Polish Academy of Sciences

Basic notions

of Multiple Criteria Decision Aiding/Making

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Decision problem

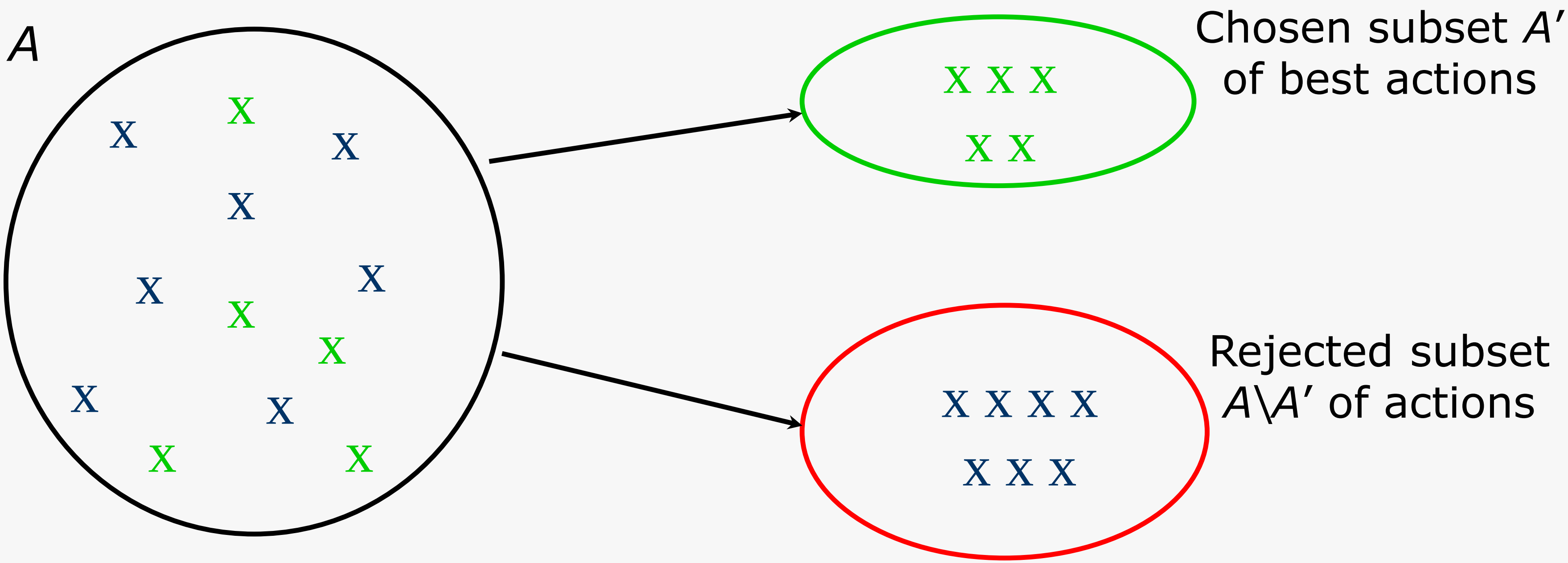
- There is an objective or objectives to be attained
- There are many alternative ways for attaining the objective(s) – they consititute a set of actions *A* (alternatives, solutions, objects, acts, …)
- A decision maker (DM) may have one of following questions with respect to set *A*:

  P_α: *How to choose the best action* ?

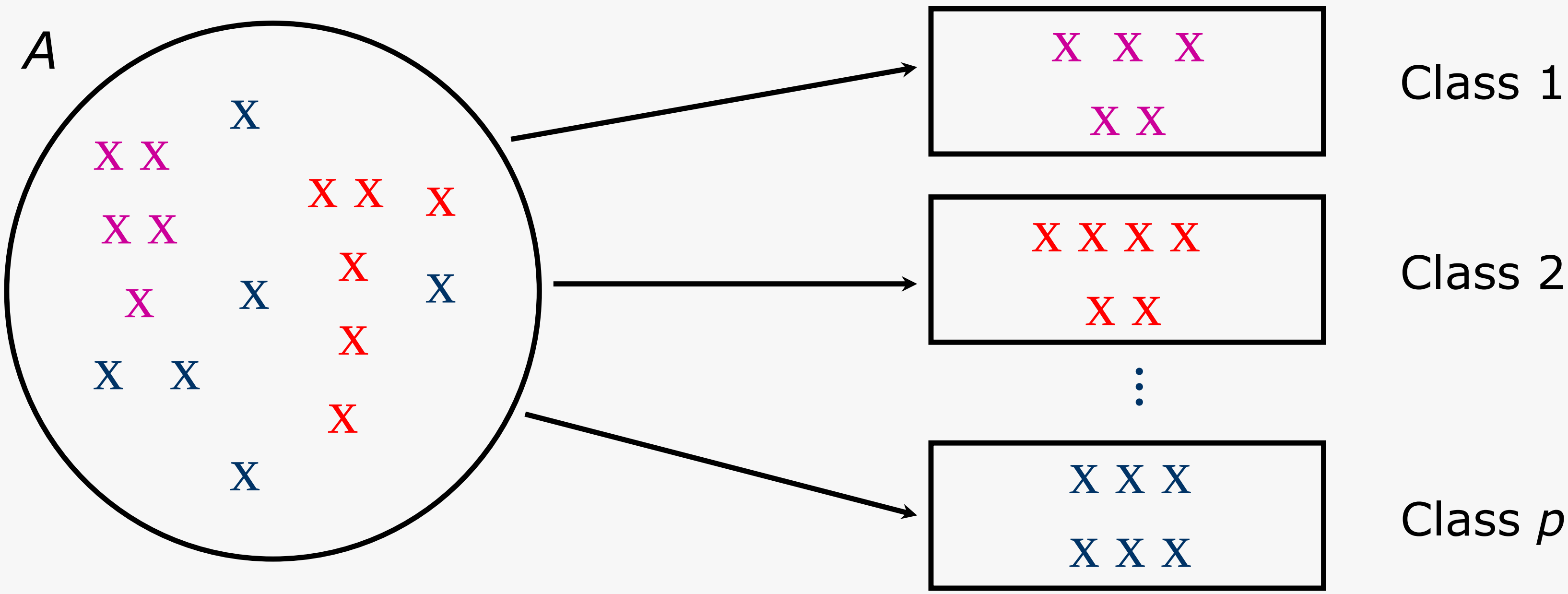
  P_β : *How to classify actions into pre-defined decision classes* ?

  P_γ : *How to order actions from the best to the worst* ?

P_α : Choice problem (optimization)



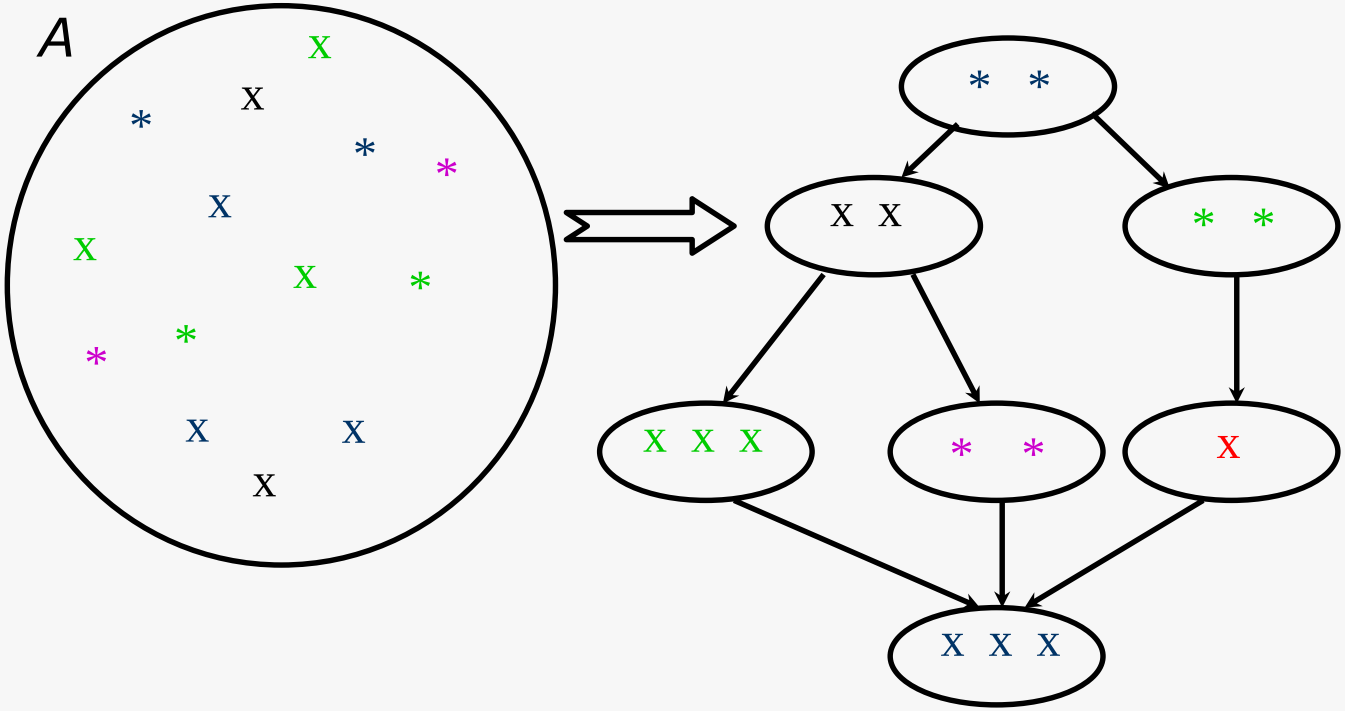
P_β : Classification problem (sorting)



$$\text{Class } 1 \succ \text{Class } 2 \succ \ldots \succ \text{Class } p$$

P_γ : Ordering problem (ranking)

Partial or complete ranking
of actions



What can one reasonably expect from Decision Aiding? (Roy, 1985)

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- Analyzing the decision making context by identifying the actors, the various possibilities of action, their consequences, and the stakes
- Organizing and structuring how the decision making process will unfold, to increase consistency between the values underlying the objectives, and the quality of the final decision
- Drawing up recommendations based on results from models and computational procedures designed with respect to some working hypotheses
- Participating in the process to legitimate the final decision
- Actors of a decision process:
  - Stakeholders (actors concerned by the decision)
  - Single or multiple Decision Makers (DM)
  - Analyst (facilitator)

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Coping with multiple dimensions in decision aiding

- Questions P_α, P_β , P_γ are followed by new questions:

DM: who is the decision maker and how many they are ?

The decision making process is generally a multi-actor process.

MC: what are the criteria for assessing the quality of actions ?

A decision maker (DM) seldom has a single clear criterion in mind. Seldom is there a common unit between the scales, which are rather heterogeneous. That is why it may be very difficult to define a priori a unique criterion able to take account all the relevant points of view.

RU: what are the consequences of actions and are they deterministic or uncertain (single state of nature with $P=1$ or multiple states of nature with different $P \leq 1$) ?

	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ	P_α P_β P_γ
DM	1	m	1	1	m	m	1	m
MC	1	1	n	1	n	1	n	n
RU	1	1	1	RU	1	RU	RU	RU

Optimization
Sorting
Ranking

Theory of
Social
Choice (TSC)

Multi-Criteria
Decision
Aiding (MCDA)

Decision under
Risk and
Uncertainty (DRU)

Solution
„philosophically"
simple

For a conflict between dimensions DM, MC, RU, the decision problem has no solution at this stage (*ill-posed problem*)

„Multi-dimensional" decision problems

	Theory of Social Choice	Multiple Criteria Decision Aiding	Decision under Risk and Uncertainty
Element of set *A*	Candidate	Action	Act
Dimension of evaluation space	Voter	Criterion	Probability of an outcome
Objective information about comparison of elements from *A*	Dominance relation	Dominance relation	Stochastic dominance relation

- The only objective information one can draw from the statement of a multi-dimensional decision problem is the **dominance relation**

TSC | MCDA | DRU

	Voters	
Cand.	V_1	V_2
a	3	1
b	1	2
c	2	3

	Criteria	
Action	Time	Cost
a	3	1
b	1	2
c	2	3

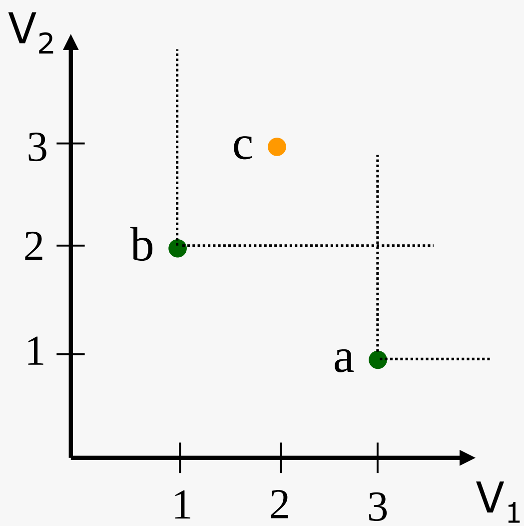
	Probability of gain	
Act	Gain$\geq G_1$	Gain$\geq G_2$
a	0.7	0.6
b	1.0	0.5
c	0.8	0.4

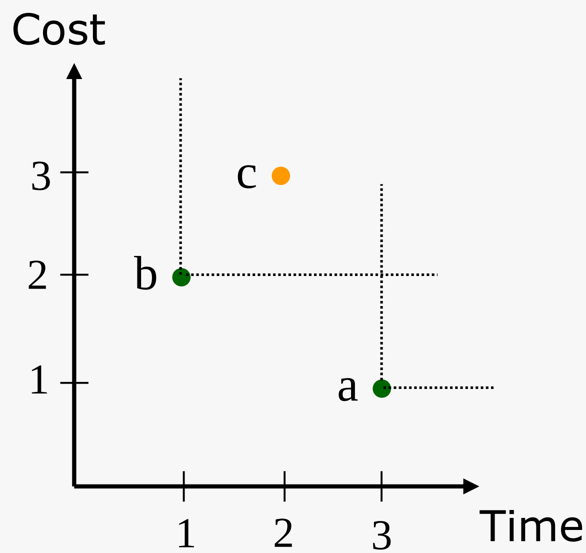
$V_1 : b \succ c \succ a$

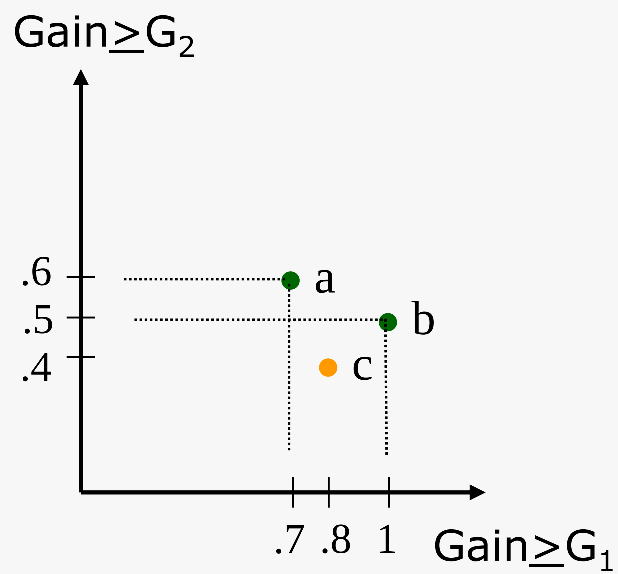
$V_2 : a \succ b \succ c$

● non-dominated
● dominated

$G_1 < G_2$

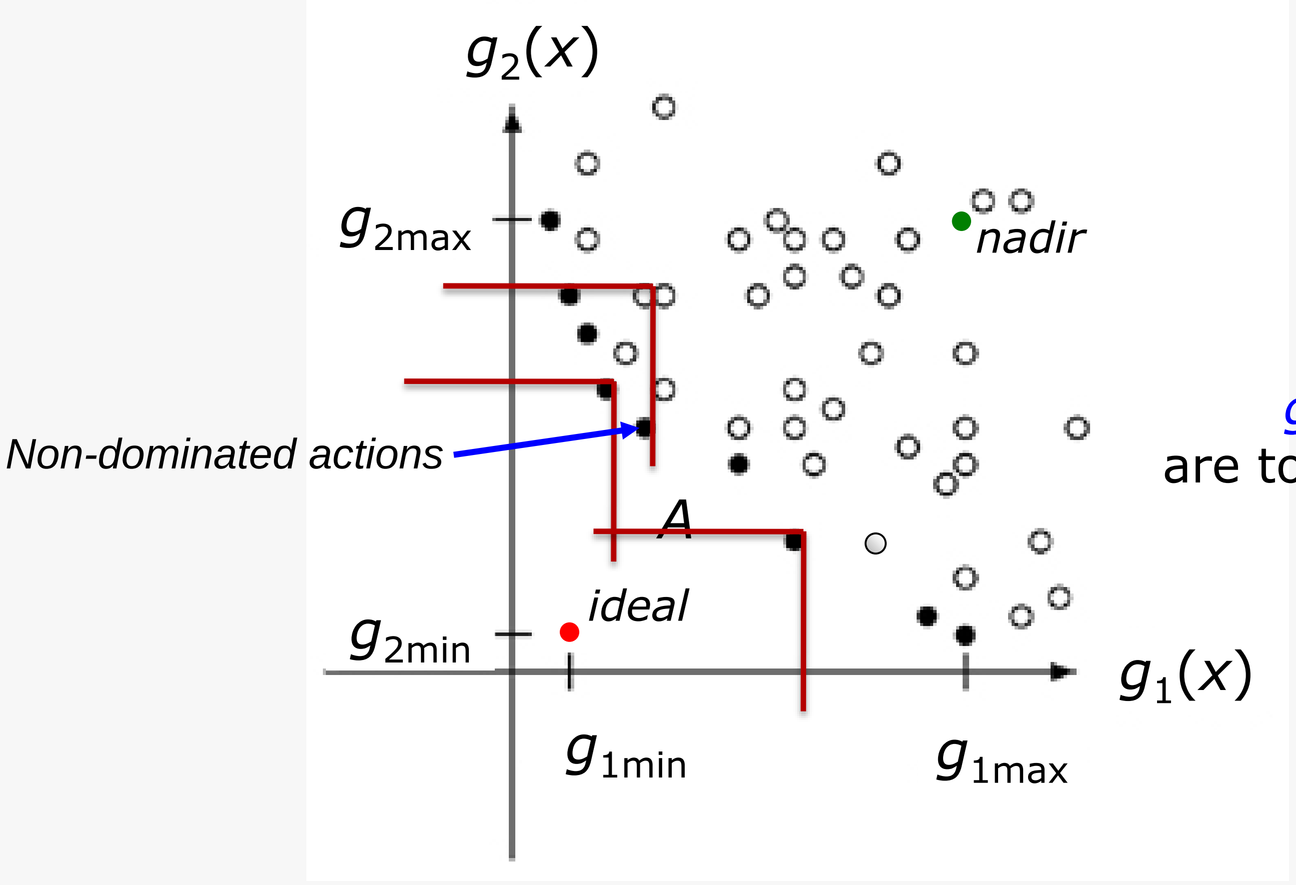






Dominance relation

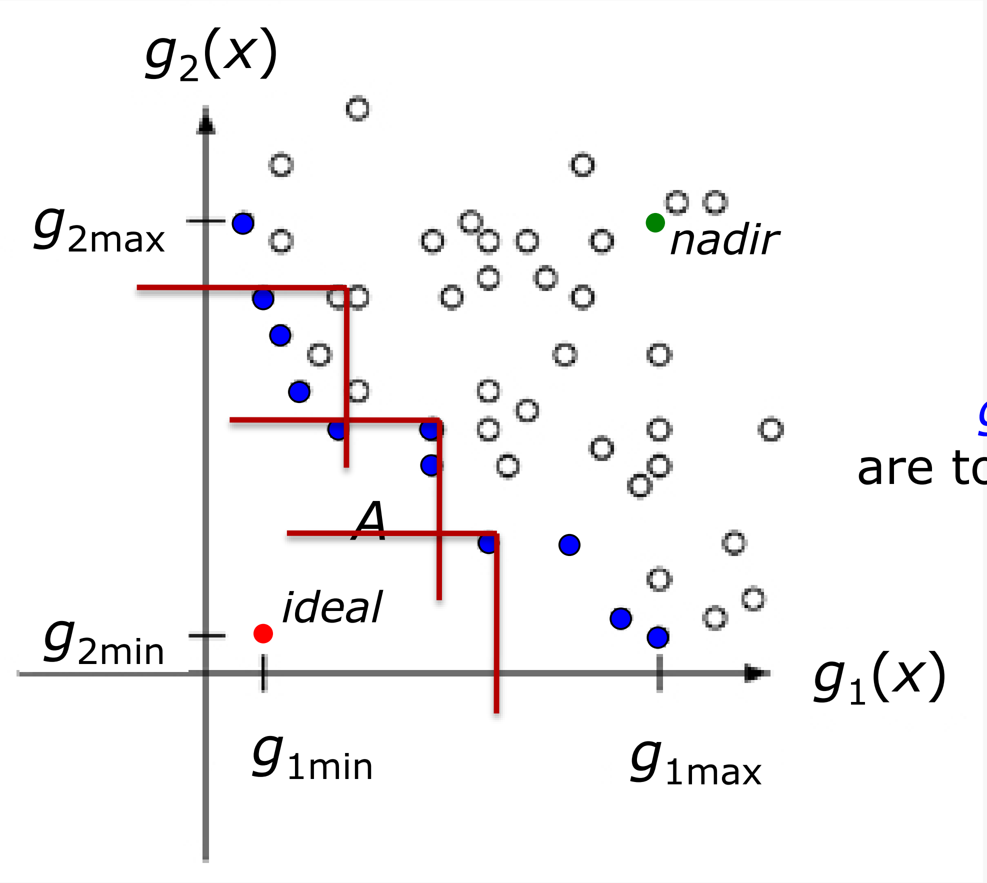
- Action $a \in A$ is non-dominated (Pareto-optimal) if and only if there is no other action $b \in A$ such that $g_i(b) \succeq g_i(a)$, $i=1,\ldots,n$, and on at least one dimension $j=\{1,\ldots,n\}$, $g_j(b) \succ g_j(a)$



g_1 and g_2 are to be minimized

Dominance relation

- Action $a \in A$ is weakly non-dominated (weakly Pareto-optimal) if and only if there is no other action $b \in A$ such that $g_i(b) \succ g_i(a)$, $i=1,\ldots,n$,



g_1 and g_2 are to be minimized

Enriching dominance relation – preference modeling

- Dominance relation is too poor – it leaves many actions non-comparable

1

- One can „enrich" the dominance relation, using preference information elicited from the DM

2

- Preference information is an input to learn/build a preference model that aggregates the vector evaluations of objects

- The preference model induces a preference relation in set *A*, richer than the dominance relation (the elements of *A* become more comparable)

3

- A proper exploitation of the preference relation in *A* leads to a recommendation in terms of choice, classification or ranking

- We will concentrate on **Multi-Criteria Decision Aiding (MCDA)**, i.e., dimension = criterion

Multi-criteria approach over mono-criterion approach (Roy, 1985)

- It facilitates taking account of a broad spectrum of points of view liable to structure a decision making process for all the relevant actors
- By making a family of criteria explicit, it preserves the original concrete meaning of the corresponding evaluations for each actor, without resorting to any fabricated conversion
- It clears the way for a discussion on the respective roles that each criterion may play during the decision aiding process, e.g., weight, veto, aspiration level, rejection level

Can the MCDA always be totally objective? No, because…

- The borderline between <u>what is</u> and <u>what is not</u> feasible is often fuzzy in the real decision-making process
- Preferences of a DM are very seldom clearly formed: firm convictions evolve in a nebula of uncertainty, half-held beliefs, or even conflict and contradiction (the more in multi-actor context)
- Any interaction and questioning between the analyst and the DM may have several unpredictable or imperceptible effects
- Much of the data is imprecise, uncertain or ill-defined
- It is impossible to say that a decision is good or bad only by referring to a mathematical model
- The organisational, pedagogical and cultural aspects of the entire decision-aiding process also contribute to its quality and success

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All this makes that Decision Aiding is
Art & Science
and includes an inherent subjective component

- It is impossible to say that a decision is good or bad only by referring to a mathematical model
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What is a criterion ?

- Criterion is a real-valued function g_i defined on A, reflecting a value of each action from a particular point of view, such that in order to compare any two actions $a,b \in A$ from this point of view it is sufficient to compare two values: $g_i(a)$ and $g_i(b)$, called performances
- Scales of criteria:
  - Ordinal scale – only the order of values matters; a distance in ordinal scale has no meaning of intensity, so one cannot compare differences of performances (e.g. school marks, customer satisfaction, earthquake scales)
  - Cardinal scales – a distance in ordinal scale has a meaning of intensity:
    - Interval scale – „zero" in this scale has no absolute meaning, but one can compare differences of evaluations (e.g. Fahrenheit scale)
    - Ratio scale – „zero" in this scale has an absolute meaning, so a ratio of evaluations has a meaning (e.g. weight, Kelvin scale)

From a single criterion to a family of criteria

- In MCDA, knowing which type of scale we are working with is critical to be sure that its degrees are used in a meaningful way
- The building of the family *F* of criteria is an important step in MCDA
- In the DA process, the role of family *F* is to:
  - serve for communication and discussussion in the decision process
  - build the convictions and the feeling of satisfaction/dissatisfaction
  - contribute to rendering the decision legitimate

Family of criteria should satisfy logical requirements (be consistent)

- A family of criteria $F=\{g_1,...,g_n\}$ is consistent if it is:
  - Exhaustive – if two actions have the same performances on all criteria, then they have to be indifferent, i.e.

    if for any $a,b \in A$, there is $g_i(a) \sim g_i(b)$, $i=1,\ldots,n$, then $a \sim b$
  - Monotonic – if action a is preferred to action b ($a \succ b$), and there is action c, such that $g_i(c) \succeq g_i(a)$, $i=1,\ldots,n$, then $c \succ b$
  - Non-redundant – elimination of any criterion from the family F should violate at least one of the above properties

  [$\sim$ indifference (I), $\succ$ strict preference (P), $\succeq$ weak preference (S)]
- None of the above requirements implies that the criteria of F must be independent

Main sources of imperfect knowledge and ill determination (Roy 1985)

- Imperfect knowledge & ill determination about the decision process make that:
- DA carries a non-avoidable part of arbitrariness and ignorance that has an impact on:
  - the way the problem is addressed
  - the model that is built
  - the data that is acquired for an operational model
  - the way results are obtained and analyzed
- Main sources of imperfect knowledge and ill determination are closely linked to some of the limitations to objectivity, namely:

Main sources of imperfect knowledge and ill determination (Roy 1985)

1. Some phenomena, factual quantities or qualities are imprecise, uncertain and, more generally, poorly understood or ill determined

2. The decision aiding process will be carried out in a real life context that may not correspond exactly to the model on which the decision aiding is based (*the map is not the territory*)

3. The system of values used for evaluating the feasibility and relative interest of diverse potential actions is usually fuzzy, incomplete and influenceable

4. Hesitation of the DM, instability of her preferences, absence of some hardly expressible criteria in the family *F* (leads to inversion of dominance)

Main tools for dealing with imperfect knowledge and ill determination

- *Probability theory*: used for instance in the multi-attribute utility theory (MAUT) and, more generally, for building criteria
- *Indifference and preference thresholds*: used for instance in ELECTRE methods for working with pseudo-criteria
- *Possibility theory and fuzzy logic*
- *Rough sets and multi-valued logics*
- *Statistics*: used in data analytics and preference learning
- Regardless of the tools, the analyst must seek to obtain robust solutions and/or robust conclusions (valid in the whole range of ignorance)

Aggregation of multi-criteria evaluations

- Three families of **preference modelling (aggregation) methods**:
  - Multiple Attribute Utility Theory (MAUT) using a value function,

    e.g. $U(a)=\sum_{i=1}^{n}k_i g_i(a)$, $U(a)=\sum_{i=1}^{n}u_i[g_i(a)]$, Choquet/Sugeno integral
  - Outranking methods using an outranking relation S

    $a\ S\ b$ = „a is at least as good as b"
  - Decision rule approach using a set of decision rules

e.g. "If $g_i(a)\succeq r_i$ & $g_j(a)\succeq r_j$ & ... $g_h(a)\succeq r_h$, then $a \rightarrow$ *Class t* or higher"

"If $g_i(a)\succeq_i^{\geq h(i)} g_i(b)$ & $g_j(a)\succeq_j^{\geq h(j)} g_j(b)$ & ... $g_p(a)\succeq_p^{\geq h(p)} g_p(b)$, *then aSb*"

  - Decision rule model is the most general of all three

R.Słowiński, S.Greco, B.Matarazzo: Axiomatization of utility, outranking and decision-rule preference models for multiple-criteria classification problems under partial inconsistency with the dominance principle, *Control and Cybernetics*, 31 (2002) no.4, 1005-1035

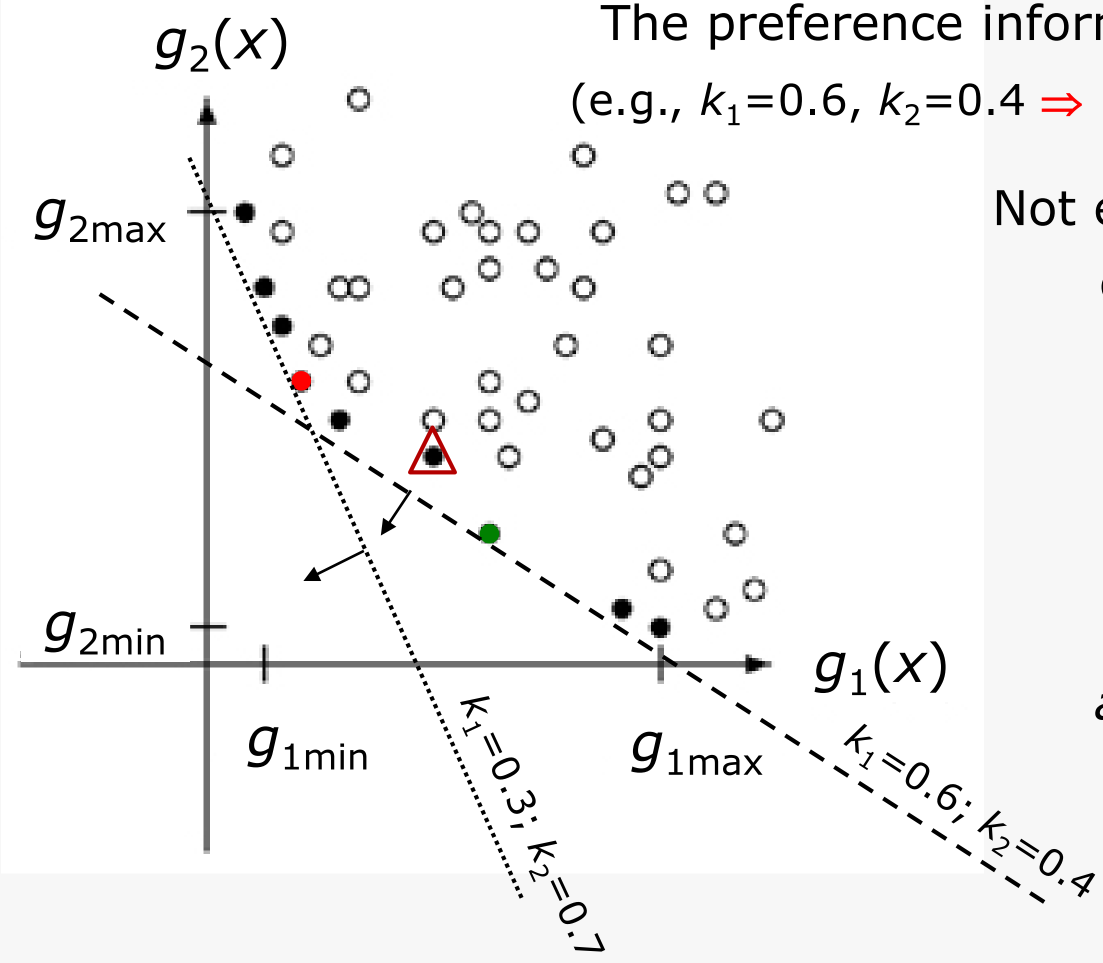
Preference modeling by dominance-based decision rules

- Dominance-based *„if…, then…"* decision rules are the only aggregation operators that:
  - give account of most complex interactions among criteria,
  - are non-compensatory,
  - accept ordinal evaluation scales and do not convert ordinal evalautions into cardinal ones,
- Rules identify values that drive DM's decisions – each rule is a scenario of a causal relationship between evaluations on a subset of criteria and a comprehensive judgment

S.Greco, B.Matarazzo, R.Słowiński: Axiomatic characterization of a general utility function and its particular cases in terms of conjoint measurement and rough-set decision rules. *European J. of Operational Research*, 158 (2004) 271-292

Aggregation using a „weighted-sum" utility function U

- The most intuitive model: $U(a) = \sum_{i=1}^{n} k_i g_i(a)$



The preference information = *trade-off weights* k_i

(e.g., k_1=0.6, k_2=0.4 $\Rightarrow$ 1.5 of g_2 compensates 1.0 of g_1)

Not easy to elicit and, moreover, criteria must be independent

Easy exploitation of the preference relation induced by U in A

$$a \succeq b \Leftrightarrow \sum_{i=1}^{n} k_i g_i(a) \geq \sum_{i=1}^{n} k_i g_i(b)$$

Other properties of a „weighted sum"

- The weights and thus the trade-offs are constant for the whole range of variation of criteria values
- The „weighted sum" and, more generally, an additive utility function requires that criteria are independent in the sense of preferences, i.e., $u_i(a)=g_i \times k_i$ does not change with a change of $g_j(a)$, $j=1,\ldots,n$; $j \neq i$
- In other words, this model cannot represent the following preferences:

Car	(↓) Gas consumption	(↓) Price	(↑) Comfort
a	5	90	5
b	9	90	9
c	5	50	5
d	9	50	9

$b \succ a$ while $c \succ d$

It requires that:

if $b \succ a$ then $d \succ c$

The property of mutual preferential independence among criteria

- If we have n criteria $g_1,\ldots,g_n$, then we have to assess mutual preferential independence between any subsets of criteria, e.g., for n=5:

$$[\alpha, x, \beta, y, \gamma] \succeq [\alpha, w, \beta, z, \gamma] \Leftrightarrow [\delta, x, \varepsilon, y, \phi] \succeq [\delta, w, \varepsilon, z, \phi]$$

- In general, there are $\binom{n}{i}$ subsets of i criteria, i=1,…,n-1;

  e.g., $\binom{n}{1} = n$ assessments of preferential independence must be made for individual criteria g_i with their complements

- In total, one must make $\sum_{i=1}^{n-1}\binom{n}{i} = 2^n - 2$ pairwise assesments of preferential independence; e.g., for n=5, 30 checks

- Indifference curves of any pair of criteria are unaffected by the fixed levels of remaining criteria

Why additive value function needs preferential indepenence?

- For $n \geq 3$, a necessary and sufficient condition for a proper representation of preferences by an additive value function U is mutual preferential independence among criteria g_i , $i=1,\ldots,n$:

  assume the following preference relation

$$[\alpha, x, \beta] \succ [\alpha, y, \beta], \text{ i.e., } U(\alpha, x, \beta) > U(\alpha, y, \beta)$$

$$\text{or } u_1(\alpha)+u_2(x)+u_3(\beta) > u_1(\alpha)+u_2(y)+u_3(\beta)$$

  changing α,β by γ,δ we must have the same inequality, so

$$u_1(\alpha)+u_2(x)+u_3(\beta) > u_1(\alpha)+u_2(y)+u_3(\beta)$$

$$\Leftrightarrow$$

$$u_1(\gamma)+u_2(x)+u_3(\delta) > u_1(\gamma)+u_2(y)+u_3(\delta)$$

Preference modeling using more genral utility function U

- Additive difference model (Tversky 1969, Fishburn 1991)

$$a \succeq b \Leftrightarrow \sum_{i=1}^{n} \varphi_i \{u_i[g_i(a)] - u_i[g_i(b)]\} \geq 0$$

- Transitive decomposable model (Krantz et al. 1971)

$$a \succeq b \Leftrightarrow f\{u_1[g_1(a)],\ldots,u_n[g_n(a)]\} \geq f\{u_1[g_1(b)],\ldots,u_n[g_n(b)]\}$$

f: $\boldsymbol{R}^n \rightarrow \boldsymbol{R}$, non-decreasing in each argument

- Non-transitive additive model (Bouyssou 1986, Fishburn 1990, Vind 1991)

$$a \succeq b \Leftrightarrow \sum_{i=1}^{n} v_i[g_i(a), g_i(b)] \geq 0$$

v_i: $\boldsymbol{R}^2 \rightarrow \boldsymbol{R}$, $i=1,\ldots,n$, non-decreasing in the first and non-increasing in the second argument

- Non-transitive non-additive model (Fishburn 1992, Bouyssou & Pirlot 1997)

$$a \succeq b \Leftrightarrow f\{v_1[g_1(a), g_1(b)],\ldots,v_n[g_n(a), g_n(b)]\} \geq 0$$

Interaction between criteria and the Choquet integral

Consider the following example [Grabisch 1996] :

- Director of a scientific high school has to evaluate students according to their level in mathematics, physics and literature
- As the school has a scientific profile, mathematics and physics are more important than literature
- However, there is a risk of over-evaluation of students being good in mathematics and physics, because students good in mathematics are usually good also in physics, i.e., there is a **redundancy (or negative synergy)** between mathematics and physics
- Moreover, the director would like to give a bonus to students that, besides mathematics and physics, are also good in literature, i.e., there is a **complementarity (or positive synergy)** between mathematics and physics on one hand and literature on the other

Choquet integral [Choquet 1954]

The Choquet integral replaces the usual weight of weighted sum with a weight for each subset of criteria, e.g.:

- $\mu(\emptyset)=0$,
- $\mu(\{\text{Mathematics}\})= \mu(\{\text{Physics}\})= 0.45$,
- $\mu(\{\text{Literature}\})=0.3$,
- $\mu(\{\text{Mathematics, Physics}\})=0.5$,

  negative interaction (redundancy)
- $\mu(\{\text{Mathematics, Literature}\})=\mu(\{\text{Physics, Literature}\})=0.9$,

  positive interaction (synergy)
- $\mu(\{\text{Mathematics, Physics, Literature}\})=1$.

This permits to take into account the interaction between criteria

Choquet integral – additive vs. non-additive aggregation

- Instead of weights k_i for each criterion $g_i \in F$ in a weighted sum:
  $\mu(G)$ – joint weight of criteria from a subset $G \subseteq F$

- $\mu : 2^F \rightarrow [0, 1]$ – non-additive measure (capacity):
  - $\mu(\emptyset)=0$, $\mu(F)=1$
  - for $G' \subset G \subseteq F$, $\mu(G') \leq \mu(G)$
  - in general, $\mu(G' \cup G) \neq \mu(G') + \mu(G)$
  - positive interaction (synergy): $\mu(G' \cup G) > \mu(G') + \mu(G)$
  - negative interaction (redundancy): $\mu(G' \cup G) < \mu(G') + \mu(G)$

Choquet integral [Choquet 1954]

Given evaluations on n criteria (gain-type, with the same scale):

$$g_1,\ldots,g_n \text{ with } g_i \geqslant 0, \quad \forall i = 1,\ldots,n,$$

the Choquet integral of $(g_1,\ldots,g_n)$ is computed as follows:

$$Ch_\mu(g_1,\ldots,g_n) = \sum_{i=1}^{n} \mu(G_i)\left(g_{(i)} - g_{(i-1)}\right)$$

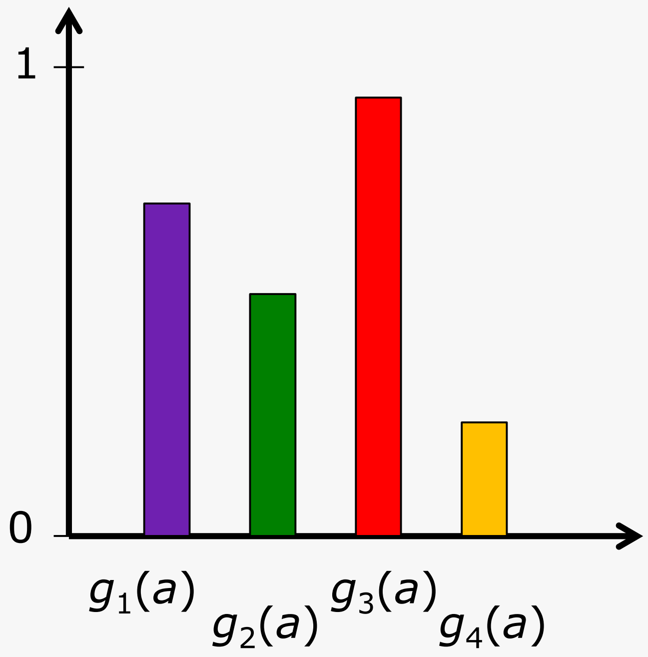
where: $g_{(0)} = 0,$

$(\cdot)$ index permutation: $g_{(i-1)} \leq g_{(i)}, \ i = 1,\ldots,n$

$G_i = \{g_{(i)},\ldots,g_{(n)}\}$

All criteria must have the same cardinal evaluation scale

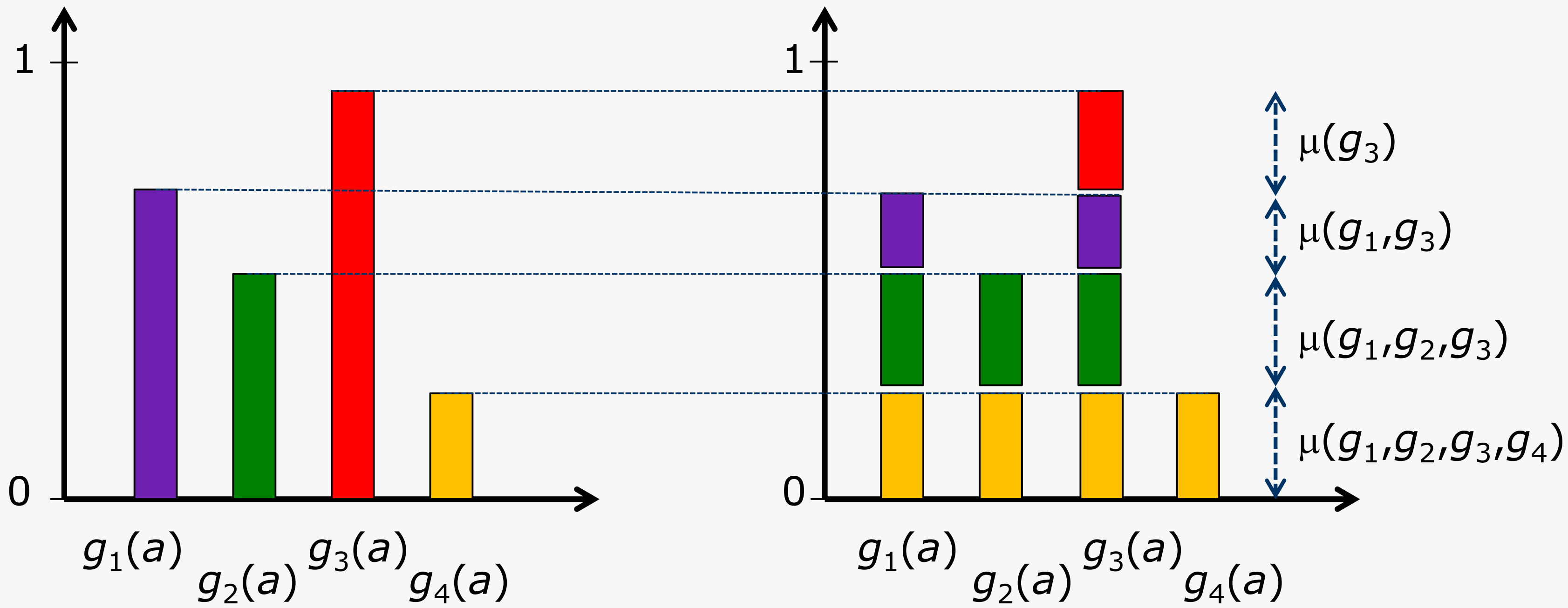
Weighted sum vs. discrete Choquet integral



Weighted sum:

$$U(a) = \sum_{i=1}^{n} k_i \, g_i(a) = \sum_{i=1}^{n} \mu(\{g_i\}) \, g_i(a)$$

Weighted sum vs. discrete Choquet integral



Weighted sum:

$$U(a) = \sum_{i=1}^{n} k_i \, g_i(a) = \sum_{i=1}^{n} \mu(\{g_i\}) \, g_i(a)$$

Choquet integral:

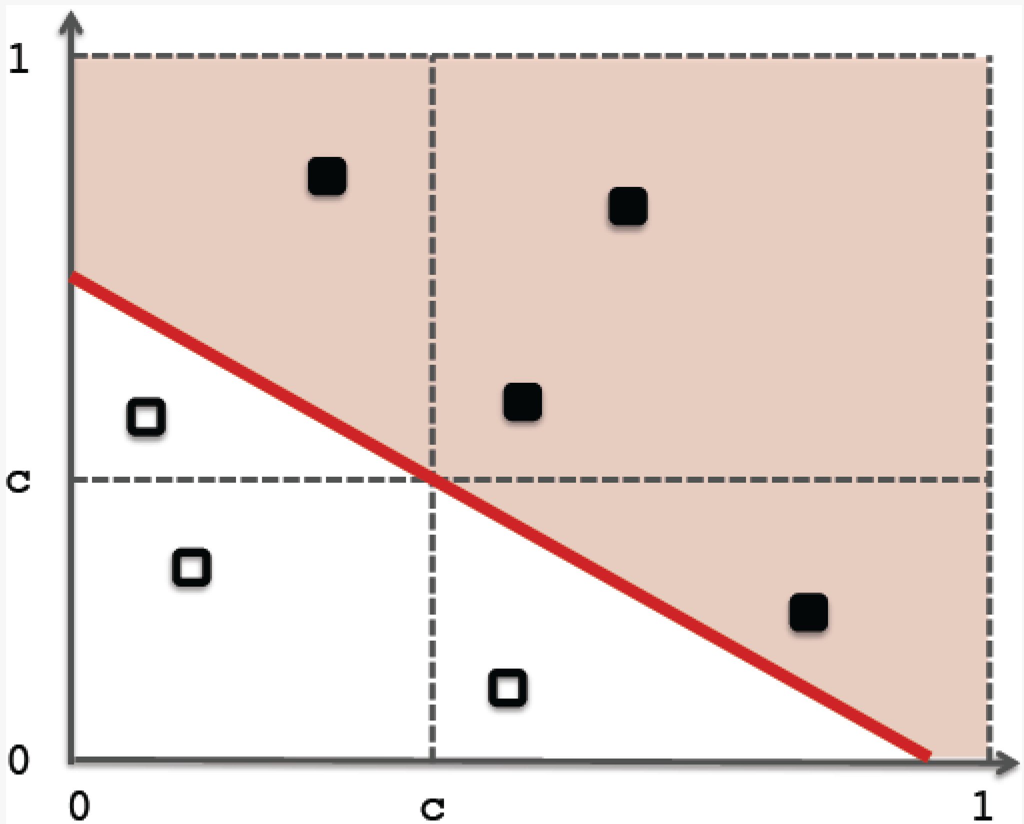
$$U(a) = \sum_{i=1}^{n} \mu(G_i) \left(g_{(i)}(a) - g_{(i-1)}(a)\right)$$

where $_{(\cdot)}$ is a permutation of $\{1,\ldots,n\}$, such that $0 \le g_{(1)}(a) \le g_{(2)}(a) \le \ldots \le g_{(n)}(a)$,

$G_i = \{g_{(i)},\ldots,g_{(n)}\}$, $g_{(0)} = 0$; $g_4(a) \le g_2(a) \le g_1(a) \le g_3(a) \rightarrow (1)=4, (2)=2, (3)=1, (4)=3$

Isoquants of the Choquet integral for two criteria – special cases

- Weighted sum (linear additive) – no interaction

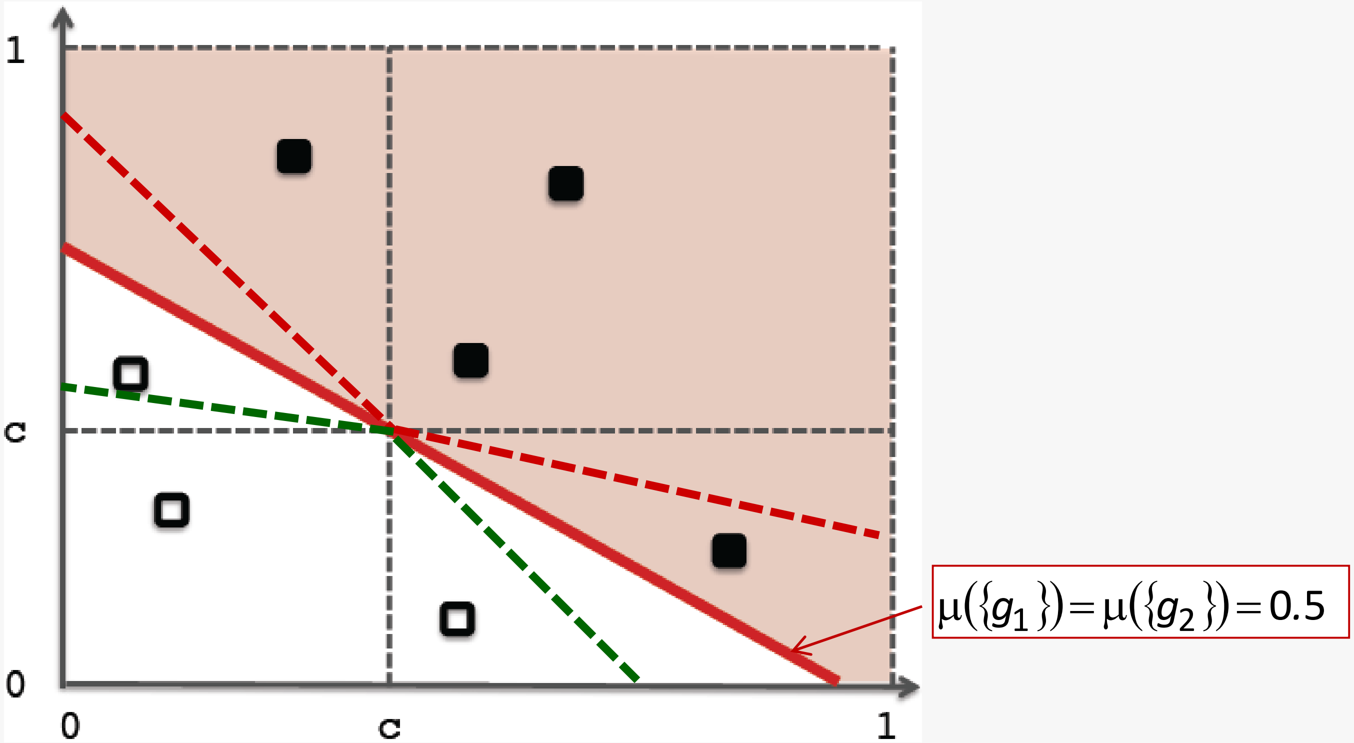


$$U(a) = \mu(\{g_1\})g_1(a) + \mu(\{g_2\})g_2(a) \geq c, \quad \mu(\{g_1\}) + \mu(\{g_2\}) = 1$$

Isoquants of the Choquet integral for two criteria – special cases

- Ordered Weighted Average (OWA) – positive interaction if $\mu(\{g_1\})=\mu(\{g_2\})<0.5$

  – negative interaction if $\mu(\{g_1\})=\mu(\{g_2\})>0.5$

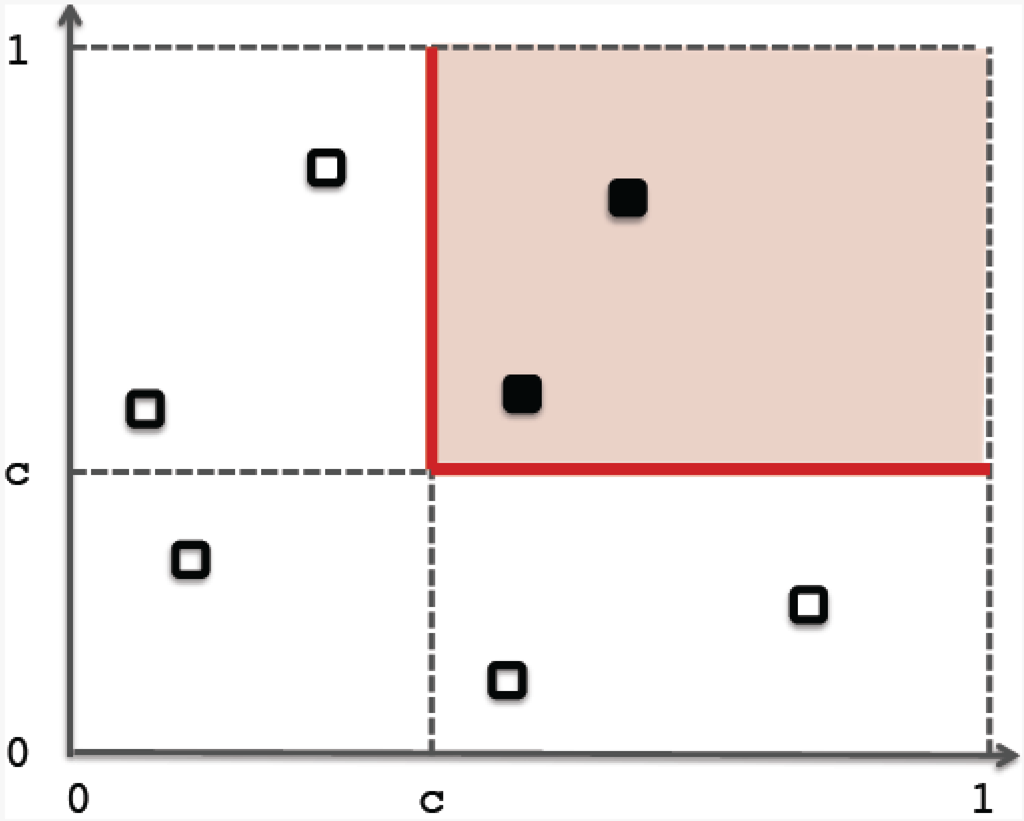


$$U(a) = k_1 g_{(1)}(a) + k_2 g_{(2)}(a) = (1-\mu(\{g_1\}))\, g_{(1)}(a) + \mu(\{g_2\})\, g_{(2)}(a) \geq c,$$

with $\mu(\{g_1\})=\mu(\{g_2\})$ and $\mu(\{g_1,g_2\})=1$

Isoquants of the Choquet integral for two criteria – special cases

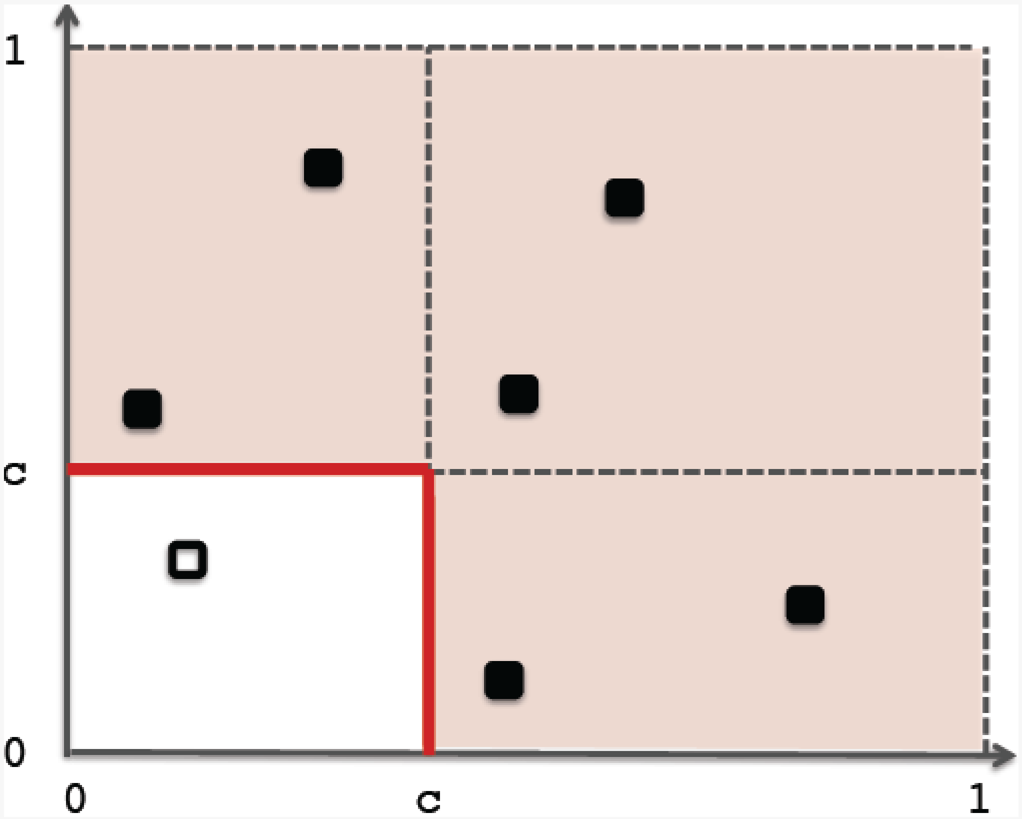
- Min – maximum negative interaction (redundancy)



$$U(a) = min\{g_1(a), g_2(a)\} \geq c, \ \mu(\{g_1\}) = 0, \ \mu(\{g_2\}) = 0, \ \mu(\{g_1, g_2\}) = 1$$

Isoquants of the Choquet integral for two criteria – special cases

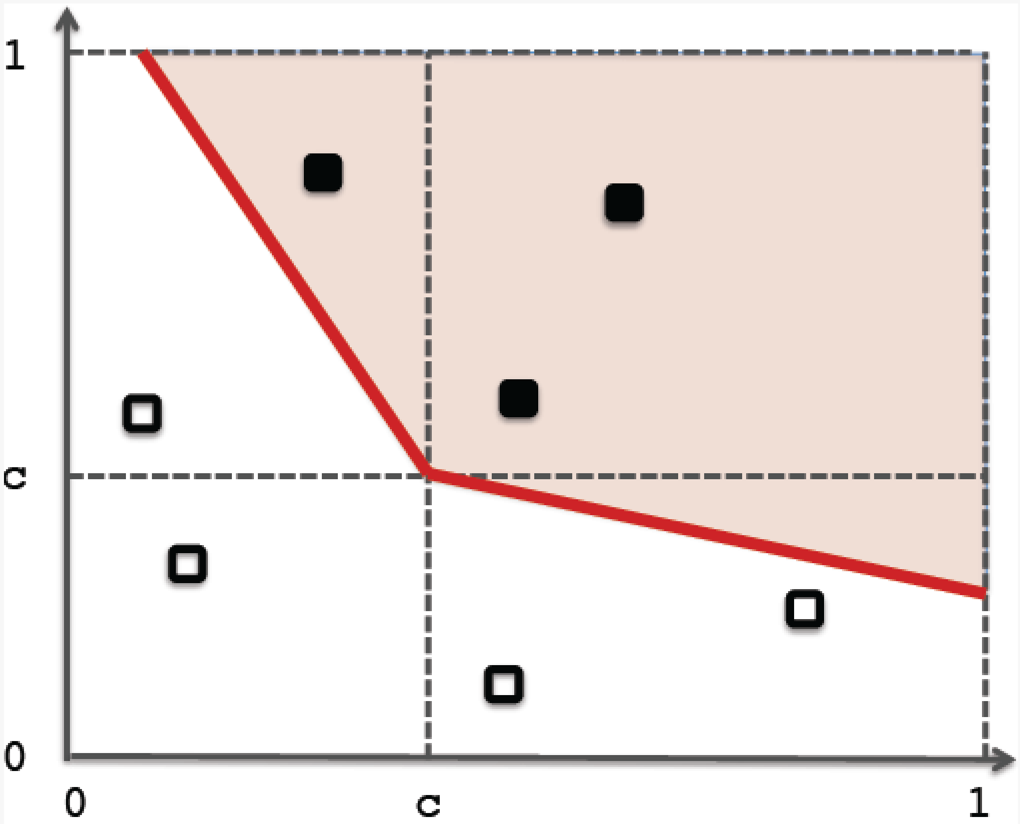
- Max – maximum positive interaction (synergy)



$$U(a) = max\{g_1(a), g_2(a)\} \geq c,\ \ \mu(\{g_1\}) = 1,\ \mu(\{g_2\}) = 1,\ \mu(\{g_1, g_2\}) = 1$$

Isoquants of the Choquet integral for two criteria – special cases

- 2-additive Choquet – positive interaction (synergy)



$$U(a) = \mu(\{g_1\})g_1(a) + \mu(\{g_2\})g_2(a) + [\mu(\{g_1, g_2\}) - \mu(\{g_1\}) - \mu(\{g_2\})]\,min\{g_1(a), g_2(a)\} \geq c$$

positive interaction when $\mu(\{g_1, g_2\}) > \mu(\{g_1\}) + \mu(\{g_2\})$

Isoquants of the Choquet integral for two criteria – special cases

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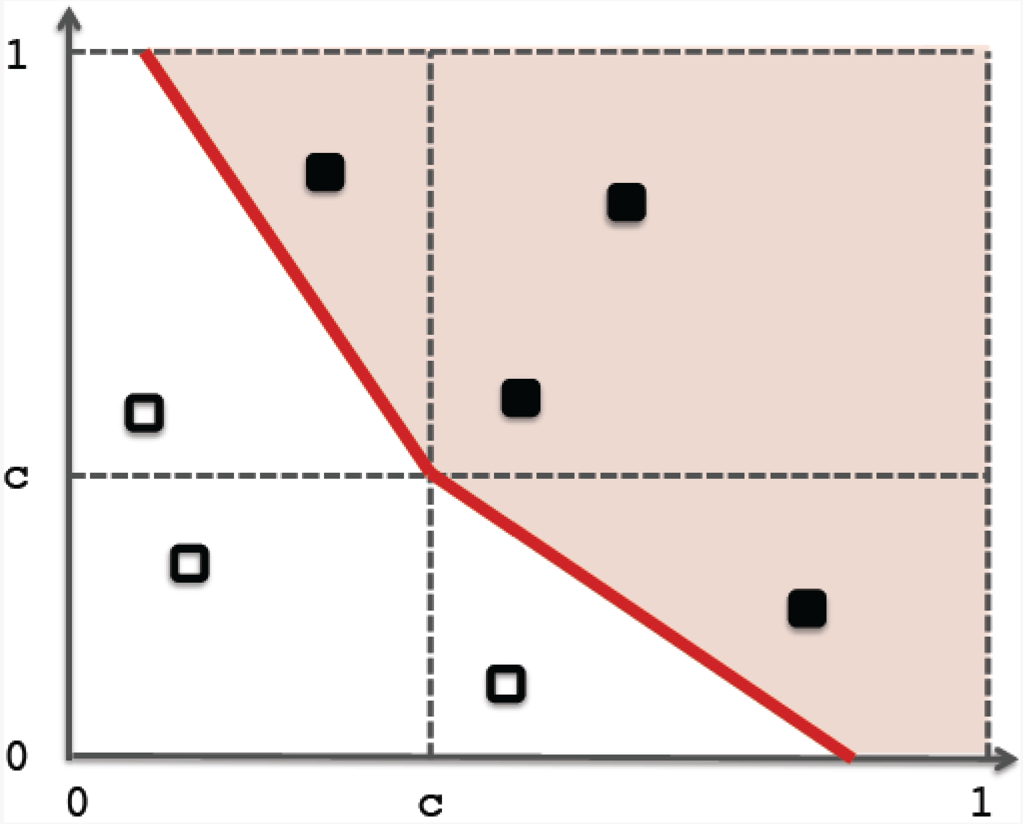


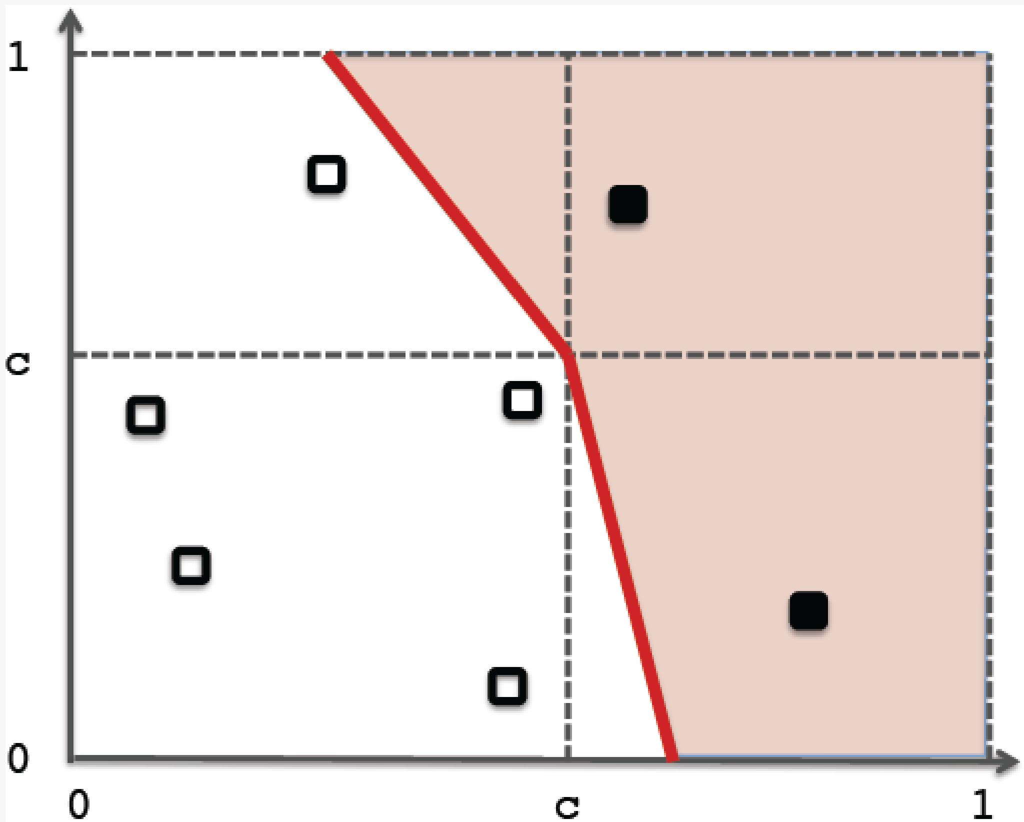
greater capacity=weight of g_1 than before

$$U(a) = \mu(\{g_1\})g_1(a) + \mu(\{g_2\})g_2(a) + [\mu(\{g_1, g_2\}) - \mu(\{g_1\}) - \mu(\{g_2\})]\,min\{g_1(a), g_2(a)\} \geq c$$

positive interaction when $\mu(\{g_1, g_2\}) > \mu(\{g_1\}) + \mu(\{g_2\})$

Isoquants of the Choquet integral for two criteria – special cases

- 2-additive Choquet – negative interaction (redundancy)



$U(a) \ = \mu(\{g_1\})g_1(a) + \mu(\{g_2\})g_2(a) + [\mu(\{g_1, g_2\}) - \mu(\{g_1\}) - \mu(\{g_2\})] min\{g_1(a), g_2(a)\} \geq c$

negative interaction when $\mu(\{g_1, g_2\}) < \mu(\{g_1\}) + \mu(\{g_2\})$

Choquet integral - Möbius representation

- By considering the Möbius representation of 2-additive capacity μ:

$$\mu(G) = \sum_{i \in G} m(\{i\}) + \sum_{\{i,j\} \subseteq G} m(\{i,j\}), \ \forall G \subseteq F,$$

  - monotonicity:

$$\begin{cases} m(\{i\}) \geq 0, \ \forall i \in F, \\ m(\{i\}) + \sum_{j \in G} m(\{i,j\}) \geq 0, \ \forall i \in F, \text{ and } \forall G \subseteq F \setminus \{i\}, \ G \neq \varnothing \end{cases}$$

  - normalization:

$$m(\varnothing) = 0, \qquad \sum_{i \in F} m(\{i\}) + \sum_{\{i,j\} \subseteq F} m(\{i,j\}) = 1$$

  - we get:

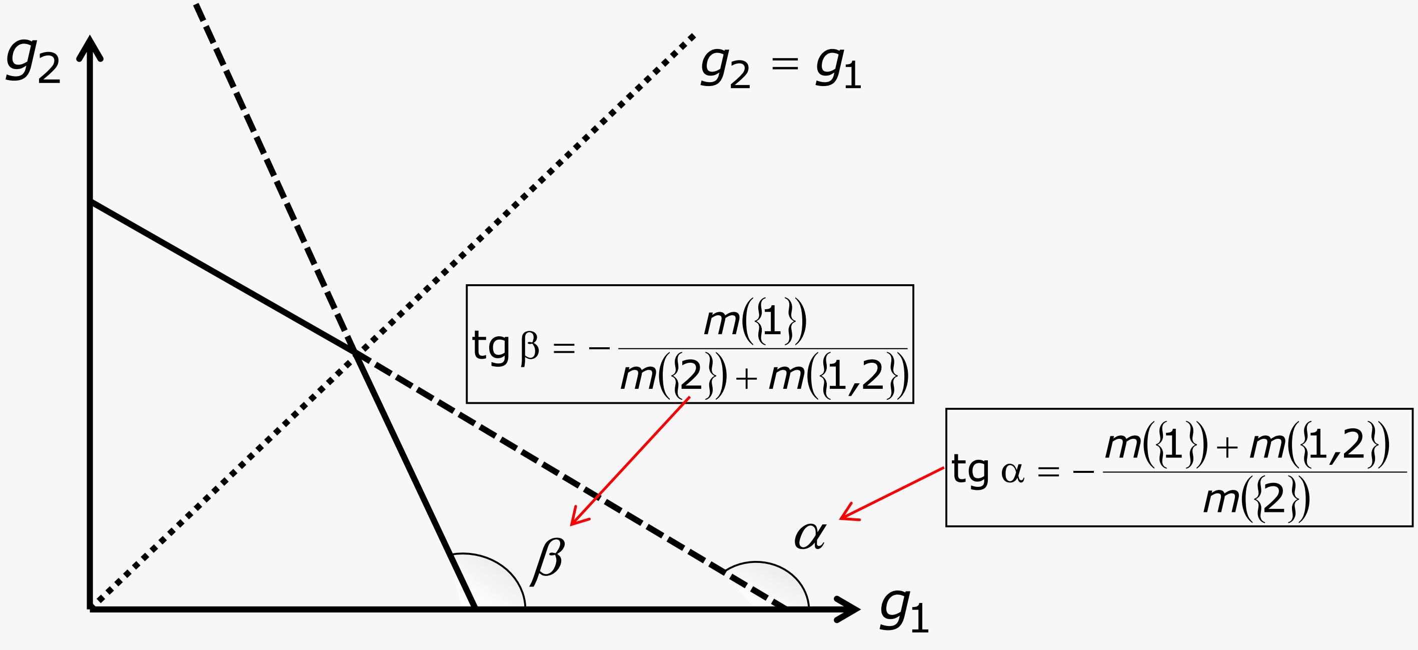
$$Ch_\mu(g_1, \ldots, g_n) = \sum_{i \in F} m(\{i\})\, g_i + \sum_{\{i,j\} \subseteq F} m(\{i,j\}) \min\left\{ g_i, g_j \right\}$$

A particular case of the Choquet integral: n=2

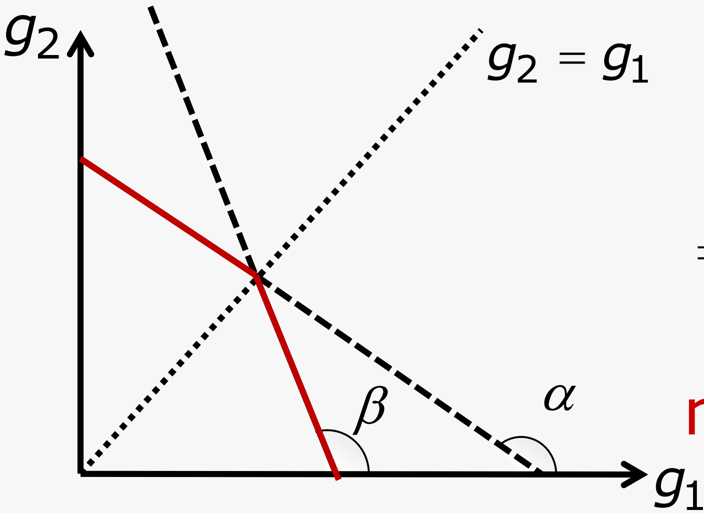
If n=2, then…

$$Ch_\mu(g_1, g_2) = m(\{1\})\, g_1 + m(\{2\})\, g_2 + m(\{1,2\}) \min\{g_1, g_2\} =$$

$$= \begin{cases} (m(\{1\}) + m(\{1,2\}))\, g_1 + m(\{2\})\, g_2 & \text{if } g_1 \le g_2 \\ m(\{1\})\, g_1 + (m(\{2\}) + m(\{1,2\}))\, g_2 & \text{if } g_1 \ge g_2 \end{cases}$$



The Choquet integral isoquants ('wings')



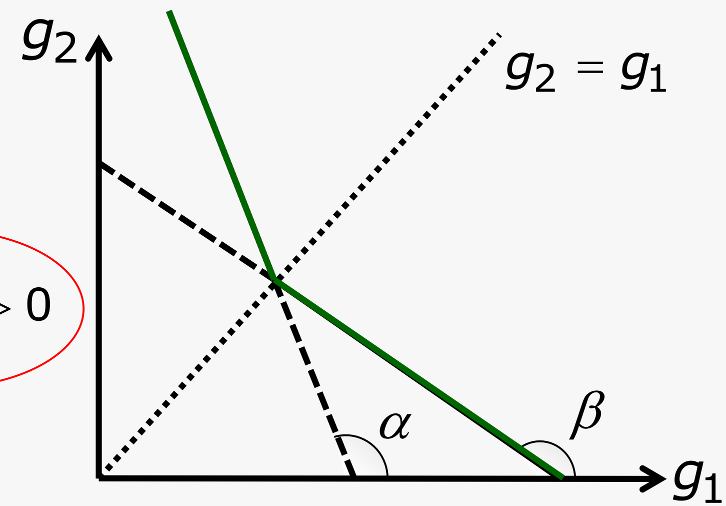
$$\alpha > \beta \Rightarrow \mathrm{tg}\,\alpha > \mathrm{tg}\,\beta \Rightarrow$$

$$\Rightarrow -\frac{m(\{1\}) + m(\{1,2\})}{m(\{2\})} > -\frac{m(\{1\})}{m(\{2\}) + m(\{1,2\})} \Rightarrow m(\{1,2\}) < 0$$

negative interaction - redundancy

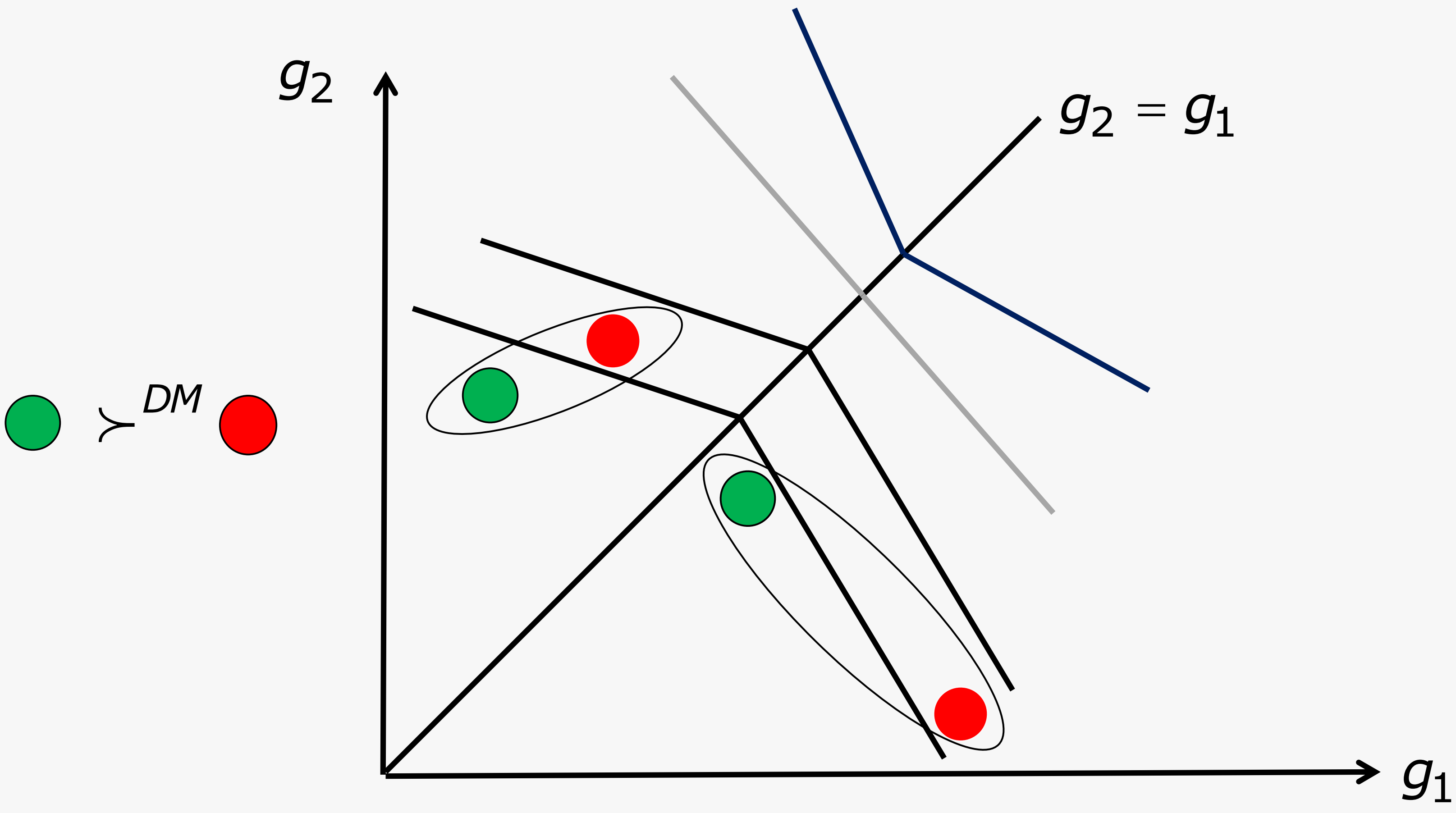
$$\alpha < \beta \Rightarrow \mathrm{tg}\,\alpha < \mathrm{tg}\,\beta \Rightarrow$$

$$-\frac{m(\{1\}) + m(\{1,2\})}{m(\{2\})} < -\frac{m(\{1\})}{m(\{2\}) + m(\{1,2\})} \Rightarrow m(\{1,2\}) > 0$$

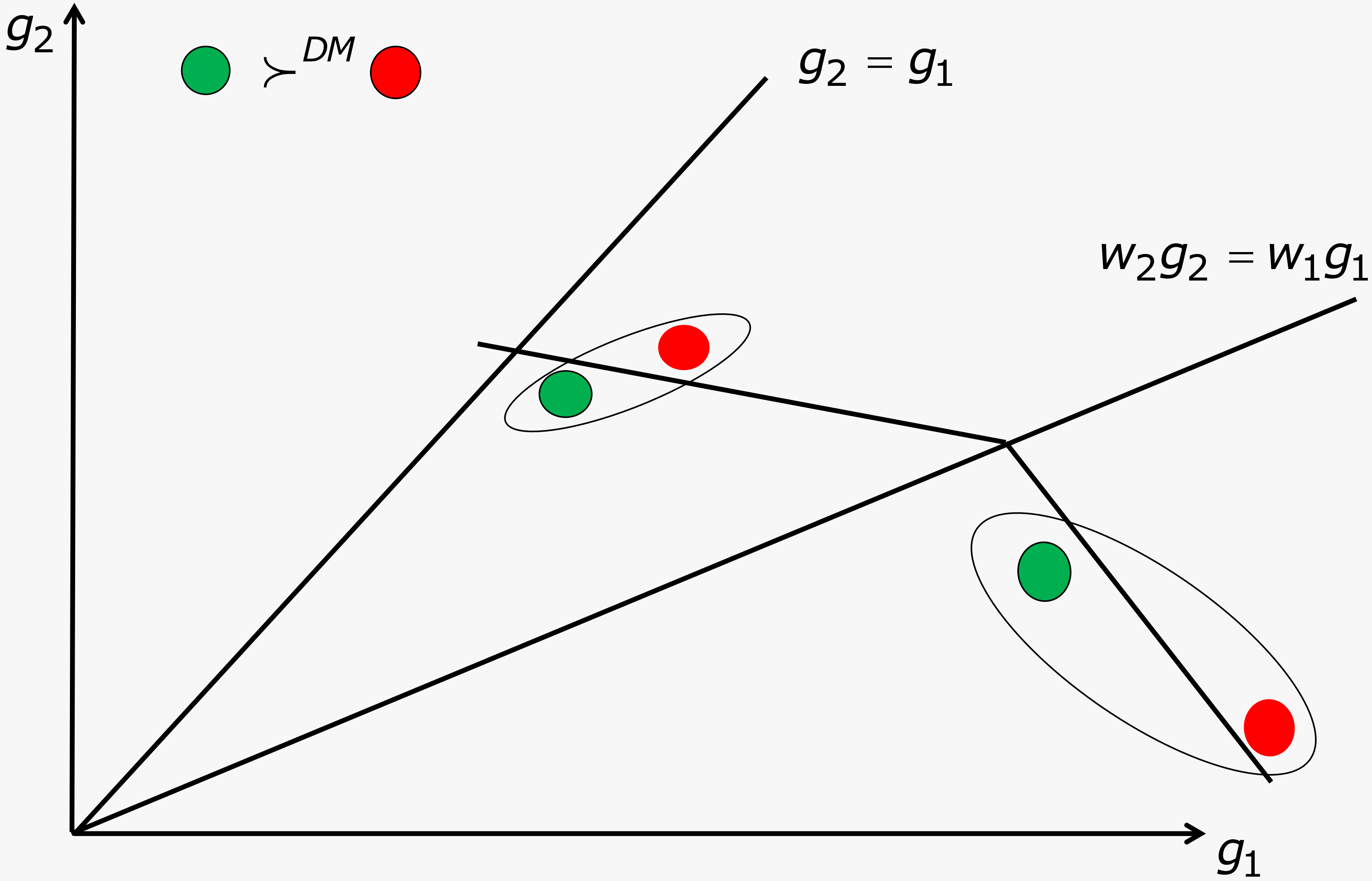


positive interaction - synergy

Graphical interpretation



Scaling of objectives



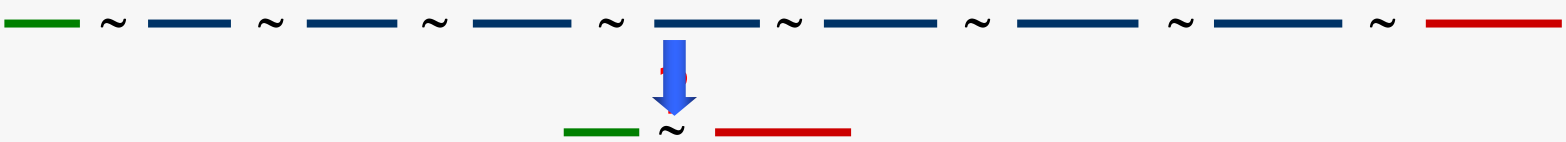
Weak points of the aggregation using utility (value) function

- Utility function distinguishes only 2 possible relations between actions:

  preference relation: $a \succ b \;\Leftrightarrow\; U(a) > U(b)$

  indifference relation: $a \sim b \;\Leftrightarrow\; U(a) = U(b)$

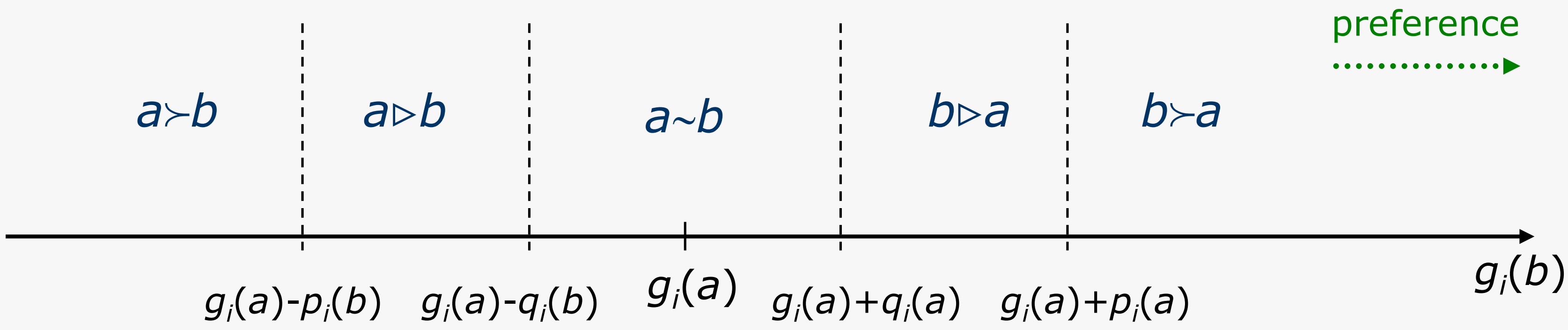
- $\succ$ is asymmetric (antisymmetric and irreflexive) and transitive
- $\sim$ is symmetric, reflexive and transitive
- Transitivity of indifference is troublesome, e.g.



- In consequence, a non-zero indifference threshold q_i is necessary
- An immediate transition from indifference to preference is unrealistic, so a preference threshold $p_i \geq q_i$ and a weak preference relation $\triangleright$ are desirable
- Another realistic situation which is not modelled by U is incomparability, so a good model should include also an incomparability relation „?"

Four basic preference relations and an outranking relation S

- Four basic preference relations are: $\{\sim, \triangleright, \succ, ?\}$



- Criterion with thresholds $p_i(a) \geq q_i(a) \geq 0$ is called pseudo-criterion

- The four basic situations of indifference, strict preference, weak preference, and incomparability are sufficient for establishing a realistic model of Decision Maker's (DM's) preferences

Four basic preference relations and an outranking relation S

- Axiom of limited comparability (Roy 1985):

  Whatever the actions considered, the criteria used to compare them, and the information available, one can develop a satisfactory model of DM's preferences by assigning one, or a grouping of two or three of the four basic situations, to any pair of actions.

- Outranking relation S groups three basic preference relations:

  $S = \{\sim, \rhd, \succ\}$ – reflexive and non-transitive

  aSb means: „action a is at least as good as action b"

- For each couple $a,b \in A$:

$$aSb \wedge \text{non } bSa \Leftrightarrow a \rhd b \vee a \succ b$$

$$aSb \wedge bSa \Leftrightarrow a \sim b$$

$$\text{non } aSb \wedge \text{non } bSa \Leftrightarrow a?b$$

The old challenge and some new aspirations of MCDA

- Aggregation of vector evaluations, i.e., preference modeling:
  - till early 80’s: „model-centric”
    (model first, then preference info in terms of model parameters)
  - since 80’s: more and more „human-centric”
    (PC allowed human-computer interaction – „trial-an-error”)
  - in XXI century: „knowledge driven”
    (more data about human choices;
    holistic preference information first, then model building;
    explanation of past decisions, and prediction of future decisions)

The old challenge and some new aspirations of MCDA

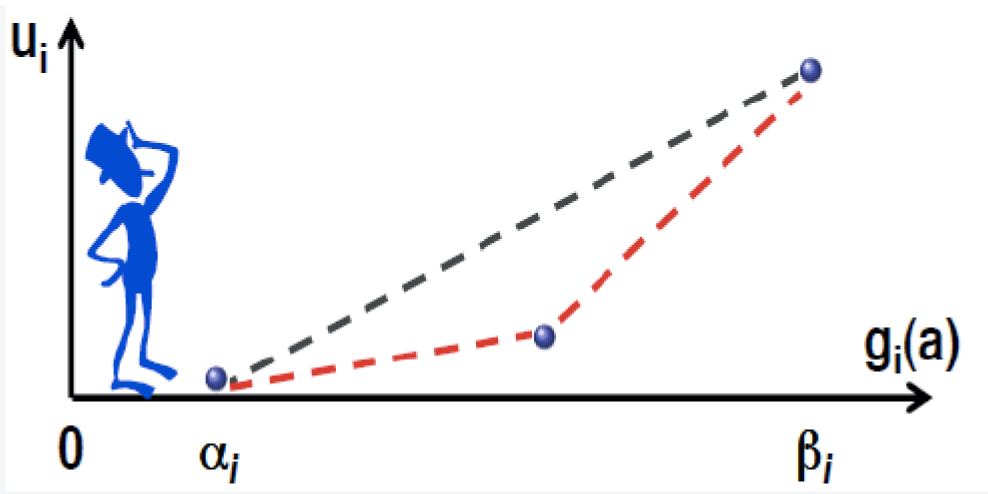
- Focus on „fair" aggregation:
  - Ensure faithful representation of a value system of the DM
  - Act in a constructive and transparent way, in interaction with the DM, so that she elicits preference information reflecting adequately her evolving preferences
  - Handle „imperfect" information: partial, inconsistent, unstable, uncertain,…

Elicitation of preference information by the Decision Maker (DM)

- Direct or indirect ?
- Direct elicitation of numerical values of model parameters by DMs demands much of their cognitive effort:

  weights, indifference, preference and veto thresholds,…

Value function model



substitution rates or shapes
of marginal value functions

P.C.Fishburn (1967): Methods of Estimating Additive Utilities. *Management Science*, 13(7), 435-453
(twenty-four methods
of estimating additive utilities
are listed and classified)

$$U(a) = \sum_{i=1}^{n} w_i g_i(a) \text{ or } \sum_{i=1}^{n} u_i[g_i(a)]$$

Elicitation of preference information by the Decision Maker (DM)

- Indirect elicitation: through holistic judgments, i.e., decision examples
- Decision aiding based on decision examples is gaining importance because:
  - Decision example is a relatively „easy" preference information
  - Decisions can also be observed without active participation of DMs
  - Psychologists confirm that DMs are more confident exercising their decisions than explaining them (J.G.March 1978; P.Slovic 1977)
- Related paradigms:
  - Revealed preference theory in economics (P.Samuelson 1938), is a method of analyzing choices made by individuals: preferences of consumers can be revealed by their purchasing habits
  - Learning from examples in AI/ML (knowledge discovery)
- Conclusion: indirect elicitation of preferences is more user-friendly

Indirect elicitation of preference information by the DM

[TIME=24, COST=56, RISK=75]
$\succ$
[TIME=28, COST=67, RISK=25]

 $\succ$ 

$\succ$

Pairwise preferences between alternatives

characterized by cardinal and/or ordinal features (criteria)

[MATH=18, PHYS=16, LIT=15] $\Rightarrow$ Class „MEDIUM”
[MATH=17, PHYS=16, LIT=18] $\Rightarrow$ Class „GOOD”

Classification examples

A is preferred to **Z** more than **C** is preferred to **K**

Intensity of preference

Alternative **F** should be among **5%** of the best ones

Rank related

Indirect preference information – example of technical diagnostics

- 176 buses (objects)
- 8 symptoms (attributes)
- Decision = technical state:

  3 – good state (in use)

  2 – minor repair

  1 – major repair (out of use)
- Aggregation = finding relationships between symptoms & technical state
- The model explains expert's decisions and supports diagnosis of new buses

Examples:

	MaxSpeed	ComprPressure	Blacking	Torque	SummerCons	WinterCons	OilCons	HorsePower	State
1.	90	2	38	481	21	26	0	145	3
2.	76	2	70	420	22	25	2	110	1
3.	63	1	82	400	22	24	3	101	1
4.	90	2	49	477	21	25	1	138	3
5.	85	2	52	460	21	25	1	130	2
6.	72	2	73	425	23	27	2	112	1
7.	88	2	50	480	21	24	1	140	3
8.	87	2	56	465	22	27	1	135	3
9.	90	2	16	486	26	27	0	150	3
10.	60	1	95	400	23	24	4	96	1
11.	80	2	60	451	21	26	1	125	1
12.	78	2	63	448	21	26	1	120	2
13.	90	2	26	482	22	24	0	148	3
14.	62	1	93	400	22	28	3	100	1
15.	82	2	54	461	22	26	1	132	2
16.	65	2	67	402	22	23	2	103	1
17.	90	2	51	468	22	26	1	138	3
18.	90	2	15	488	20	23	0	150	3
19.	76	2	65	428	27	33	2	116	1
20.	85	2	50	454	21	26	1	129	2
21.	85	2	58	450	22	25	1	126	2
22.	88	2	48	458	22	25	1	130	3
23.	60	1	90	400	24	28	4	95	1
24.	64	2	71	420	23	25	2	105	1
25.	75	2	64	432	22	25	1	114	2
26.	74	2	64	420	21	25	1	110	2
27.	68	2	70	400	22	26	2	100	1

Attributes: 9 of 10 | Examples: 76 | Decision: State | Missing Values: No

Indirect preference information – „Thierry's choice"

(data from [Bouyssou et al. 2006])

- reference actions ranked by the DM: $11 \succ 3 \succ 13 \succ 9 \succ 14$

Pairwise Comparison Table (PCT):

	obj1	obj2	diff_price	diff_accel	diff_pick_up	diff_brakes	diff_road_h	relation
1.	11	11	0	0	0	0	0	S
2.	11	3	564	-0,7	-0,1	-0,33	0,25	S
3.	11	13	318	-1,9	-2,1	0,67	1,5	S
4.	11	9	-2263	-1,1	0,1	0,33	1	S
5.	11	14	-3797	-0,6	-1,9	0,33	0,5	S
6.	3	11	-564	0,7	0,1	0,33	-0,25	Sc
7.	3	3	0	0	0	0	0	S
8.	3	13	-246	-1,2	-2	1	1,25	S
9.	3	9	-2827	-0,4	0,2	0,66	0,75	S
10.	3	14	-4361	0,1	-1,8	0,66	0,25	S
11.	13	11	-318	1,9	2,1	-0,67	-1,5	Sc
12.	13	3	246	1,2	2	-1	-1,25	Sc
13.	13	13	0	0	0	0	0	S
14.	13	9	-2581	0,8	2,2	-0,34	-0,5	S
15.	13	14	-4115	1,3	0,2	-0,34	-1	S
16.	9	11	2263	1,1	-0,1	-0,33	-1	Sc
17.	9	3	2827	0,4	-0,2	-0,66	-0,75	Sc
18.	9	13	2581	-0,8	-2,2	0,34	0,5	Sc
19.	9	9	0	0	0	0	0	S
20.	9	14	-1534	0,5	-2	0	-0,5	S
21.	14	11	3797	0,6	1,9	-0,33	-0,5	Sc
22.	14	3	4361	-0,1	1,8	-0,66	-0,25	Sc
23.	14	13	4115	-1,3	-0,2	0,34	1	Sc
24.	14	9	1534	-0,5	2	0	0,5	Sc
25.	14	14	0	0	0	0	0	S

$S = \succeq$

$Sc = \text{not} \succeq$

The model explains DM's preferences & supports comparison of new cars

Representation of preferences

- Scoring function: $U(a) = \sum_{i=1}^{n} k_i g_i(a)$ or $U(a) = \sum_{i=1}^{n} u_i[g_i(a)]$

  like in *MAUT*, *Discriminant Analysis*, *Logistic Regression* or *Perceptron*,

  e.g. $\boxed{U(a) = 0.21 \times g_{\text{Speed}}(a) + 0.03 \times g_{\text{Compr}}(a) + \ldots + 0.18 \times g_{Power}(a) = 0.45}$

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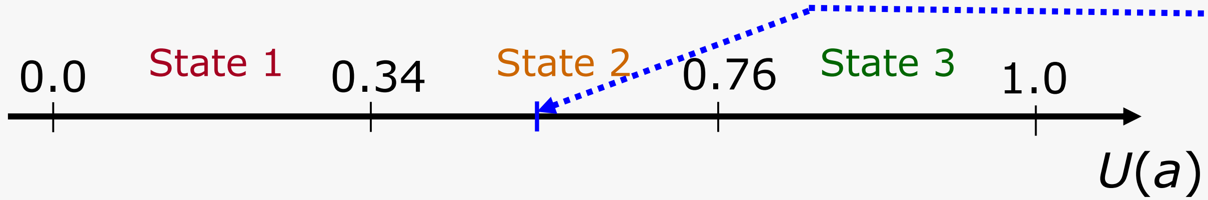

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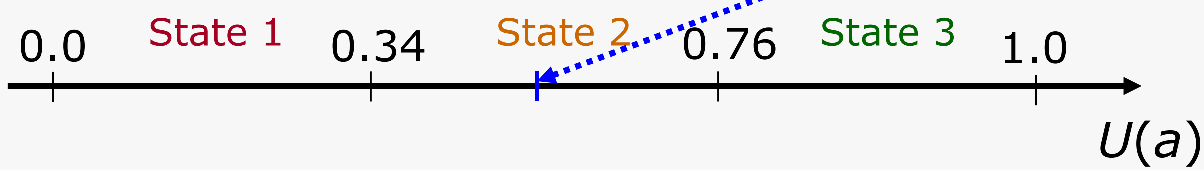

- Decision rules or trees,

  like in *Artificial Intelligence*, *Data Minining* or *Learning from Examples*,

  e.g. *if* OilCons $\leq$ 1 & WinterGasCons $\leq$ 25, *then* State $\succeq$ 2

  if MaxSpeed $\leq$ 85 & WinterGasCons $\geq$ 25, *then* State $\preceq$ 2

Representation of preferences

- Scoring function: $U(a)=\sum_{i=1}^{n}k_i g_i(a)$ or $U(a)=\sum_{i=1}^{n}u_i[g_i(a)]$

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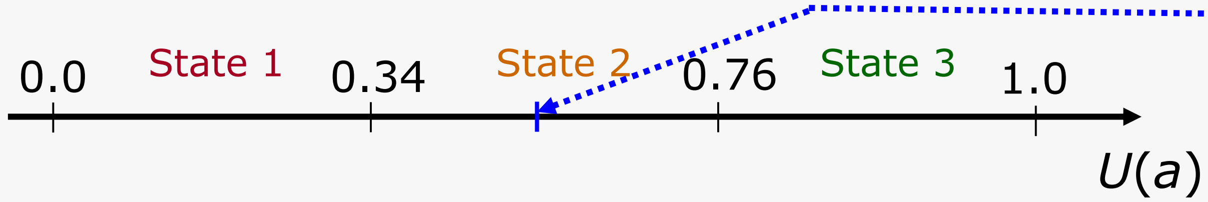

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  - Natural interpretability and great ability of representation

Operations Research, Decision Analysis, Decision Aiding, Analytics

- Operations Research (OR): „the science of better”; OR is the discipline of applying advanced analytical methods to help make better decisions
- Decision Analysis (normative & prescriptive) includes tools for identifying, representing, and formally assessing important aspects of a decision, for prescribing a recommended course of action maximizing the expected utility
- Decision Aiding (constructive) is a process involving the Decision Maker (DM) in co-construction of her preferences by exploring, interpreting, debating and arguing, with the aim of recommending a course of action that increases the consistency between the evolution of the process and DM’s objectives and value system
- Analytics is the scientific process of transforming data into insight for making better decisions

Operational Research, Decision Analysis, Decision Aiding, Analytics

- Knowledge-based Decision Support, Intelligent Decision Support, Machine Preference Learning also aim to recommend better decisions

- Decision Analysis (Bell, Raiffa & Tversky 1988) assumes an ideal rationality, and aims at giving an „objectively" best recommendation. Decision Analysis is based on 3 pillars:
  - *normative approach* defines basic principles of rationality and deduces its consequences,
  - *descriptive approach* verifies if these principles of rationality are respected in real life decisions,
  - *prescriptive approach* suggests how to avoid the violations of the same principles of rationality.

- Decision Aiding (Roy 1985) assumes that preferences of the DM with respect to considered alternatives do not pre-exist in the DM's mind

Operational Research, Decision Analysis, Decision Aiding, Analytics

- While all agree that the goal is to help managers to make better decisions, there is no agreement for a unique meaning of „better”
- This meaning depends on the operational approaches used for guiding the DA process, and on the way the recommendation is finally reached
- Combined with the above-mentioned limitations to objectivity, this shows that – in a decision-making context – we cannot scientifically prove that the recommended decision is „the best one”
- This implies that the concepts, models and methods, which will be presented during this course, must not be considered as a means of discovering a pre-existing truth, which would be universally accepted
- They have to be seen as keys to doors giving access to elements of knowledge contributing to acceptance of a final recommendation

Multiobjective Optimization

$$\begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{bmatrix} \rightarrow \text{Min (or Max)}$$

$$\text{subject to the constraints:}$$

$$g_1(\boldsymbol{x}) \, \{\leq, =, \geq\} \, b_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$g_m(\boldsymbol{x}) \, \{\leq, =, \geq\} \, b_m$$

where $\boldsymbol{x}=[x_1,\ldots,x_k]$ - vector of decision variables (continuous/integer)

$f_j(x)$, $j=1,\ldots,n$ - real-valued objective functions

$g_i(x)$, $i=1,\ldots,m$ - real-valued functions of the constraints

b_i, $i=1,\ldots,m$ - constant RHS of the constraints

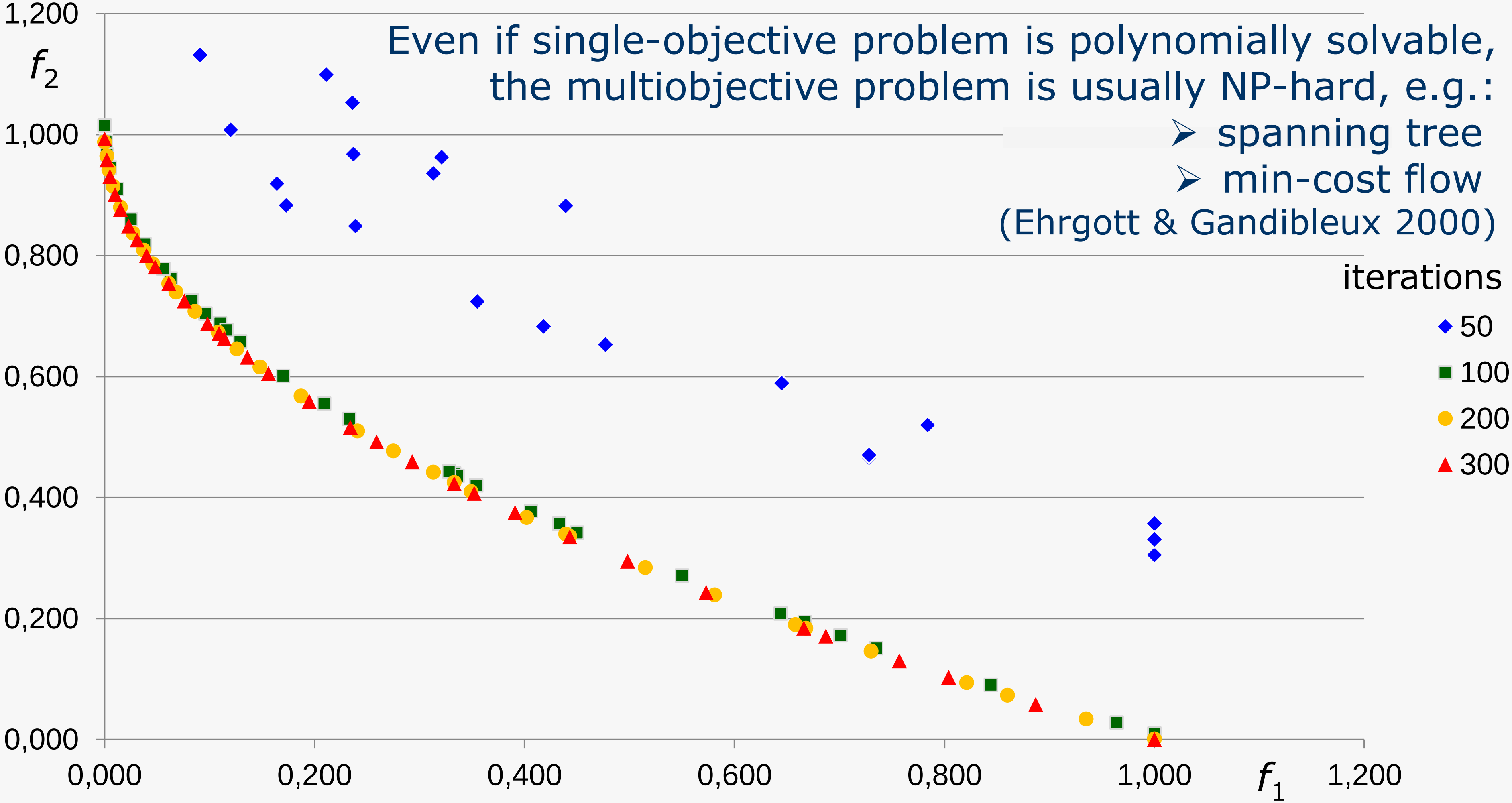
Multiobjective Combinatorial Optimization (MOCO)

MOCO problems are NP-hard, #P-hard → intractable

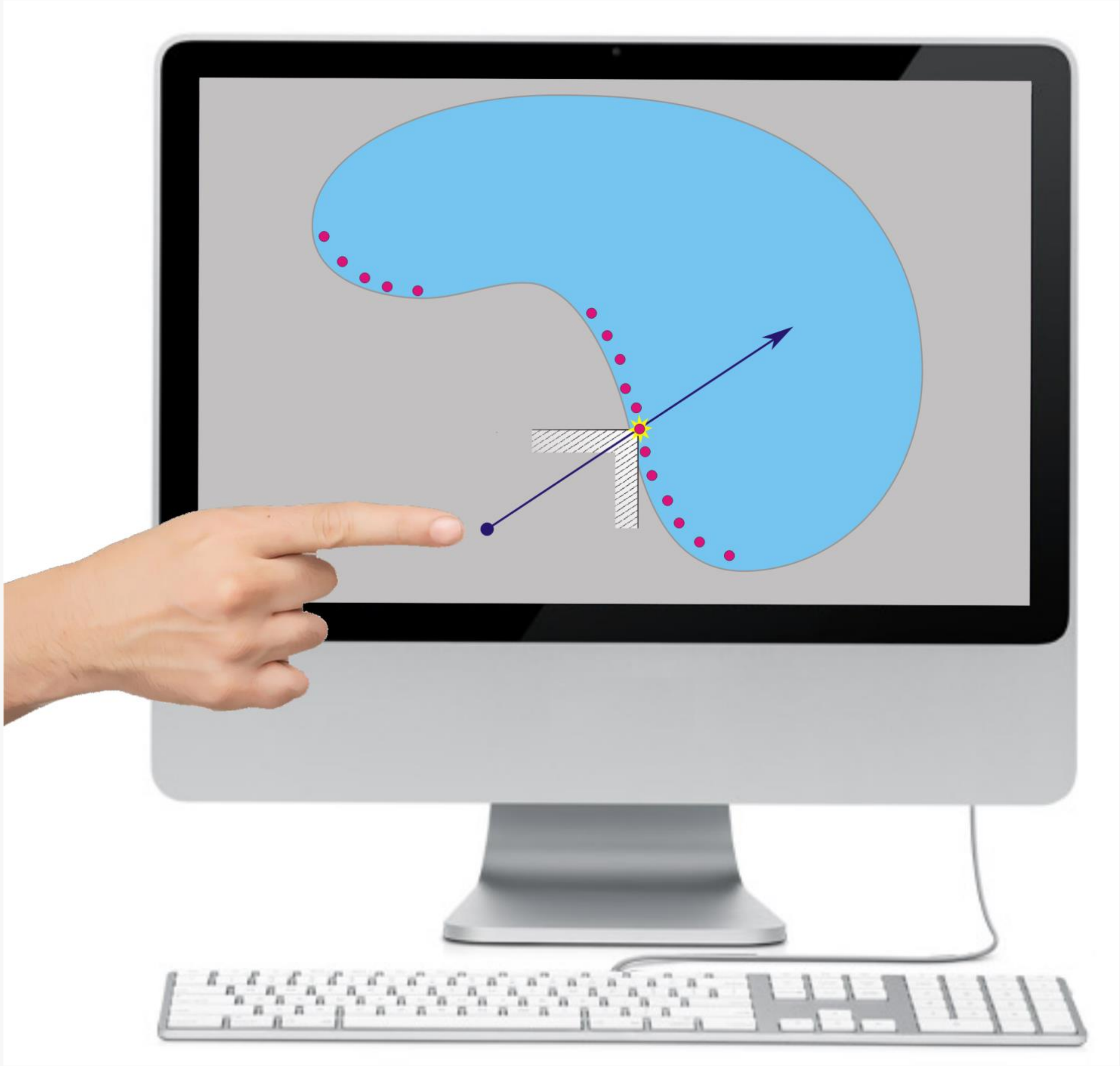
Even if single-objective problem is polynomially solvable, the multiobjective problem is usually NP-hard, e.g.:

- spanning tree
- min-cost flow

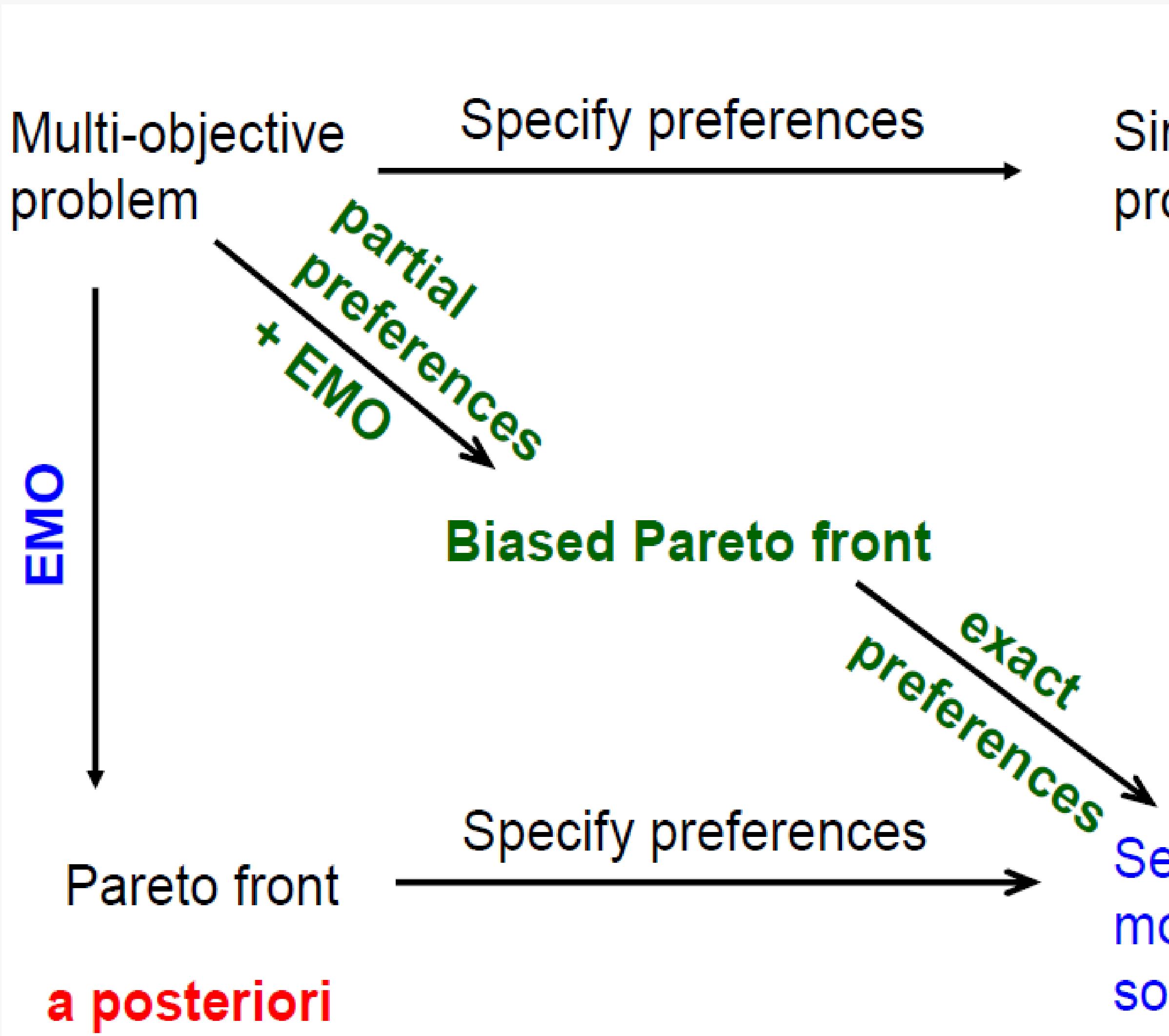
(Ehrgott & Gandibleux 2000)

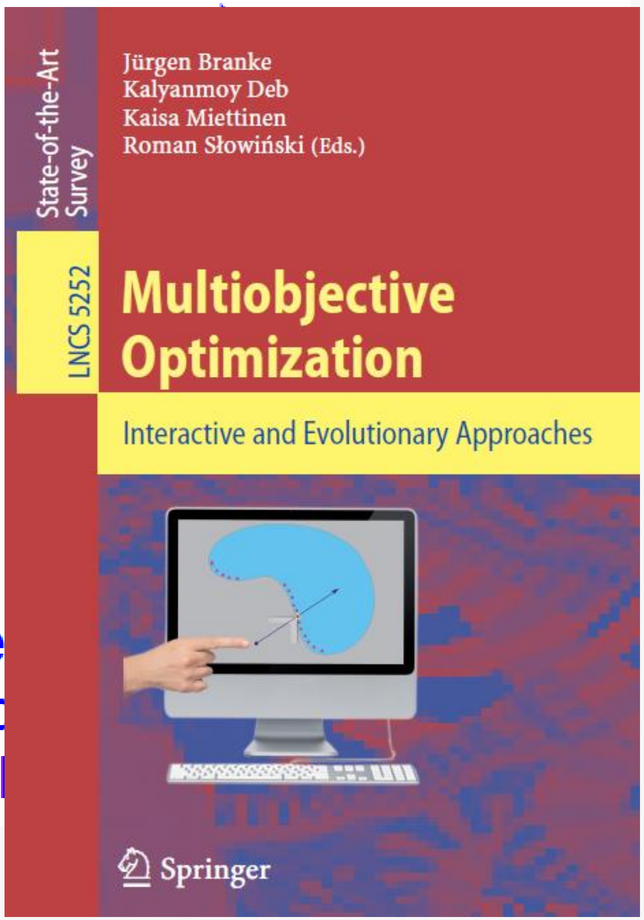


Multiobjective Optimization – „most preferred" solution



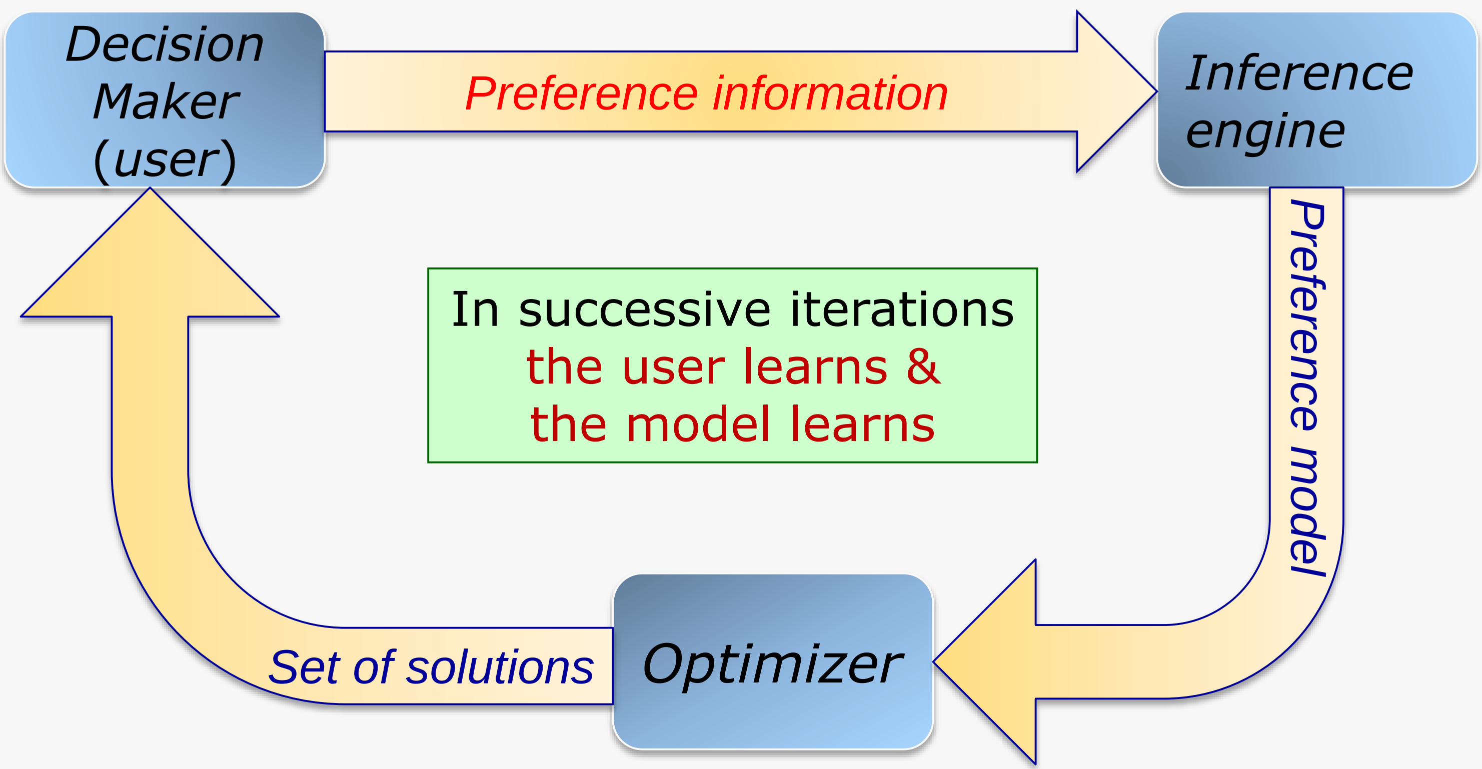
Interactive Multiobjective Optimization & EMO





Interactive optimization – constructive learning

- DM looks at intermediate results from optimization
- DM provides preference information
- Optimizer uses DM's preferences to focus the search on most promising solutions



When you will face the choice of an MCDA method you may refer to:

EURO J Decis Process (2013) 1:69–97
DOI 10.1007/s40070-013-0004-7

ORIGINAL ARTICLE

Questions guiding the choice of a multicriteria decision aiding method

Bernard Roy · Roman Słowiński

We formulate some questions that may help an analyst to choose a multicriteria decision aiding method well adapted to the decision context. These questions take into account several aspects of the decision process & of the cooperation between the analyst and the DM. We present these questions in a hierarchical order.

The initial question is what type of results the method is expected to bring.
The next questions concern requirements on preference scales, acquisition of preference information, handling of imperfect knowledge, acceptance of compensation among criteria, and existence of interaction among criteria.
The last questions are about intelligibility, axiomatic characterization, and weaknesses of the considered methods.

1. Do not confuse: realities of the first order and realities of the second (Watzlawick, 1977)

- REALITY OF THE FIRST ORDER IS the one involving some physical and objective properties of things that may be verified by repeated experiments.
- REALITY OF THE SECOND ORDER IS the one constructed by assigning essentially subjective meaning, significance or value to the reality of the first order. It involves a reality where the consensus is no longer based on an objective perception of things, nor on the possibility of experimental refutation, but on the acceptance of working hypotheses.

2. Do not confuse: describing-discovering with fabricating-constructing

- In decision aiding, there are often interaction protocols between the analyst and decision maker. Protocol design is based on the claim that the protocols will enable the discovery of a very rich & complex reality that is supposedly in the DM's mind (e.g., a utility function that is supposed to guide the DM's decisions).
- What is actually in the DM's mind is much poorer and not necessarily compliant with the model underlying the interaction protocol.
- This source of confusion may be connected to the first because it could possibly stem from a search for illusory objectivity. The concern for objectivity leads some to claim they are describing and discovering, whereas actually they are fabricating and constructing at least partially, which is often the right approach in decision aiding.

3. Do not confuse: uncertainty with indeterminacy

- Uncertainty implies that there is or will be certainty somewhere, called "the realization of uncertainty". When the said realization is feasible, talking about uncertainty would be correct.
  This is the case of uncertainty wrt realities of the first order.
- On the contrary, using the term uncertainty wrt realities of the second order seems absolutely inappropriate. This is the case when attempting to grasp the comfort level of a car, the inconvenience caused by a noise, the aesthetic quality of a landscape or landmark, or the borderline between the acceptable and the unacceptable.
- In these cases, the relevant issue is not defined precisely enough to claim that at a later time, it will be clearly characterized. That is why it is more appropriate to talk about indeterminacy instead of uncertainty.

4. Do not confuse: indifference with incomparability

- During decades the decision theory only considered three ways of comparing two actions *a* and *b*: *a* is preferred to *b*; *b* is preferred to *a*; *a* is indifferent to *b*.
- When some critical factual information is lacking, when the arbitration between conflicting arguments raises questions, or when the conclusion is conditioned by the way of taking account of realities of the second order, it should be possible to say: *a* is incomparable to *b*.

5. Do not confuse: numerical with quantitative

- The successive levels on a qualitative scale are usually number-coded. These numbers are then often used for calculating averages or distances. The compensations applied to this type of calculation implicitly assume that these numbers have a quantitative meaning, which is wrong.
- By referring to the type of scales involving numbers, one can avoid making the results to say more than they actually do.

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6. Do not confuse: the procedure for formulating a well-posed problem with the procedure aiming at integrating a decision making process

- The first consists in defining formally a problem with all information required to find a solution. This implies that the solution can be found based only on the information in the problem statement.
- The solutions stemming from this type of procedure often appear to be practically unfeasible because they are based on a formal problem that only little match reality.
- The second procedure consists in favoring an aiding process that adapts to a changing reality and identifies the questions to which the aiding process must provide properly formulated answers.
- Rather than optimal solutions, robust solutions likely to adapt to the changing reality should be sought.

7. Do not confuse: legitimacy based on realism and objectivity with legitimacy based on procedural rationality and communication

- The search for legitimacy in realism and objectivity prompts analysts to reach the truth and give an objective recommendation for decision.
- To do so, analysts will rely on procedures of the first type. They will claim they are describing & discovering rather than fabricating & constructing, and they will neglect realities of the second order. They may also rely on past decisions.
- Legitimacy based on procedural rationality and communication will lead analysts to use models considered as tools, constructed jointly with the DM, likely to take account of realities of the first as well as of the second order.
- Legitimacy is then based on the DM’s understanding of the analyst’s procedure of the second type, and on the communication that should make the working hypotheses and their ensuing results intelligible.
- Once an analyst is the co-builder of the produced knowledge, (s)he can no longer be considered as outside the decision-aiding process.