2,1)}, g_{(2,2,2)} \} \]
We wish to consider preference relation $\succeq_r$ in each node of the hierarchy tree, e.g.:

- $a \succeq_{(2)} b$ iff $U_{(2)}(a) \geq U_{(2)}(b)$
- $c \succeq_{(1, 3)} d$ iff $U_{(1, 3)}(c) \geq U_{(1, 3)}(d)$
- $e \succeq_{(2, 1)} f$ iff $U_{(2, 1)}(e) \geq U_{(2, 1)}(f)$
Multiple Criteria Hierarchy Process (MCHP) – main idea

- In case of preferentially independent criteria, preference relation $\succeq_r$ should enjoy some intuitive properties, e.g.:

\[
a \succeq_{(r,j)} b \text{ for all } j = 1, \ldots, n(r) \Rightarrow a \succeq_r b
\]

\[
a \succeq_{(1,1)} b, \ a \succeq_{(1,2)} b, \ a \succeq_{(1,3)} b \Rightarrow a \succeq_{(1)} b
\]
In case of preferentially independent criteria, preference relation $\succeq_r$ should enjoy some intuitive properties, e.g.:

- $a \succeq_{(r,j)} b$ for all $j = 1, \ldots, n(r) \Rightarrow a \succeq b$
- $a \succeq_{(1,1)} b$, $a \succeq_{(1,2)} b$, $a \succeq_{(1,3)} b \Rightarrow a \succeq_{(1)} b$
- $\neg(a \succeq_{(r,j)} b)$ for all $j = 1, \ldots, n(r) \Rightarrow \neg(a \succeq b)$
- $\neg(a \succeq_{(2,1)} b)$, $\neg(a \succeq_{(2,2)} b) \Rightarrow \neg(a \succeq_{(2)} b)$
- $a \succeq_r b \Rightarrow a \succeq_{(r,j)} b$ for at least one $j \in \{1, \ldots, n(r)\}$
- $a \succeq_{(2)} b \Rightarrow a \succeq_{(2,1)} b$ or $a \succeq_{(2,2)} b$
Multiple Criteria Hierarchy Process (MCHP) – main idea

- Any MCDM method could be used to construct preference relation $\succeq_r$ in particular nodes using the available preference information:
  - MAVT
  - ELECTRE/PROMETHEE
  - UTA
  - ...
- The choice depends on type of aggregation & preference information

Multiple Criteria Hierarchy Process (MCHP) – main idea

- Consider the simplest preference model – weighted sum

- For any \( a \in A \), the value of \( a \) is:

\[
U(a) = w_{(1,1)} g_{(1,1)}(a) + w_{(1,2)} g_{(1,2)}(a) + w_{(1,3)} g_{(1,3)}(a) + w_{(2,1)} g_{(2,1)}(a) + w_{(2,2)} g_{(2,2)}(a) + w_{(2,3)} g_{(2,3)}(a)
\]

The unknown model parameters are weights of elementary criteria only

- \( w_r = \sum_{t \in E(G_r)} w_t \)
- \( t \) - index of elementary criterion \( g_t \)

\[
w_{(1,1)} = \sum_{t \in E(G_{(1,1)})} w_t = w_{(1,1,1)} + w_{(1,1,2)}
\]
Multiple Criteria Hierarchy Process (MCHP) – main idea

- Each node is associated with a marginal value function
- Elicitation of preferences and analysis of recommendation in tree nodes:
  weights adapt to preferences & preference relations follow from weights, e.g.:

\[ a \succeq (1,3) b \iff U_{(1,3)}(a) \geq U_{(1,3)}(b) , \text{ i.e.,} \]

\[ w_{(1,3,1)} g_{(1,3,1)}(a) + w_{(1,3,2)} g_{(1,3,2)}(a) \geq w_{(1,3,1)} g_{(1,3,1)}(b) + w_{(1,3,2)} g_{(1,3,2)}(b) \]
MCHP with additive value function

- Marginal value function associated with node $r$
  $$U_r(a) = \sum_{t \in E(G_r)} u_t(a)$$

$u_t(\cdot)$ – marginal value function monotonically dependent on elementary criterion $g_t$

$$U_1(a) = \sum_{t \in E(G_1)} u_t(a) = u_{(1,1,1)}(a) + u_{(1,1,2)}(a) + u_{(1,2,1)}(a) + u_{(1,2,2)}(a) + u_{(1,2,3)}(a) + u_{(1,3,1)}(a) + u_{(1,3,2)}(a)$$
MCHP with additive value function

- Total value function associated with the root

\[ U(a) = \sum_{t \in EL} u_t(a) \]

\[ U(a) = u_{(1,1,1)}(a) + u_{(1,1,2)}(a) + u_{(1,2,1)}(a) + u_{(1,2,2)}(a) + u_{(1,2,3)}(a) + u_{(1,3,1)}(a) + u_{(1,3,2)}(a) + u_{(2,1,1)}(a) + u_{(2,1,2)}(a) + u_{(2,1,3)}(a) + u_{(2,2,1)}(a) + u_{(2,2,2)}(a) = \]

\[ = U_{(1,1)}(a) + U_{(1,2)}(a) + U_{(1,3)}(a) + U_{(2,1)}(a) + U_{(2,2)}(a) = \]

\[ = U_1(a) + U_2(a) \]

- How to construct marginal value functions \( u_t(\cdot), t \in EL \) ?
MCHP with *additive value function* - preference elicitation by DM

- **Direct** or **indirect**?
- **Direct** elicitation of numerical values of model parameters by DMs demands much of their cognitive effort
- **Indirect** = through decision examples
- Decision aiding based on decision examples is gaining importance because:
  - Decision example is a relatively „easy” preference information
  - Decisions can also be observed without active participation of DMs
  - Psychologists confirm that DMs are more confident exercising their decisions than explaining them
MCHP with additive value function - preference elicitation by DM

- Types of indirect preference information in particular nodes of the tree:
  - **Pairwise comparison**: \( a \) is at least as good as \( b \) on criterion \( G_r \)
    \[
    a \succeq_r b \iff U_r(a) \geq U_r(b)
    \]
  - **Intensity of preference**: considering criterion \( G_r \) or \( g_t \),
    \( a \) is preferred to \( b \) at least as much as \( c \) is preferred to \( d \)
    \[
    (a, b) \succeq^*_r (c, d) \iff U_r(a) - U_r(b) \geq U_r(c) - U_r(d) \geq 0
    \]
    \[
    (a, b) \succeq^*_t (c, d) \iff u_t(a) - u_t(b) \geq u_t(c) - u_t(d) \geq 0
    \]

Checking for the existence of a compatible value function in node $r$

$$\varepsilon^* = \max \varepsilon, \text{ subject to :}$$

- $$U_r(a^*) \geq U_r(b^*) + \varepsilon \quad \text{if } a^* \succ_r b^*$$
- $$U_r(a^*) = U_r(b^*) \quad \text{if } a^* \sim_r b^*$$
- $$U_r(a^*) - U_r(b^*) \geq U_r(c^*) - U_r(d^*) + \varepsilon \quad \text{if } (a^*, b^*) \succ_r (c^*, d^*)$$
- $$U_r(a^*) - U_r(b^*) = U_r(c^*) - U_r(d^*) \quad \text{if } (a^*, b^*) \sim_r (c^*, d^*)$$

$$u_t(x_t^k) - u_t(x_t^{k-1}) \geq 0, \quad t \in EL, \quad k = 2, \ldots, m_t(A^R)$$

$$u_t(x_t^0) = 0, \quad t \in EL \quad (EL : \text{set of elementary criteria})$$

$$\sum_{t \in EL} u_t(x_t^m) = 1$$

Since $$U_r(a) = \sum_{t \in E(G_r)} u_t(a)$$, the only unknown of this LP problem are marginal value functions of elementary criteria $$u_t$$ and threshold $$\varepsilon$$.
Checking for the existence of a compatible value function in node \( r \)

\[ \varepsilon^* = \max \varepsilon, \text{ subject to:} \]

\[ U_r(a^*) \geq U_r(b^*) + \varepsilon \quad \text{if } a^* \succ_r b^* \]

\[ U_r(a^*) = U_r(b^*) \quad \text{if } a^* \sim_r b^* \]

\[ U_r(a^*) - U_r(b^*) \geq U_r(c^*) - U_r(d^*) + \varepsilon \quad \text{if } (a^*, b^*) \succ_r (c^*, d^*) \]

\[ U_r(a^*) - U_r(b^*) = U_r(c^*) - U_r(d^*) \quad \text{if } (a^*, b^*) \sim_r (c^*, d^*) \]

\[ u_t(x_t^k) - u_t(x_t^{k-1}) \geq 0, \quad t \in EL, \quad k = 2, \ldots, m_t(A^R) \]

\[ u_t(x_t^0) = 0, \quad t \in EL \quad (EL: \text{set of elementary criteria}) \]

\[ \sum_{t \in EL} u_t(x_t^m) = 1 \]

If \( E^{AR} \) is feasible and \( \varepsilon^* > 0 \), then there exists at least one value function compatible with the preference information.
Checking for the existence of a compatible value function in node $r$

If for the given preference information there is no compatible value function, the user can:

- identify and eliminate „troublesome” pieces of preference information (Mousseau et al. 2003),

- continue to use „not completely compatible” set of value functions with an acceptable approximation error (Jacquet-Lagrèze & Siskos 1982),

- augment the complexity of the value function, e.g., pass from additive linear to additive monotonic, or to Choquet integral
Calculating necessary and possible preference relations in node $r$

- For all pairs of alternatives $a, b \in A$, their performances on elementary criteria $g_t(a), g_t(b)$ add to $m_t$ characteristic points of marginal value function $u_t$, $t \in EL$; then $E^{A^R}$ becomes $E(a,b)$

- Consider constraints:

$$U_r(b) \geq U_r(a) + \varepsilon$$

$$E(a,b)$$

$$U_r(a) \geq U_r(b)$$

$$E(a,b)$$

$$(E_r^N(a,b))$$

$$(E_r^P(a,b))$$

- The necessary and the possible preference relations (LP problems):

$$a \geq_r^N b \iff$$ if $E_r^N(a,b)$ is infeasible or $\varepsilon_r^N(a,b) = \max \varepsilon$, s.t. $E_r^N(a,b)$ is $\leq 0$

$$a \geq_r^P b \iff$$ if $E_r^P(a,b)$ is feasible and $\varepsilon_r^P(a,b) = \max \varepsilon$, s.t. $E_r^P(a,b)$ is $> 0$
One can also work with a „representative” value function

- It may be desirable to have a total order and scores of alternatives.
- The idea is to select among compatible value functions that value function which better highlights the necessary ranking, i.e., maximizes the difference of values for pairs of alternatives $a$ and $b$, such that $a \succeq^N r b$ while not($b \succeq^N r a$).
- As secondary objective, we minimize the difference of values for pairs of alternatives for which no necessary relation holds, i.e., such that not($a \succeq^N r b$) and not($b \succeq^N r a$).
- Lexicographic sequence of $G_r$’s may underline their relative importance ($G_r$ is the root criterion or any level criterion, excluding those from $EL$).


Properties of necessary and possible preference relations in node $r$

- Given two alternatives $a,b \in A$, and any non-elementary criterion $G_r$:

  (i) \[ a \succeq_{(r,j)}^N b \quad \text{for all } j = 1,\ldots,n(r) \implies a \succeq_r^N b \]

  (ii) \[ a \succeq_{(r,j)}^N b \quad \text{for all } j = 1,\ldots,n(r), \; j \neq w, \text{ and } a \succeq_{(r,w)}^P b \implies a \succeq_r^P b \]

  \[ \text{not}(a \succeq_{(r,j)}^P b) \quad \text{for all } j = 1,\ldots,n(r) \implies \text{not}(a \succeq_r^P b) \]

  (iii) \[ a \succeq_r^P b \implies a \succeq_{(r,j)}^P b \quad \text{for at least one } j \in \{1,\ldots,n(r)\} \]

- **Remark**: hierarchical properties are expressed in terms of preference
  - necessary (i)
  - necessary & possible (ii)
  - possible (iii)
Example:

Ranking of students wrt. hierarchical criteria of Mathematics & Chemistry

15 students: A, B, C, D, E, F, H, I, L, M, N, O, P, Q, R
Performances of students on elementary criteria

<table>
<thead>
<tr>
<th>Elementary criterion vs. Student</th>
<th>$g_{(1,1,1)}$</th>
<th>$g_{(1,1,2)}$</th>
<th>$g_{(1,2,1)}$</th>
<th>$g_{(1,2,2)}$</th>
<th>$g_{(2,1,1)}$</th>
<th>$g_{(2,1,2)}$</th>
<th>$g_{(2,2,1)}$</th>
<th>$g_{(2,2,2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Very bad</td>
<td>Very good</td>
<td>Very bad</td>
<td>Good</td>
<td>Very good</td>
<td>Very good</td>
<td>Very bad</td>
<td>Bad</td>
</tr>
<tr>
<td>B</td>
<td>Bad</td>
<td>Very good</td>
<td>Medium</td>
<td>Very good</td>
<td>Very bad</td>
<td>Bad</td>
<td>Very bad</td>
<td>Very bad</td>
</tr>
<tr>
<td>C</td>
<td>Very good</td>
<td>Medium</td>
<td>Medium</td>
<td>Very bad</td>
<td>Very good</td>
<td>Good</td>
<td>Bad</td>
<td>Medium</td>
</tr>
<tr>
<td>D</td>
<td>Medium</td>
<td>Very bad</td>
<td>Bad</td>
<td>Very bad</td>
<td>Bad</td>
<td>Medium</td>
<td>Very bad</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>Very good</td>
<td>Very good</td>
<td>Medium</td>
<td>Medium</td>
<td>Bad</td>
<td>Very good</td>
<td>Bad</td>
<td>Very bad</td>
</tr>
<tr>
<td>F</td>
<td>Good</td>
<td>Bad</td>
<td>Bad</td>
<td>Medium</td>
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<td>Very bad</td>
<td>Very good</td>
<td>Very good</td>
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<tr>
<td>H</td>
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<td>Bad</td>
<td>Bad</td>
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<td>Very bad</td>
<td>Very bad</td>
<td>Very bad</td>
</tr>
<tr>
<td>I</td>
<td>Good</td>
<td>Good</td>
<td>Good</td>
<td>Medium</td>
<td>Medium</td>
<td>Bad</td>
<td>Good</td>
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</tr>
<tr>
<td>L</td>
<td>Good</td>
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<td>Bad</td>
<td>Good</td>
<td>Good</td>
<td>Very bad</td>
<td>Very good</td>
<td>Good</td>
</tr>
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<td>Bad</td>
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<td>Medium</td>
<td>Very good</td>
<td>Good</td>
</tr>
<tr>
<td>N</td>
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<td>Medium</td>
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<td>Very good</td>
<td>Very good</td>
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<tr>
<td>O</td>
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<td>Bad</td>
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<td>Very good</td>
<td>Very bad</td>
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<tr>
<td>P</td>
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<td>Bad</td>
<td>Medium</td>
<td>Bad</td>
<td>Very good</td>
<td>Medium</td>
<td>Very bad</td>
</tr>
<tr>
<td>Q</td>
<td>Very good</td>
<td>Very good</td>
<td>Medium</td>
<td>Very bad</td>
<td>Bad</td>
<td>Medium</td>
<td>Medium</td>
<td>Bad</td>
</tr>
<tr>
<td>R</td>
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<td>Good</td>
<td>Bad</td>
<td>Very bad</td>
<td>Bad</td>
<td>Bad</td>
<td>Medium</td>
<td>Medium</td>
</tr>
</tbody>
</table>
Dominance relation in the set of students
On Chemistry, student I is preferred to student H

\[ U_{(2)}(I) > U_{(2)}(H) \iff \]

\[ \iff U_{(2,1)}(I) + U_{(2,2)}(I) > U_{(2,1)}(H) + U_{(2,2)}(H) \iff \]

\[ \iff u_{(2,1,1)}(I) + u_{(2,1,2)}(I) + u_{(2,2,1)}(I) + u_{(2,2,2)}(I) > \]

\[ > u_{(2,1,1)}(H) + u_{(2,1,2)}(H) + u_{(2,2,1)}(H) + u_{(2,2,2)}(H) \]

Necessary preference relation after the 1\textsuperscript{st} piece of preference information
Multiple Criteria Hierarchy Process (MCHP) – value function model

- On Analytical Chemistry, student E is preferred to student H

\[ U_{(2,1)}(E) > U_{(2,1)}(H) \iff \]

\[ \iff u_{(2,1,1)}(E) + u_{(2,1,2)}(E) > u_{(2,11)}(H) + U_{(2,1,2)}(H) \]

- Necessary preference relation after the 2\(^{nd}\) piece of preference information
On Mathematics, student N is preferred to student Q

\[ U_{(1)}(N) > U_{(1)}(Q) \iff \]

\[ \iff U_{(1,1)}(N) + U_{(1,2)}(N) > U_{(1,1)}(Q) + U_{(1,2)}(Q) \iff \]

\[ \iff u_{(1,1,1)}(N) + u_{(1,1,2)}(N) + u_{(1,2,1)}(N) + u_{(1,2,2)}(N) > u_{(1,1,1)}(Q) + u_{(1,1,2)}(Q) + u_{(1,2,1)}(Q) + u_{(1,2,2)}(Q) \]

Necessary preference relation after the 3\textsuperscript{rd} piece of preference information
On Chemistry, student L is preferred to student P

\[ U_{(2)}(L) > U_{(2)}(P) \iff \]

\[ \iff U_{(2,1)}(L) + U_{(2,2)}(L) > U_{(2,1)}(P) + U_{(2,2)}(P) \iff \]

\[ \iff u_{(2,1,1)}(L) + u_{(2,1,2)}(L) + u_{(2,2,1)}(L) + u_{(2,2,2)}(L) > u_{(2,1,1)}(P) + u_{(2,1,2)}(P) + u_{(2,2,1)}(P) + u_{(2,2,2)}(P) \]

Necessary preference relation after the 4th piece of preference information
- Necessary preference relation on Mathematics

<table>
<thead>
<tr>
<th>Necessary</th>
<th>Mathematics</th>
<th>Algebra</th>
<th>Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>preference</td>
<td>≿_{N(1)}</td>
<td>≿_{N(1,1)}</td>
<td>≿_{N(1,2)}</td>
</tr>
<tr>
<td>vs. Student</td>
<td>Mathematics</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>D</td>
<td>D,F,H,L,M,N,O,P</td>
<td>D,Q,R</td>
</tr>
<tr>
<td>D</td>
<td>H,P</td>
<td></td>
<td>R</td>
</tr>
<tr>
<td>H</td>
<td>D</td>
<td>D,P</td>
<td>D,O,R</td>
</tr>
<tr>
<td>M</td>
<td>D,H</td>
<td>D,H,P</td>
<td>C,D,H,O,Q,R</td>
</tr>
<tr>
<td>N</td>
<td>C,D,F,H,P,Q,R</td>
<td></td>
<td>C,D,E,F,H,I,L,M,O,P,Q,R</td>
</tr>
<tr>
<td>P</td>
<td></td>
<td></td>
<td>D,F,H,O,R</td>
</tr>
<tr>
<td>Q</td>
<td>C,D,R</td>
<td>A,B,C,D,F,H,I,L,M,N,O,P,R</td>
<td>C,D,R</td>
</tr>
<tr>
<td>R</td>
<td>D</td>
<td>D,F,H,I,L,M,N,O,P</td>
<td>D</td>
</tr>
</tbody>
</table>

- Remark: the necessary preference relation $N ≿_{N(1)} C$, $N ≿_{N(1)} Q$, $N ≿_{N(1)} R$ is true on Mathematics but it is not true at the level below on Algebra.
### Necessary preference relation on Chemistry

<table>
<thead>
<tr>
<th>Necessary preference vs. Student</th>
<th>Chemistry $\succeq^N_{(2)}$</th>
<th>Analytical Chemistry $\succeq^N_{(2,1)}$</th>
<th>Organic Chemistry $\succeq^N_{(2,2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B,H</td>
<td>B,C,D,E,F,H,I,L,M,N,O,P,Q,R</td>
<td>B,H</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>D,F</td>
<td>H</td>
</tr>
<tr>
<td>C</td>
<td>B,H</td>
<td>B,D,F,H,I,L,M,O,Q,R</td>
<td>A,B,E,H</td>
</tr>
<tr>
<td>D</td>
<td>B</td>
<td>B,F</td>
<td>B,E,H,P</td>
</tr>
<tr>
<td>E</td>
<td>B,H</td>
<td>B,D,F,H,L,M,P,Q,R</td>
<td>B,H</td>
</tr>
<tr>
<td>F</td>
<td></td>
<td></td>
<td>A,B,C,D,E,H,I,L,M,N,O,P,Q,R</td>
</tr>
<tr>
<td>H</td>
<td></td>
<td>F,L</td>
<td>B</td>
</tr>
<tr>
<td>I</td>
<td>B,D,H</td>
<td>B,D,F,O,R</td>
<td>B,D,E,H,P</td>
</tr>
<tr>
<td>L</td>
<td>B,D,H,P</td>
<td>F</td>
<td>A,B,C,D,E,H,I,M,N,O,P,Q,R</td>
</tr>
<tr>
<td>N</td>
<td>B,D,E,H,P,Q,R</td>
<td>B,D,E,F,H,L,P,Q,R</td>
<td>A,B,C,D,E,H,I,O,P,Q,R</td>
</tr>
<tr>
<td>O</td>
<td>B,D,H,I</td>
<td>B,D,F,I,R</td>
<td>B,D,E,H,I,P</td>
</tr>
<tr>
<td>P</td>
<td>B,D,E,H</td>
<td>B,D,E,F,H,L,N,Q,R</td>
<td>B,D,E,H</td>
</tr>
<tr>
<td>Q</td>
<td>B,D</td>
<td>B,D,F,R</td>
<td>A,B,D,E,H,P</td>
</tr>
<tr>
<td>R</td>
<td>B,D</td>
<td>B,D,F</td>
<td>A,B,C,D,E,H,P,Q</td>
</tr>
</tbody>
</table>

*Remark*: the necessary preference relation $I \succeq^N_{(2)} H$, $L \succeq^N_{(2)} B,D,H,P$, $O \succeq^N_{(2)} H$ is true on Chemistry but it is not true at the level below on Analytical Ch.
## Multiple Criteria Hierarchy Process (MCHP) – value function model

- Ranking of students by representative value functions

<table>
<thead>
<tr>
<th>Rank</th>
<th>Student</th>
<th>Value Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>M</td>
<td>0.8808</td>
</tr>
<tr>
<td>2</td>
<td>N</td>
<td>0.8622</td>
</tr>
<tr>
<td>3</td>
<td>F</td>
<td>0.6690</td>
</tr>
<tr>
<td>3</td>
<td>L</td>
<td>0.6690</td>
</tr>
<tr>
<td>3</td>
<td>A</td>
<td>0.6690</td>
</tr>
<tr>
<td>6</td>
<td>I</td>
<td>0.5426</td>
</tr>
<tr>
<td>7</td>
<td>C</td>
<td>0.4915</td>
</tr>
<tr>
<td>8</td>
<td>O</td>
<td>0.4893</td>
</tr>
<tr>
<td>9</td>
<td>R</td>
<td>0.4654</td>
</tr>
<tr>
<td>10</td>
<td>Q</td>
<td>0.4617</td>
</tr>
<tr>
<td>11</td>
<td>P</td>
<td>0.4190</td>
</tr>
<tr>
<td>11</td>
<td>E</td>
<td>0.4190</td>
</tr>
<tr>
<td>13</td>
<td>B</td>
<td>0.3808</td>
</tr>
<tr>
<td>14</td>
<td>D</td>
<td>0.2117</td>
</tr>
<tr>
<td>15</td>
<td>H</td>
<td>0.1690</td>
</tr>
</tbody>
</table>
Multiple Criteria Hierarchy Process (MCHP) – value function model

- Extensions of MCHP applied to value function model

  - Gradual credibility of provided $n$ pieces of preference information: for any $G_r$, nested relations $\succeq_{r,1} \subseteq \ldots \subseteq \succeq_{r,n}$ and $\succeq_{r,1} \supseteq \ldots \supseteq \succeq_{r,n}$

  - Extreme ranking analysis: for any $G_r$, one can see the best and the worst rank of each alternative assigned by compatible value functions

  - Ordinal classification using $UTADIS^{GMS}$: preference information in terms of exemplary assignments wrt any $G_r$; recommendation in terms of possible & necessary assignments $C^P_r(a)$, $C^N_r(a)$, $\forall a,r$

  - Group decision: for any subset $D$ of DMs, and for any $G_r$, one gets 4 types of preference relation - $\succeq^{N,N}_r(D)$, $\succeq^{N,P}_r(D)$, $\succeq^{P,N}_r(D)$, $\succeq^{P,P}_r(D)$

For each non-elementary criterion $G_r$, located at level $h$ of the tree, the set of descending criteria located at level $k > h$ is denoted by $G^k_r$:

For $r = (1)$, $h = 1$, $k = 2$

$G^2_1 = \{ G_{(1,1)}, G_{(1,2)}, G_{(1,3)} \}$,

For $r = (2)$, $h = 1$, $k = 3$

$G^3_2 = \{ g_{(2,1,1)}, g_{(2,1,2)}, g_{(2,2,1)}, g_{(2,2,2)} \}$,

To each alternative $a \in A$, there corresponds a performance vector

$[g_{t_1}(a),...,g_{t_n}(a)]$, where $n = |EL|$.

MCHP and Choquet integral preference model

- Given capacity $\mu$ defined on the power set of $\text{EL}$ (elementary criteria), a capacity $\mu^k_r$ on the power set of $G^k_r$:

$$\mu^k_r : 2^{G^k_r} \rightarrow [0,1]$$

such that

for all $F \subseteq G^k_r$, $\mu^k_r(F) = \frac{\mu(E(F))}{\mu(E(G_r))}$

- Considering non-elementary criterion $G_r$ (in node $r$) and alternative $a \in A$, given capacity $\mu$ defined on the power set of $\text{EL}$, the Choquet integral score of $a$:

$$Ch_{\mu_r}(a) = \frac{Ch_\mu(a_r)}{\mu(E(G_r))}$$

where $a_r$ is such that $g_t(a_r) = \begin{cases} g_t(a) & \text{if } g_t \in E(G_r) \\ 0 & \text{otherwise} \end{cases}$
MCHP and Choquet integral preference model

- Example:
  Ranking of students wrt. hierarchical criteria of Science & Humanities

- 9 students: a, b, c, d, e, f, g, h, k
MCHP and Choquet integral preference model

- Evaluation of students on elementary criteria

<table>
<thead>
<tr>
<th>Student</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
<th>Philosophy</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>18</td>
<td>18</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>b</td>
<td>16</td>
<td>16</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>c</td>
<td>14</td>
<td>14</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td>d</td>
<td>18</td>
<td>12</td>
<td>16</td>
<td>16</td>
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<tr>
<td>e</td>
<td>15</td>
<td>15</td>
<td>18</td>
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<td>f</td>
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<td>18</td>
</tr>
<tr>
<td>g</td>
<td>15</td>
<td>17</td>
<td>18</td>
<td>16</td>
</tr>
<tr>
<td>h</td>
<td>10</td>
<td>20</td>
<td>10</td>
<td>20</td>
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<tr>
<td>i</td>
<td>14</td>
<td>14</td>
<td>14</td>
<td>14</td>
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MCHP and Choquet integral preference model

- Möbius measures on elementary criteria

<table>
<thead>
<tr>
<th>Subject Pair</th>
<th>Measure</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics</td>
<td>$m({g_{(1,1)}})$</td>
<td>0.29</td>
</tr>
<tr>
<td>Physics</td>
<td>$m({g_{(1,2)}})$</td>
<td>0.19</td>
</tr>
<tr>
<td>Literature</td>
<td>$m({g_{(2,1)}})$</td>
<td>0.29</td>
</tr>
<tr>
<td>Philosophy</td>
<td>$m({g_{(2,2)}})$</td>
<td>0.19</td>
</tr>
<tr>
<td>Mathematics &amp; Physics</td>
<td>$m({g_{(1,1)},g_{(1,2)}})$</td>
<td>-0.1</td>
</tr>
<tr>
<td>Mathematics &amp; Literature</td>
<td>$m({g_{(1,1)},g_{(2,1)}})$</td>
<td>0</td>
</tr>
<tr>
<td>Mathematics &amp; Philosophy</td>
<td>$m({g_{(1,1)},g_{(2,2)}})$</td>
<td>0</td>
</tr>
<tr>
<td>Physics &amp; Literature</td>
<td>$m({g_{(1,2)},g_{(2,1)}})$</td>
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<tr>
<td>Physics &amp; Philosophy</td>
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<tr>
<td>Literature &amp; Philosophy</td>
<td>$m({g_{(2,1)},g_{(2,2)}})$</td>
<td>-0.1</td>
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</tbody>
</table>
### MCHP and Choquet integral preference model

- **Choquet integral score**

wrt Science ($Ch_{\mu_1}$), Humanities ($Ch_{\mu_2}$), and Overall ($Ch_{\mu}$)

<table>
<thead>
<tr>
<th></th>
<th>$Ch_{\mu_1}(a)$</th>
<th>$Ch_{\mu_2}(a)$</th>
<th>$Ch_{\mu}(a)$</th>
<th>$Ch_{\mu_1}(f)$</th>
<th>$Ch_{\mu_2}(f)$</th>
<th>$Ch_{\mu}(f)$</th>
<th>Overall:</th>
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<td>12</td>
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<td>15.92</td>
<td>$g&gt;b&gt;e&gt;c&gt;d&gt;h&gt;a&gt;k$</td>
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<td>15.26</td>
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<tr>
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MCHP and Choquet integral preference model

- Shapley values & interaction indices
- Shapley value of each elementary criterion \( \text{wrt parent criterion } G_r \)

<table>
<thead>
<tr>
<th>( G_r, r=(1),(2) )</th>
<th>Science</th>
<th>Humanities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_{(r,w)} )</td>
<td>Mathematics</td>
<td>Physics</td>
</tr>
<tr>
<td>( w=1,2,3,4 )</td>
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</tr>
<tr>
<td>( \varphi^2_r(G_{(r,w)}) )</td>
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<td>0.37</td>
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- Shapley value of each elementary criterion \( \text{wrt root criterion } G_{(0)} \)

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<th>Physics</th>
<th>Literature</th>
<th>Philosophy</th>
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<tr>
<td>( w=1,2,3,4 )</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( \varphi^2_r(G_{(r,w)}) )</td>
<td>0.24</td>
<td>0.26</td>
<td>0.24</td>
<td>0.26</td>
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</table>

- Shapley value & interaction index for Science \((G_{(1)})\) & Humanities \((G_{(2)})\)

<table>
<thead>
<tr>
<th>( G_{(r,w)}, r=(0), w=1,2 )</th>
<th>Science</th>
<th>Humanities</th>
</tr>
</thead>
<tbody>
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<td>( \varphi^1_r(G_{(r,w)}) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \varphi^1_r({G_{(1)}, G_{(2)}}) )</td>
<td>0.24</td>
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</tr>
</tbody>
</table>
A friend of the Einstein family, Max Talmey, tutored Albert and recommended him the “Critique of pure reason” when he was 12 years old. This became a new Bible for him and he called it his "holy geometry book".
MCHP and Choquet integral assessed using ROR

- Types of indirect preference information supplied by the DM:
  - Considering some reference alternatives $a, b, c, d \in A$
    - Pairwise comparison: $a$ is at least as good as $b$ on criterion $G_r$
      $$a \succeq_r b \iff Ch_r(a) \geq Ch_r(b)$$
    - Intensity of preference: considering criterion $G_r$, $a$ is preferred to $b$ at least as much, as $c$ is preferred to $d$
      $$(a, b) \succeq_r^* (c, d) \iff Ch_r(a) - Ch_r(b) \geq Ch_r(c) - Ch_r(d) \geq 0$$
  - Considering some criteria $G_{r1}, G_{r2}, G_{r3}, G_{r4} \in G^k_r$:
    - $G_{r1}$ is at least as important as $G_{r2}$
      $$\phi^k_r(G_{r1}) \geq \phi^k_r(G_{r2})$$
    - $G_{r1}$ and $G_{r2}$ are positively (negatively) interacting
      $$\phi^k_r(G_{r1}, G_{r2}) \geq \varepsilon \quad (\phi^k_r(G_{r1}, G_{r2}) \leq -\varepsilon)$$
    - $G_{r1}$ is preferred to $G_{r2}$ at least as much as $G_{r3}$ is preferred to $G_{r4}$
      $$\phi^k_r(G_{r1}) - \phi^k_r(G_{r2}) \geq \phi^k_r(G_{r3}) - \phi^k_r(G_{r4}) \geq 0$$
MCHP and Choquet integral assessed using ROR

Information provided to the DM:

- Applying Robust Ordinal Regression to Choquet integral (NAROR), i.e., solving a LP problem for each pair of alternatives \( a, b \in A \) in each node \( r \), one gets 2 partial relations (necessary & possible):
  
  - \( a \) is weakly preferred to \( b \) wrt \( G_r \) for all compatible capacities
    \[
    a \succeq^N_r b
    \]
  
  - \( a \) is weakly preferred to \( b \) wrt \( G_r \) for at least one compatible capacity
    \[
    a \succeq^P_r b
    \]

- Remark: in the LP problem of ROR, the only decision variables are 2-additive Möbius function values representing capacities \( \mu \) defined on the power set of elementary criteria

MCHP and Choquet integral assessed using ROR & SMAA

- MCHP & ROR & Choquet integral identify a set of capacities compatible with preference information.

- To explore the set of compatible capacities, we use the Hit-And-Run method (Tervonen et al. 2013) & Stochastic Multiobjective Acceptability Analysis (SMAA - Lahdelma, Hokkanen, Salminen 1998) that yields:
  - the rank acceptability index \( b_{k,r}^i \) – probability that alternative \( a_k \) gets position \( i \) in the ranking obtained wrt criterion \( G_r \).
  - the pairwise winning index \( p_r(a,b) \) – probability of preference of \( a \) over \( b \) on criterion \( G_r \).


### Elementary subcriterion

<table>
<thead>
<tr>
<th>Subcriterion</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>Student-Staff Ratio (SSR)</td>
<td>The number of students per member of the academic staff</td>
</tr>
<tr>
<td>Graduating on Time (GT)</td>
<td>The percentage of graduates that graduated within the time expected for their bachelor program</td>
</tr>
<tr>
<td>Academic Staff with Doctorates (ASD)</td>
<td>The percentage of academic staff holding a doctorate</td>
</tr>
<tr>
<td>Contact with Work Environment (CWE)</td>
<td>A composite measure representing at bachelor level: (1) the inclusion of internships or phases of practical experience in the curriculum; (2) the percentage of students doing an internship (3) teaching by practitioners from outside the university departments</td>
</tr>
<tr>
<td>Research Publications (RP)</td>
<td>The number of research publications indexed in the Web of Science database, where at least one author is affiliated to the university</td>
</tr>
<tr>
<td>Citation Rate (CR)</td>
<td>The average number of times that the university department’s research publications (over the period 2008-2011) get cited in other research, adjusted (normalized) at the global level for the field of science and the year in which a publication appeared</td>
</tr>
<tr>
<td>Research Orientation of Teaching (ROT)</td>
<td>The degree to which the education is informed by research in the field (based on a survey of students in the program)</td>
</tr>
<tr>
<td>Post-Doc Positions (PDP)</td>
<td>The number of post-doc positions relative to the full-time equivalent number of academic staff</td>
</tr>
</tbody>
</table>
Performances of the universities on the considered criteria

<table>
<thead>
<tr>
<th>University</th>
<th>Country</th>
<th>TL</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>WHU School of Management (U₁)</td>
<td>Germany</td>
<td>SSR GT ASD CWE</td>
<td>RP CR ROT PDP</td>
</tr>
<tr>
<td>Aarhus U (U₂)</td>
<td>Denmark</td>
<td>2  5  5  3</td>
<td>5  5  5  2</td>
</tr>
<tr>
<td>U Tampere (U₃)</td>
<td>Finland</td>
<td>2  2  5  2</td>
<td>5  3  3  5</td>
</tr>
<tr>
<td>Lille Catholic U (U₄)</td>
<td>France</td>
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<td>5  4  3  1</td>
</tr>
<tr>
<td>U Paris West (U₅)</td>
<td>France</td>
<td>3  4  5  4</td>
<td>5  2  2  1</td>
</tr>
<tr>
<td>Polytech, U Milan (U₆)</td>
<td>Italy</td>
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<td>5  4  3  5</td>
</tr>
<tr>
<td>U Trento (U₇)</td>
<td>Italy</td>
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<td>4  5  3  5</td>
</tr>
<tr>
<td>Vilnius Gediminas Technical U (U₈)</td>
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<td>5  5  4  2</td>
</tr>
<tr>
<td>U Porto (U₉)</td>
<td>Portugal</td>
<td>5  2  4  2</td>
<td>5  4  4  2</td>
</tr>
<tr>
<td>Bucharest U Economic Studies Marketing (U₁₀)</td>
<td>Romania</td>
<td>4  4  4  3</td>
<td>5  2  2  5</td>
</tr>
<tr>
<td>Bucharest Faculty of Business Administration (U₁₁)</td>
<td>Romania</td>
<td>5  4  5  3</td>
<td>5  2  3  3</td>
</tr>
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</table>
Preference information provided by the DM

- SSR and GT are positively interacting
- GT and CWE are positively interacting
- ASD and ROT are negatively interacting
- RP and ROT are negatively interacting
- RP and CR are negatively interacting
- CWE is more important than PDP when they are referred to the root criterion $G_0$
- The difference of importance between GT and SSR is greater than the difference of importance between CWE and ASD when they are referred to the root criterion $G_0$
- With respect to R, university $U_3$ is preferred to university $U_8$
- With respect to TL, $U_{10}$ is preferred to $U_8$
MCHP and NAROR results

(a) Teaching and Learning (TL)

(b) Research (R)
MCHP and SMAA (Rank Acceptability Indices)

(q) Teaching and Learning

<table>
<thead>
<tr>
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<th>(a_{ij}^4)</th>
<th>(a_{ij}^5)</th>
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(r) Research

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<td>0.01</td>
<td>2.01</td>
<td>10.96</td>
<td>63.47</td>
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</tbody>
</table>

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RAI at comprehensive level

\[
\begin{array}{cccccccccc}
 & G_0 & a_1^0 & a_2^0 & a_3^0 & a_4^0 & a_5^0 & a_6^0 & a_7^0 & a_8^0 \\
\hline
\text{WHU} & 1.12 & 22.25 & 45.71 & 134.4 & 10.87 & 7.89 & 0 & 0 & 0 \\
\text{Aarhus} & 0 & 0 & 0 & 0 & 0 & 0 & 8.35 & 22.85 & 20.82 \\
\text{Tampere} & 0 & 0 & 0 & 0 & 0 & 0 & 26.41 & 29.08 & 22.94 \\
\text{Lille} & 0 & 0 & 0 & 0 & 0 & 0 & 1.54 & 8.72 & 17.94 \\
\text{Paris West} & 0 & 0 & 0 & 0 & 0 & 0 & 1.21 & 53.69 & 16.15 \\
\text{Milan Politech} & 0 & 1.51 & 10.81 & 40.31 & 27.57 & 16.8 & 0 & 0 & 0 \\
\text{Trento} & 93.23 & 5.5 & 0.8 & 0.47 & 0 & 0 & 0 & 0 & 0 \\
\text{Vilnius} & 4.75 & 58.11 & 30.13 & 5.92 & 0.75 & 0.34 & 0 & 0 & 0 \\
\text{Porto} & 0 & 0 & 0 & 0 & 0 & 0 & 8.81 & 23.41 & 22.59 \\
\text{Bucharest SM} & 0 & 0.94 & 3.3 & 7.05 & 28.04 & 59.26 & 1.7 & 0.01 & 0 \\
\text{Bucharest BA} & 0.9 & 11.99 & 8.75 & 33.85 & 32.97 & 11.54 & 0 & 0 & 0 \\
\end{array}
\]
### MCHP and SMAA (Pairwise Winning Indices)

#### (s) Teaching and Learning

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<th>$U_2$</th>
<th>$U_3$</th>
<th>$U_4$</th>
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Complete rankings of universities by the most representative capacity

(c) Comprehensive
(d) Teaching & Learning
(e) Research
Interactive optimization with Robust Ordinal Regression
Interactive optimization with Robust Ordinal Regression

- Robust Ordinal Regression in a loop: preference elicitation with constructive learning
- Results are robust, because they take into account partial preference information

Decision maker

Preference information

Robustness analysis

Preference model

Necessary and possible results (ranking, sorting)

Set of compatible preference model parameters
Interactive optimization with Robust Ordinal Regression

**Input** (preference information)
- Pairwise comparisons of solutions
- Best (or worst) solution out of a set
- Ranking of several solutions
- Ordinal or cardinal intensity of preference for pairs of solutions
- Sorting of solutions into quality classes
- ...

**Output** (preference model)
- Value function
- Outranking relation
- Artificial neural network
- Decision rules
- Decision trees
- ...

Input: preference information
Output: preference model

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Input (preference information)
- Pairwise comparisons of solutions
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- ...

Output (preference model)
- Value function
- Outranking relation
- Artificial neural network
- Decision rules
- Decision trees
- ...
How complex should the preference model be?

- **Model too simple**
  - not able to represent user’s preferences
  - Example: linear model unable to capture preference information

- **Model too complex/flexible**
  - no generalization power, all solutions enter only one front, takes very long to learn all the parameters
  - Example: Dominance relation, general additive model with monotonic marginal value functions

„Everything should be made as simple as possible
– but not simpler” [Albert Einstein]
Preference information and model complexity

- Pref. info: Possibly no fully compatible model
  1. Discard preference information
  2. Find a model with minimal error

- Model complexity: Fully specified model
  - Many compatible value functions
    1. Pick "representative" value function
    2. Consider **all** compatible value functions

- **NEMO** integrates **ROR** into **NSGA-II** (Deb et al. 2000)

- Every $q$ iterations, the DM is expressing preferences by comparing pairwise some non-dominated solutions in the current population

- Preference model:
  - Linear value function  \[ U(a) = \sum_{i=1}^{n} w_i \times f_i(a) \]
  - General additive value function  \[ U(a) = \sum_{i=1}^{n} u_i [f_i(a)] \]
  - Choquet integral  \[ U(a) = \sum_{i=1}^{n} \mu(F_i) (f_i(a) - f_{i-1}(a)) \]
  - ...

- **No scaling** of objectives is necessary – NEMO handles heterogeneous objectives
The NEMO framework

- **NSGA-II**: dominance ranking of solutions from a current population
  
  Within the **same front**, order the individuals with respect to the **crowding distance**

- **NEMO-0**: in non-dominated fronts, individuals are ranked by representative value function compatible with preference information
The NEMO framework

- **NSGA-II**: dominance ranking of solutions from a current population
  
  Within the same front, order the individuals with respect to the crowding distance

- **NEMO-I**: replaces dominance relation by pairwise necessary preference relation
  
  \( O(p^2) \) LPs to solve in every iteration
The NEMO framework

- **NSGA-II**: dominance ranking of solutions from a current population
  
  Within the same front, order the individuals with respect to the crowding distance

- **NEMO-II**: put in the first front solutions that are preferred to all others in the population for at least one compatible value function
  
  only $O(p)$ LPs to solve

  Never preferred under NEMO-II-linear

Front of NEMO-II $\subseteq$ Front of NEMO-I $\subseteq$ Front of NSGA-II
Recent work: NEMO-II-Choquet

- Use **Choquet integral** as preference model
  - Well-accepted model in decision theory
  - Allows to model interaction between objectives
- **Adapt complexity** of preference model to complexity of preferences
  - Start with linear model
  - Switch to 2-additive Choquet once no linear compatible value function can be found
- Every $q$ iterations the user is expressing preferences by comparing two non-dominated solutions
- Put in the **first front** solutions that are preferred to all others in the population for at least one compatible value function
- Within the **same front**, order the individuals with respect to the crowding distance
A particular case of the Choquet integral: $n=2$

If $n=2$, then...

$$Ch_{\mu}(f_1, f_2) = m(\{1\}) f_1 + m(\{2\}) f_2 + m(\{1,2\}) \min\{f_1, f_2\} =$$

$$= \begin{cases} (m(\{1\}) + m(\{1,2\})) f_1 + m(\{2\}) f_2 & \text{if } f_1 \leq f_2 \\ m(\{1\}) f_1 + (m(\{2\}) + m(\{1,2\})) f_2 & \text{if } f_1 \geq f_2 \end{cases}$$

\[\beta = \arctan \left( \frac{m(\{1\})}{m(\{2\}) + m(\{1,2\})} \right), \quad \alpha = \arctan \left( \frac{m(\{1\}) + m(\{1,2\})}{m(\{2\})} \right)\]
2-additive Choquet – positive interaction (synergy)

\[ U(a) = \mu(\{f_1\}) f_1(a) + \mu(\{f_2\}) f_2(a) + \left[ \mu(\{f_1, f_2\}) - \mu(\{f_1\}) - \mu(\{f_2\}) \right] \text{min}\{f_1(a), f_2(a)\} \geq c \]

positive interaction when \( \mu(\{f_1, f_2\}) > \mu(\{f_1\}) + \mu(\{f_2\}) \)
2-additive Choquet – positive interaction (synergy)

\[ U(a) = \mu(\{f_1\}) f_1(a) + \mu(\{f_2\}) f_2(a) + \left[ \mu(\{f_1, f_2\}) - \mu(\{f_1\}) - \mu(\{f_2\}) \right] \min \{f_1(a), f_2(a)\} \geq c \]

Positive interaction when \( \mu(\{f_1, f_2\}) > \mu(\{f_1\}) + \mu(\{f_2\}) \)

greater capacity = weight of \( f_1 \) than before
Isoquants of the Choquet integral for two criteria – special cases

- 2-additive Choquet – negative interaction (redundancy)
Graphical interpretation

\[ f_2 = f_1 \]

\[ \gamma_{DM} \]
Scaling of objectives

\[ \begin{align*}
  f_2 & = f_1 \\
  w_2 f_2 & = w_1 f_1 
\end{align*} \]
NEMO-II-Ch main points

- Start with the linear value function as preference model
- Ask every $q$ iterations DM’s preferences by comparing two non-dominated solutions
- Order the solutions by checking if there exists at least one compatible model for which $x$ is preferred to all other solutions
- Within the same front order the solutions with respect to the crowding distance
- Switch to the 2-additive Choquet integral preference model as soon as the linear model is not able to represent the preferences of the DM anymore
Why NEMO-II-Ch? (DTLZ1-5D)

- DM compares two n-d solutions in the same front every 10 iterations.
- It is better to start with the simplest model (the linear one);
- Passing to the 2-additive Choquet integral preference model produces better results than passing to the complete Choquet integral model;
- In NEMO-II-Ch interactions between pairs of criteria are considered.
Thank you