Polish Academy of Sciences

Decision Rule Approach to Multiple Criteria Decision Aiding/Making

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Plan

- The challenge of Decision Aiding
- Syntax of decision rules
- Rough Set concept and Dominance-based Rough Set Approach (DRSA)
- Decision rule approach to multiple criteria classification
- Attractiveness measures of decision rules
- Variable-consistency and stochastic DRSA
- Decision rule approach to multiple criteria choice and ranking
- Decision rule approach to decision under risk & uncertainty
- Decision rule approach to interactive multiobjective optimization
- Decision rule approach to evolutionary multiobjective optimization
- Application of monotonic rules to non-ordinal classification
- Examples of applications of DRSA
- Conclusions & software demo
- Additional material:
  - Algebra and topology for DRSA
  - DRSA as a way of handling Fuzzy-Rough Hybridization

The challenge of Decision Aiding

- Decision Aiding aims at giving the Decision Maker (DM) a recommendation concerning a set of *objects* *A* evaluated on multiple *dimensions*
  - *objects* = alternatives, options, actions, solutions,…
  - *dimensions* = voters, criteria, probabilistic gains/losses, attributes,…
- The only objective information stemming from the statement of a decision problem is the dominance relation in set *A* (partial weak order)
- The challenge: aggregation of evaluations on multiple dimensions

Enriching dominance relation – preference modeling

- One can „enrich" the dominance relation, using preference information elicited by the DM
- Preference information is an input to learn/build a preference model that aggregates the vector evaluations of objects
- The preference model induces a preference relation in set *A*, richer than the dominance relation
- A proper exploitation of the preference relation in *A* leads to a recommendation in terms of:
  - Ordinal classification (or sorting, to pre-defined & ordered classes)
  - Ranking (ordering of alternatives from the best to the worst)
  - Choice (or multiobjective optimization; search of the best solution)

Indirect elicitation of preference information by the DM

- Examples of decisions concern some objects relatively well known to the DM, i.e. reference objects :

DATA
  - pairwise comparisons of objects
  - assignment of objects to classes
  - comparisons of pairs of objects wrt intensity of preference

- Indirect elicitation is concordant with:
  - „Posterior rationality" principle by John March (1978): emphasizes the discovery of intentions as an interpretation of actions, rather than as *a priori* position
  - AI and Machine Learning : „Learning from examples"
  - OR : „Analytics - the scientific process of transforming data into insight for making better decisions"

Aggregation of multiple criteria evaluations

- Three families of **preference modelling (aggregation) methods**:
  - Multiple Attribute Utility Theory (MAUT) using a value function,

    e.g. $U(a)=\sum_{i=1}^{n} k_i g_i(a)$, $U(a)=\sum_{i=1}^{n} u_i[g_i(a)]$, Choquet/Sugeno integral
  - Outranking methods using an outranking relation S

    $a\ S\ b$ = "a is at least as good as b"
  - Decision rule approach using a set of „*if…, then…*" decision rules
- Decision rule model is the most general of all three

R.Słowiński, S.Greco, B.Matarazzo: Axiomatization of utility, outranking and decision-rule preference models for multiple-criteria classification problems under partial inconsistency with the dominance principle, *Control and Cybernetics*, 31 (2002) no.4, 1005-1035

Syntax of decision rules

ordinal classification:

- *if* $x_{q1} \succeq_{q1} r_{q1}$ *and* $x_{q2} \succeq_{q2} r_{q2}$ *and* … $x_{qp} \succeq_{qp} r_{qp}$, *then* $x \rightarrow$ class t or better
- *if* $x_{q1} \preceq_{q1} r_{q1}$ *and* $x_{q2} \preceq_{q2} r_{q2}$ *and* … $x_{qp} \preceq_{qp} r_{qp}$, *then* $x \rightarrow$ class t or worse

choice ranking cardinal criteria:

- *if* $(x \succ_{q1}^{\geq h(q1)} y)$ *and* $(x \succ_{q2}^{\geq h(q2)} y)$ *and* … $(x \succ_{qp}^{\geq h(qp)} y)$, *then* xSy
- *if* $(x \succ_{q1}^{\leq h(q1)} y)$ *and* $(x \succ_{q2}^{\leq h(q2)} y)$ *and* … $(x \succ_{qp}^{\leq h(qp)} y)$, *then* $xS^{c}y$

choice ranking ordinal criteria:

- *if* $\boxed{x_{g1} \succeq_{g1} r_{g1} \;\&\; y_{g1} \preceq_{g1} r'_{g1}}$ & … $x_{gp} \succeq_{gp} r_{gp}$ & $y_{gp} \preceq_{gp} r'_{gp}$, *then* xSy
- *if* $\boxed{x_{g1} \preceq_{g1} r_{g1} \;\&\; y_{g1} \succeq_{g1} r'_{g1}}$ & … $x_{gp} \preceq_{gp} r_{gp}$ & $y_{gp} \succeq_{gp} r'_{gp}$, *then* $xS^{c}y$

pair of objects x,y evaluated on criterion g_1

S.Greco, B.Matarazzo, R.Słowiński: Decision rule approach. Chapter 13 [in]: S.Greco M.Ehrgott, and J.Figueira (eds.), *Multiple Criteria Decision Analysis: State of the Art Surveys*, 2nd edition, Operations Research & Management Science 233, Springer, New York, 2016, pp. 497-552.

Indirect preference information – example of technical diagnostics

- 176 buses (objects)
- 8 symptoms (attributes)
- Decision = technical state:

  3 – good state (in use)

  2 – minor repair

  1 – major repair (out of use)
- Aggregation = finding relationships between symptoms & technical state

- The model explains expert's decisions and supports diagnosis of new buses

Examples:

	MaxSpeed	ComprPressure	Blacking	Torque	SummerCons	WinterCons	OilCons	HorsePower	State
1.	90	2	38	481	21	26	0	145	3
2.	76	2	70	420	22	25	2	110	1
3.	63	1	82	400	22	24	3	101	1
4.	90	2	49	477	21	25	1	138	3
5.	85	2	52	460	21	25	1	130	2
6.	72	2	73	425	23	27	2	112	1
7.	88	2	50	480	21	24	1	140	3
8.	87	2	56	465	22	27	1	135	3
9.	90	2	16	486	26	27	0	150	3
10.	60	1	95	400	23	24	4	96	1
11.	80	2	60	451	21	26	1	125	1
12.	78	2	63	448	21	26	1	120	2
13.	90	2	26	482	22	24	0	148	3
14.	62	1	93	400	22	28	3	100	1
15.	82	2	54	461	22	26	1	132	2
16.	65	2	67	402	22	23	2	103	1
17.	90	2	51	468	22	26	1	138	3
18.	90	2	15	488	20	23	0	150	3
19.	76	2	65	428	27	33	2	116	1
20.	85	2	50	454	21	26	1	129	2
21.	85	2	58	450	22	25	1	126	2
22.	88	2	48	458	22	25	1	130	3
23.	60	1	90	400	24	28	4	95	1
24.	64	2	71	420	23	25	2	105	1
25.	75	2	64	432	22	25	1	114	2
26.	74	2	64	420	21	25	1	110	2
27.	68	2	70	400	22	26	2	100	1

Attributes: 9 of 10 | Examples: 76 | Decision: State | Missing Values: No

Indirect preference information – „Thierry’s choice”

(data from [Bouyssou et al. 2006])

- reference actions ranked by the DM: $11 \succ 3 \succ 13 \succ 9 \succ 14$

Pairwise Comparison Table (PCT):

	obj1	obj2	diff_price	diff_accel	diff_pick_up	diff_brakes	diff_road_h	relation
1.	11	11	0	0	0	0	0	S
2.	11	3	564	-0,7	-0,1	-0,33	0,25	S
3.	11	13	318	-1,9	-2,1	0,67	1,5	S
4.	11	9	-2263	-1,1	0,1	0,33	1	S
5.	11	14	-3797	-0,6	-1,9	0,33	0,5	S
6.	3	11	-564	0,7	0,1	0,33	-0,25	Sc
7.	3	3	0	0	0	0	0	S
8.	3	13	-246	-1,2	-2	1	1,25	S
9.	3	9	-2827	-0,4	0,2	0,66	0,75	S
10.	3	14	-4361	0,1	-1,8	0,66	0,25	S
11.	13	11	-318	1,9	2,1	-0,67	-1,5	Sc
12.	13	3	246	1,2	2	-1	-1,25	Sc
13.	13	13	0	0	0	0	0	S
14.	13	9	-2581	0,8	2,2	-0,34	-0,5	S
15.	13	14	-4115	1,3	0,2	-0,34	-1	S
16.	9	11	2263	1,1	-0,1	-0,33	-1	Sc
17.	9	3	2827	0,4	-0,2	-0,66	-0,75	Sc
18.	9	13	2581	-0,8	-2,2	0,34	0,5	Sc
19.	9	9	0	0	0	0	0	S
20.	9	14	-1534	0,5	-2	0	-0,5	S
21.	14	11	3797	0,6	1,9	-0,33	-0,5	Sc
22.	14	3	4361	-0,1	1,8	-0,66	-0,25	Sc
23.	14	13	4115	-1,3	-0,2	0,34	1	Sc
24.	14	9	1534	-0,5	2	0	0,5	Sc
25.	14	14	0	0	0	0	0	S

$S = \succeq$

$Sc = \text{not} \succeq$

The model explains DM’s preferences & supports comparison of new cars

Representation of preferences

- Scoring function: $U(a) = \sum_{i=1}^{n} k_i g_i(a)$ or $U(a) = \sum_{i=1}^{n} u_i[g_i(a)]$

  like in *MAUT*, *Discriminant Analysis*, *Logistic Regression* or *Perceptron*,

  e.g. $\boxed{U(a) = 0.21 \times g_{\text{Speed}}(a) + 0.03 \times g_{\text{Compr}}(a) + \ldots + 0.18 \times g_{Power}(a) = 0.45}$

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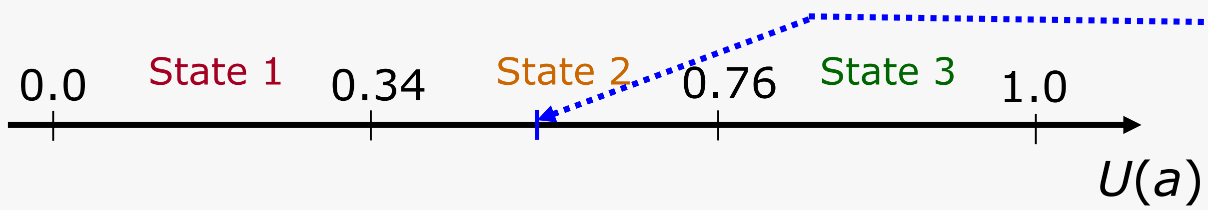

Representation of preferences

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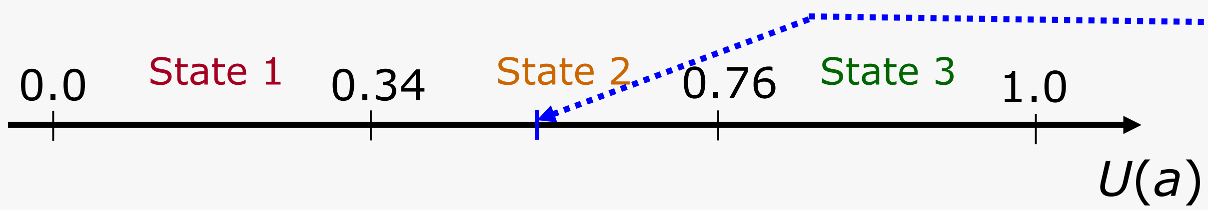


- Decision rules or trees,

  like in *Artificial Intelligence*, *Data Minining* or *Learning from Examples*,

  e.g. *if* OilCons $\leq$ 1 & WinterGasCons $\leq$ 25, *then* State $\succeq$ 2

  if MaxSpeed $\leq$ 85 & WinterGasCons $\geq$ 25, *then* State $\preceq$ 2

  - Natural interpretability and great ability of representation

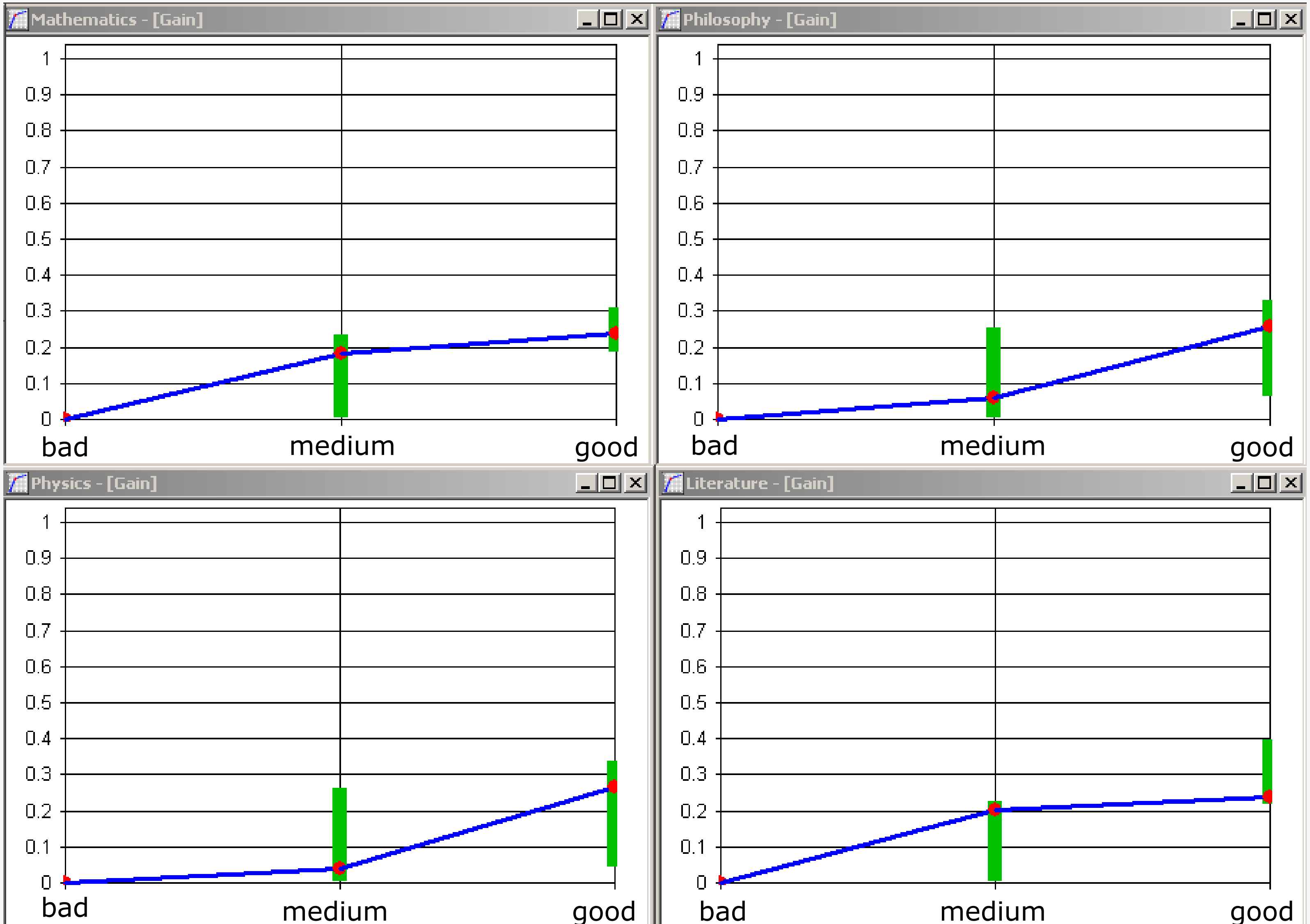
Dataset with decision examples concerning ordinal classification

Student	Mathematics	Physics	Literature	Philosophy	Overall_Eval.
***S*1**	good	medium	bad	medium	bad
***S*2**	medium	medium	bad	bad	medium
***S*3**	medium	medium	medium	bad	medium
***S*4**	good	good	medium	medium	medium
***S*5**	good	good	medium	medium	good
***S*6**	good	medium	good	good	good
***S*7**	good	good	good	medium	good
***S*8**	bad	bad	bad	bad	bad
***S*9**	bad	bad	medium	bad	bad
***S*10**	good	medium	medium	bad	medium

$S7$: 0.80 $S4$: 0.76 $S1$: 0.33
$S6$: 0.77 $S10$: 0.48 $S2$: 0.22
$S5$: 0.76 $S3$: 0.42 $S9$: 0.20
$S8$: 0.00

UTADIS

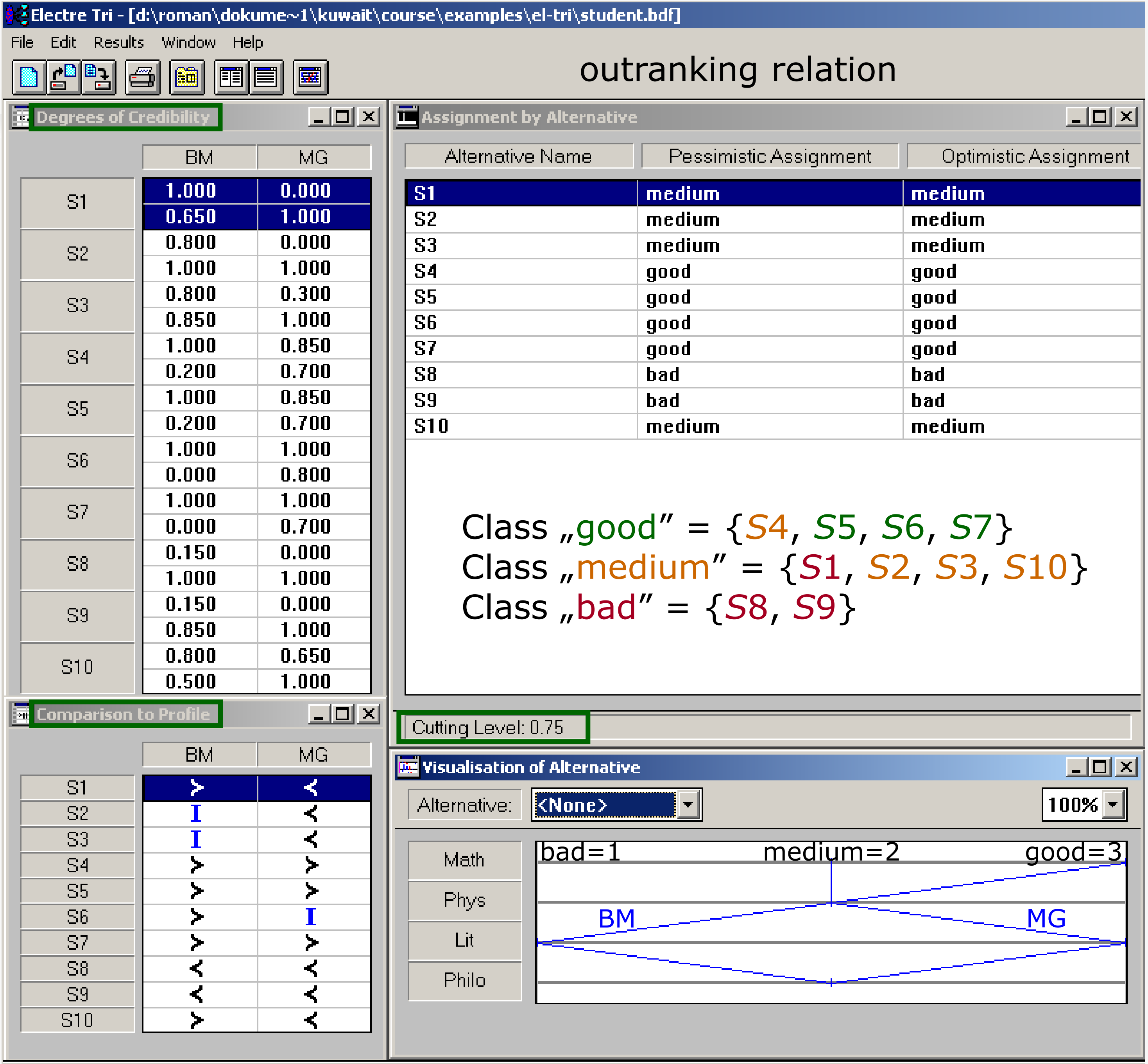
value function $U(a) = \sum_{i=1}^{n} u_i[g_i(a)]$



Inconsistent decision examples concerning ordinal classification

Student	Mathematics	Physics	Literature	Philosophy	Overall_Eval.
*S*1	good	medium	bad	medium	bad
*S*2	medium	medium	bad	bad	medium
*S*3	medium	medium	medium	bad	medium
*S*4	good	good	medium	medium	medium
*S*5	good	good	medium	medium	good
*S*6	good	medium	good	good	good
*S*7	good	good	good	medium	good
*S*8	bad	bad	bad	bad	bad
*S*9	bad	bad	medium	bad	bad
*S*10	good	medium	medium	bad	medium

outranking relation



Degrees of Credibility

	BM	MG
S1	1.000	0.000
	0.650	1.000
S2	0.800	0.000
	1.000	1.000
S3	0.800	0.300
	0.850	1.000
S4	1.000	0.850
	0.200	0.700
S5	1.000	0.850
	0.200	0.700
S6	1.000	1.000
	0.000	0.800
S7	1.000	1.000
	0.000	0.700
S8	0.150	0.000
	1.000	1.000
S9	0.150	0.000
	0.850	1.000
S10	0.800	0.650
	0.500	1.000

Comparison to Profile

	BM	MG
S1	>	<
S2	I	<
S3	I	<
S4	>	>
S5	>	>
S6	>	I
S7	>	>
S8	<	<
S9	<	<
S10	>	<

Assignment by Alternative

Alternative Name	Pessimistic Assignment	Optimistic Assignment
S1	medium	medium
S2	medium	medium
S3	medium	medium
S4	good	good
S5	good	good
S6	good	good
S7	good	good
S8	bad	bad
S9	bad	bad
S10	medium	medium

Class „good” = {*S4*, *S5*, *S6*, *S7*}
Class „medium” = {*S1*, *S2*, *S3*, *S10*}
Class „bad” = {*S8*, *S9*}

Weights:

Math=0.35
Phys=0.3
Lit=0.15
Philo=0.25

Profiles:

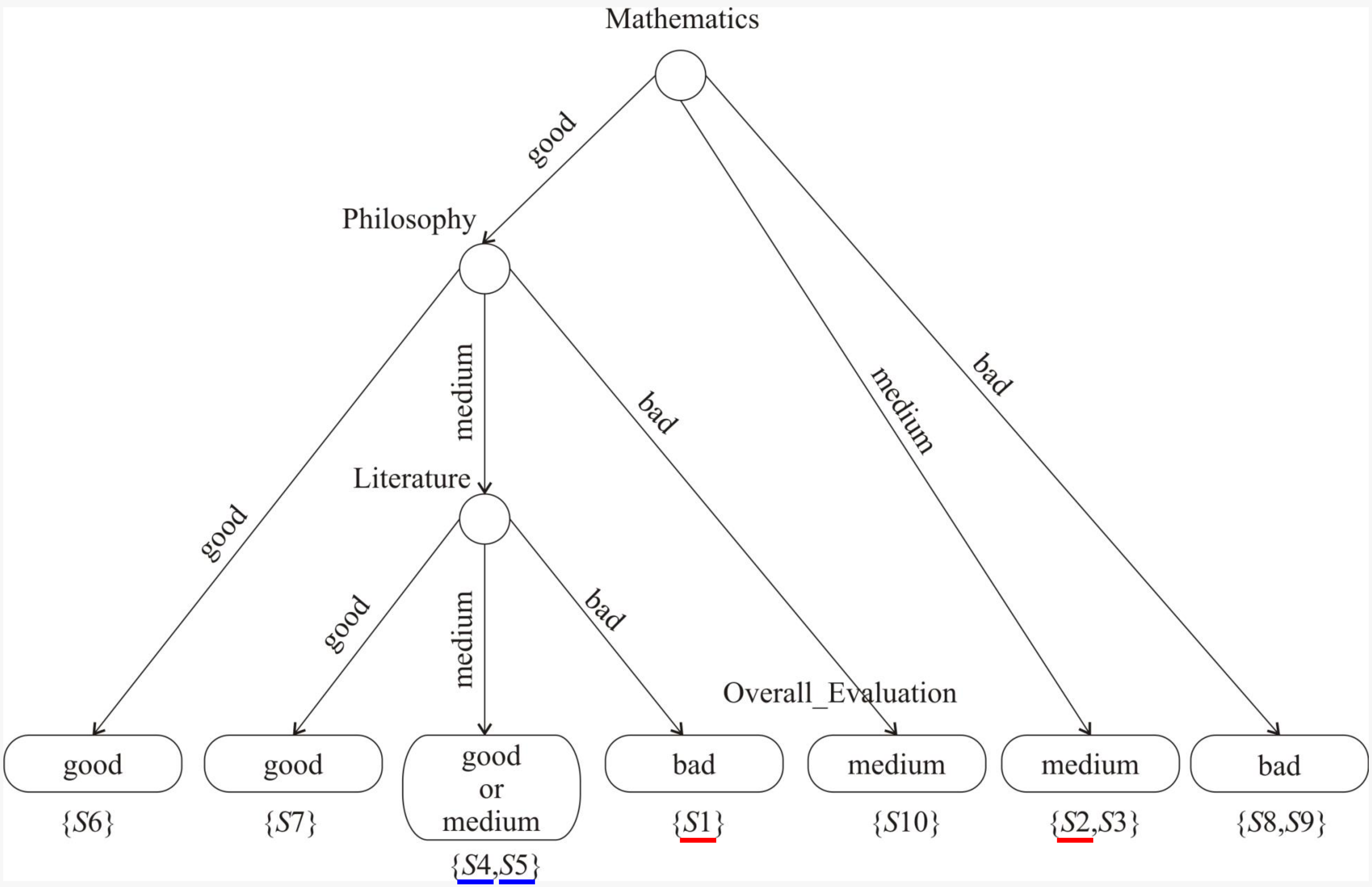
Bad-Medium BM:
Math=medium
Phys=medium
Lit=bad
Philo=medium

Medium-Good MG:
Math=good
Phys=medium
Lit=good
Philo=medium

Thresholds:

Indifference=0
Preference=1
Veto=2

Decision tree (C4.5)



Decision rules (dominance relation in premise and conclusion)

If Lit $\succeq$ good, *then* student $\succeq$ good {*S6*,*S7*}

If Phys $\succeq$ medium & Lit $\succeq$ medium, *then* student $\succeq$ medium {*S3*,*S4*,*S5*,*S6*,*S7*,*S10*}

If Phys $\succeq$ good & Lit $\preceq$ medium, *then* student is medium or good {*S4*,*S5*}

If Math $\succeq$ medium & Lit $\preceq$ bad, *then* student is bad or medium {*S1*,*S2*}

If Lit $\preceq$ bad, *then* student $\preceq$ medium {*S1*,*S2*,*S8*}

If Philo $\preceq$ bad, *then* student $\preceq$ medium {*S2*,*S3*,*S8*,*S9*,*S10*}

If Phys $\preceq$ bad, *then* student $\preceq$ bad {*S8*,*S9*}

- *„People make decisions by searching for* ***rules*** *that provide good justification of their choices"* (Slovic, 1975)

Why should we seek for rules rather than for a real-valued function ?

- Description of complex phenomena by recursive estimation techniques applied on historical data (*Int. J. Environment and Pollution, vol.12, no.2/3, 1999*)
- The dependence of the size of the mouth of a river in month k, represented by the relative tidal energy (RTE_k), from RTE_{k-1}, the river flow (F_{k-1}), the onshore wind (W_{k-1}) and the crude monthly count of storm events (S_k) (Elford et al. 1999; Murray Mouth, Australia):

$$\boxed{RTE_k = A_1 RTE_{k-1} + A_2 \frac{(F_{k-1} - 200)^{2.4}}{8RTE_{k-1} + 1} + A_3 \frac{W_{k-1}}{8RTE_{k-1} + 1} + A_4 S_k + \varepsilon_k}$$

where the exponent 2.4 was tuned by „trial and error”, coefficients A_1, A_2, A_3, A_4 were computed using a recursive least squares (RLS), and ε_k is the model error

Why should we seek for rules rather than for a real-valued function ?

- Description of complex phenomena by recursive estimation techniques applied on historical data (*Int. J. Environment and Pollution, vol.12, no.2/3, 1999*)
- The impact of urban stormwater on the quality of the receiving water (Rossi, Słowiński, Susmaga 1999; Lausanne and Genève).
- Example of rule induced from empirical observation of some sites:

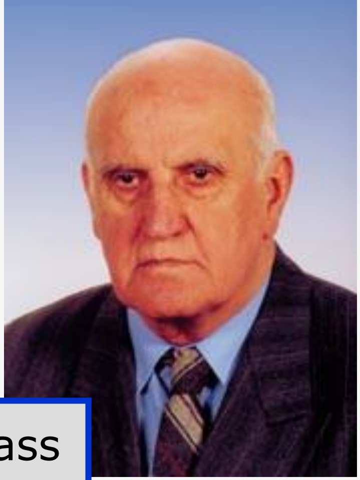
If *the site is of type 2 (residential),* **and** *total rainfall is up to 8 mm,* **and** *max intensity of rain is between 2.7 and 11.2 mm/h,* **then** *total mass of suspended solids is < 2.2 kg/ha*

- The rule is more expressive and involves heterogeneous data: nominal, qualitative and quantitative

Rough set concept

Inconsistencies in data – Rough Set Theory

- Zdzisław Pawlak (1926–2006)



Student	Mathematics	Physics	Literature	Overall class
S1	**good**	**medium**	**bad**	**bad**
S2	**medium**	**medium**	**bad**	**medium**
S3	**medium**	**good**	**medium**	**medium**
S4	**medium**	**good**	**medium**	**good**
S5	**good**	**medium**	**good**	**good**
S6	**good**	**good**	**good**	**good**
S7	**bad**	**bad**	**medium**	**bad**
S8	**bad**	**bad**	**medium**	**bad**

Inconsistencies in data – Rough Set Theory

- The granules of indiscernible objects are used to approximate classes

Student	Mathematics (**M**)	Physics (**Ph**)	Literature (**L**)	Overall class
S1	**good**	**medium**	**bad**	**bad**
S2	**medium**	**medium**	**bad**	**medium**
S3	**medium**	**good**	**medium**	**medium**
S4	**medium**	**good**	**medium**	**good**
S5	**good**	**medium**	**good**	**good**
S6	**good**	**good**	**good**	**good**
S7	**bad**	**bad**	**medium**	**bad**
S8	**bad**	**bad**	**medium**	**bad**

Inconsistencies in data – Rough Set Theory

- Lower approximation of class „good"

Student	Mathematics (**M**)	Physics (**Ph**)	Literature (**L**)	Overall class
S1	**good**	**medium**	**bad**	**bad**
S2	**medium**	**medium**	**bad**	**medium**
S3	**medium**	**good**	**medium**	**medium**
S4	**medium**	**good**	**medium**	**good**
S5	**good**	**medium**	**good**	**good**
S6	**good**	**good**	**good**	**good**
S7	**bad**	**bad**	**medium**	**bad**
S8	**bad**	**bad**	**medium**	**bad**

Lower Approximation

Inconsistencies in data – Rough Set Theory

- Lower and upper approximation of class „good"

Student	Mathematics (**M**)	Physics (**Ph**)	Literature (**L**)	Overall class
S1	**good**	**medium**	**bad**	**bad**
S2	**medium**	**medium**	**bad**	**medium**
S3	**medium**	**good**	**medium**	**medium**
S4	**medium**	**good**	**medium**	**good**
S5	**good**	**medium**	**good**	**good**
S6	**good**	**good**	**good**	**good**
S7	**bad**	**bad**	**medium**	**bad**
S8	**bad**	**bad**	**medium**	**bad**

Upper Approximation

Lower Approximation

IRSA – rules induced from rough approximations

- Certain decision rule supported by objects from lower approximation of class „good” (discriminant rule)

  If Lit=good, *then* Student is certainly good
  {S5,S6}

- Possible decision rule supported by objects from upper approximation of class „good” (partly discriminant rule)

  If Phys=good, *then* Student is possibly good
  {S3,S4,S6}

- Approximate decision rule supported by objects from the boundary of class „medium” or „good”

  If Phys=good & Lit=medium, *then* Student is medium or good
  {S3,S4}

What is missing to Indiscernibility-based Rough Set Approach?

- Classical rough set approach does not detect inconsistency w.r.t. dominance (Pareto principle) – a basic principle in decision making

Student	Mathematics (**M**)	Physics (**Ph**)	Literature (**L**)	Overall class
S1	**good**	**medium**	**bad**	**bad**
S2	**medium**	**medium**	**bad**	**medium**
S3	**medium**	**good**	**medium**	**medium**
S4	**medium**	**good**	**medium**	**good**
S5	**good**	**medium**	**good**	**good**
S6	**good**	**good**	**good**	**good**
S7	**bad**	**bad**	**bad**	**bad**
S8	**bad**	**bad**	**medium**	**bad**

Rules induced from indiscernibility-based rough approximations

- Certain decision rules based on indiscernibility are inconsistent with respect to the dominance principle (monotonicity constraints):

If Math=good & Lit=bad, *then* Student is certainly bad {S1}

If Math=medium & Lit=bad, *then* Student is certainly medium {S2}

Dominance-based Rough Set Approach: DRSA

Classical Rough Set Theory vs. Dominance-based Rough Set Theory

Classical Rough Set Theory

$\Downarrow$

Indiscernibility principle

If *x* and *y* are indiscernible with respect to all relevant **attributes**,

then *x* should classified to the same class as *y*

Dominace-based Rough Set Theory

$\Downarrow$

Dominance principle (monotonicity constraints)

If *x* is at least as good as *y* with respect to all relevant **criteria**,

then *x* should be classified at least as good as *y*

S.Greco, B.Matarazzo, R.Słowiński: Rough sets theory for multicriteria decision analysis. *European J. of Operational Research*, 129 (2001) no.1, 1-47

Dominance principle as monotonicity constraint principle

- Dominance-based Rough Set Approach (DRSA) permits representation and analysis of all phenomena involving monotonicity relationship between specific measures or perceptions, e.g.

„the more a tomato is red, and the more it is soft, the more it is ripe"

„the older the car, the more likely its breakdown" *

or

„the more similar are the causes,
*the more similar are the effects one can expect"***

*S.Greco, M.Inuiguchi, R.Słowiński: Fuzzy rough sets and multiple-premise gradual decision rules. *International Journal of Approximate Reasoning*, 41 (2005) 179-211

**S.Greco, B.Matarazzo, R.Słowiński: Case-based reasoning using gradual rules induced from dominance-based rough approximations. [In]: G.Wang et al. (eds.), *Rough Sets and Knowledge Technology* (*RSKT 2008*). LNCS 5009, Springer, Berlin, 2008, pp. 268-275.

Monotonicity and induction

- *„The procedure of induction consists in accepting as true the simplest law that can be reconciled with our experiences"*

  (L. Wittgenstein, Tractatus Logico-Philosophicus, 6.363)

- This simplest law is just monotonicity and, therefore, inductive discovery of rules can be seen as a specific way of dealing with monotonicity

- Dominance-based Rough Set concept permits data structuring wrt possible violation of dominance (lower appx, upper appx, boundary) prior to rule induction

R.Słowiński, S.Greco, B.Matarazzo: *Rough Sets in Decision Making*. [In]: R.A.Meyer (ed.): Encyclopedia of Complexity and Systems Science, Springer, NY, 2009, pp. 7753-7786.

Decision rule approach to multiple criteria classification

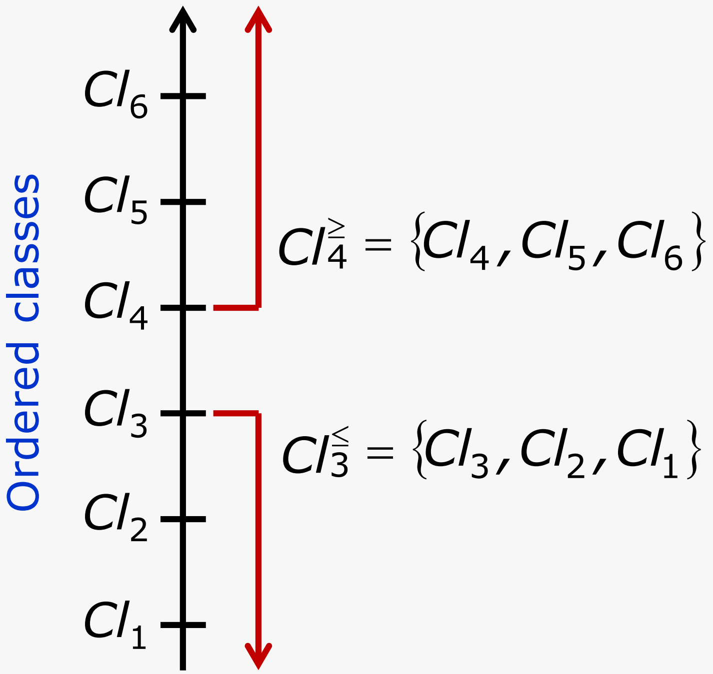
Dominance-based Rough Set Approach (DRSA)

- In order to handle monotonic dependency between conditions and decision (class assignment):

$Cl_t^{\geq} = \bigcup_{s \geq t} Cl_s$ – upward union of classes, t=2,...,m („*at least*" class Cl_t)

$Cl_t^{\leq} = \bigcup_{s \leq t} Cl_s$ – downward union of classes, t=1,...,m-1 („*at most*" class Cl_t)

- $Cl_t^{\geq}$ and $Cl_t^{\leq}$ are positive and negative dominance cones in decision space reduced to single dimension



Dominance-based Rough Set Approach (DRSA)

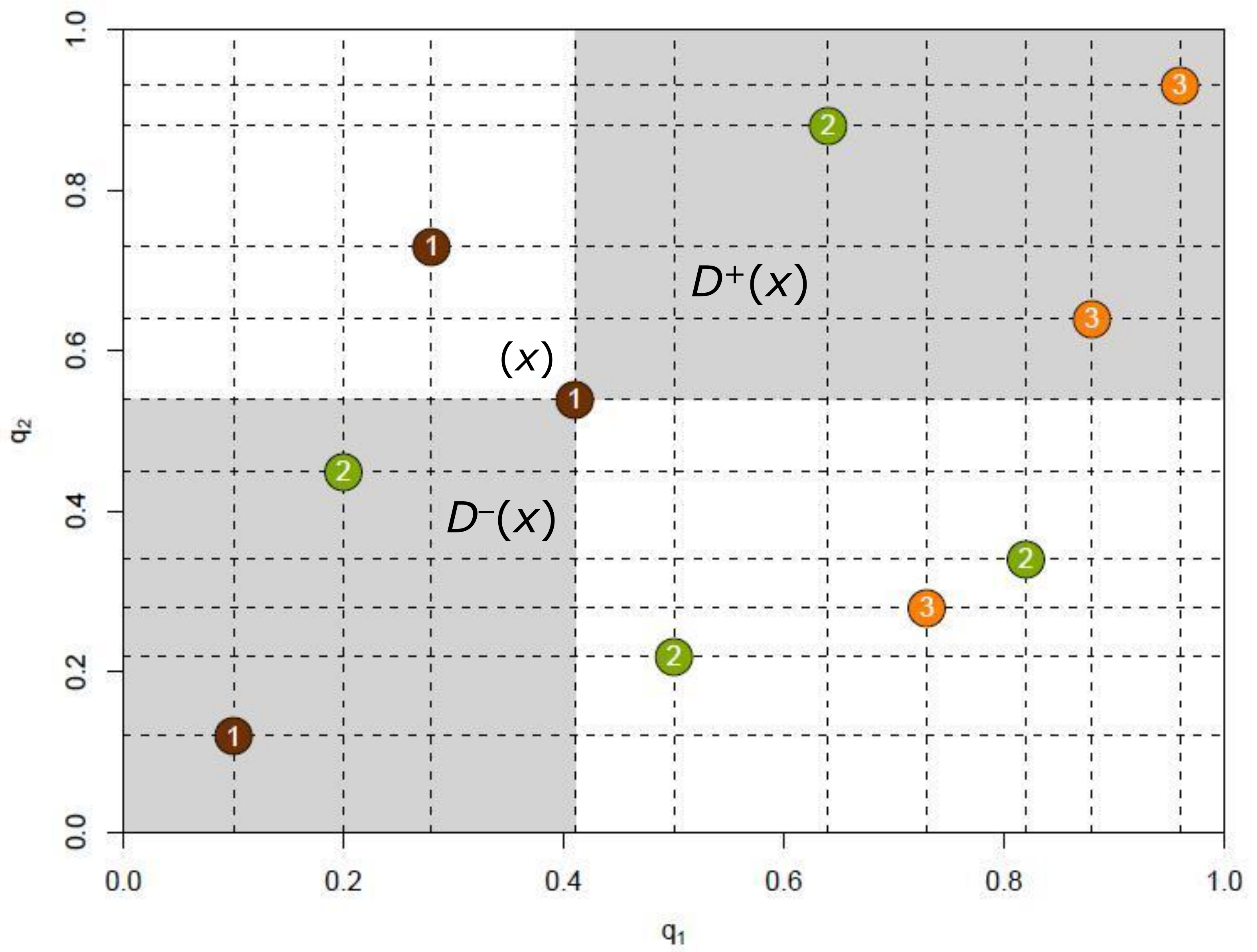
- D_P – dominance relation (partial preorder) in condition space, $P \subseteq C$
- Granules of knowledge are dominance cones in condition space

  $D_P^+(x) = \{y \in U \colon yD_Px\}$: *P-dominating set (positive cone)*

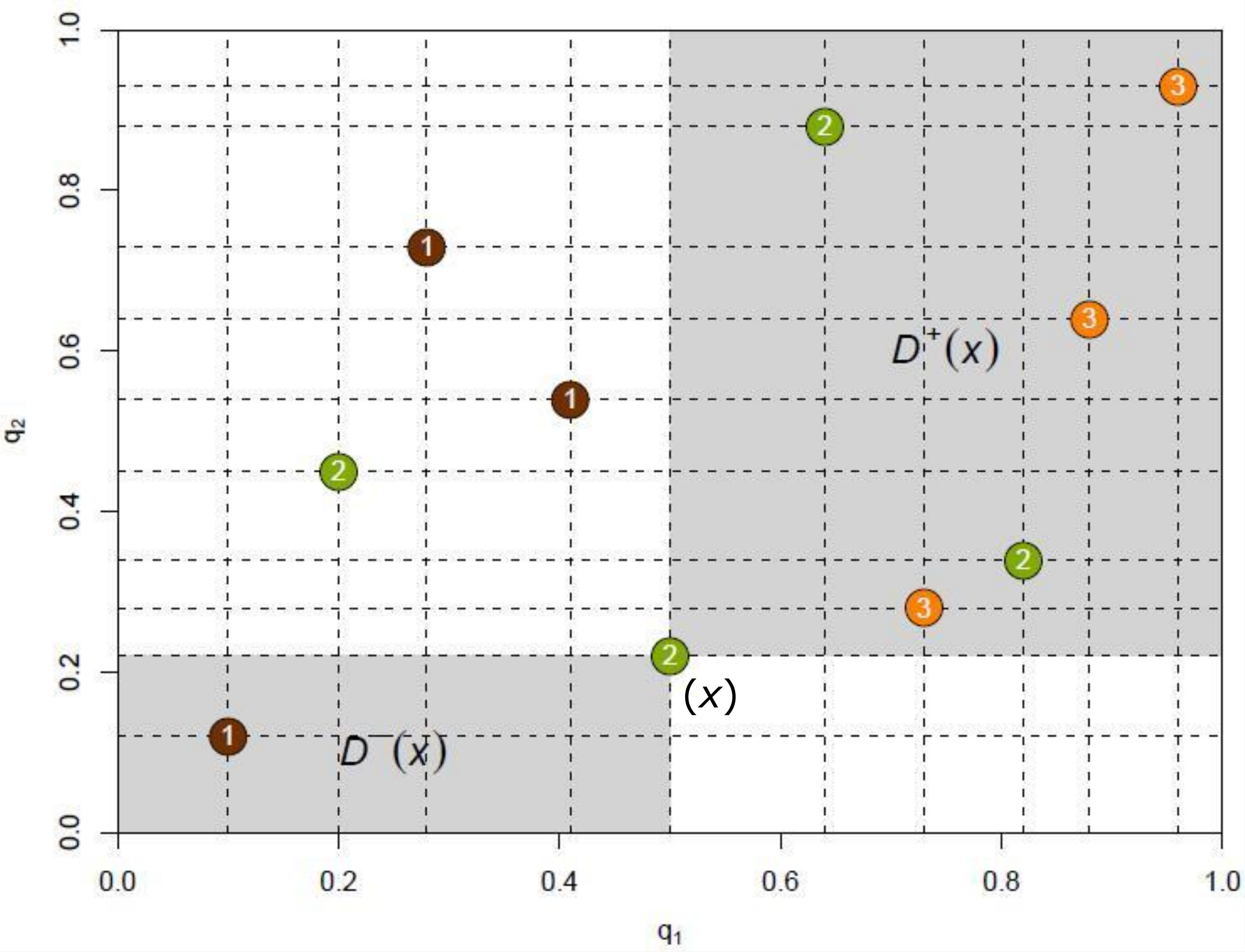
  $D_P^-(x) = \{y \in U \colon xD_Py\}$: *P-dominated set (negative cone)*

- Classification patterns to be discovered are functions representing granules $Cl_t^{\geq}$, $Cl_t^{\leq}$, by granules $D_P^+(x)$, $D_P^-(x)$

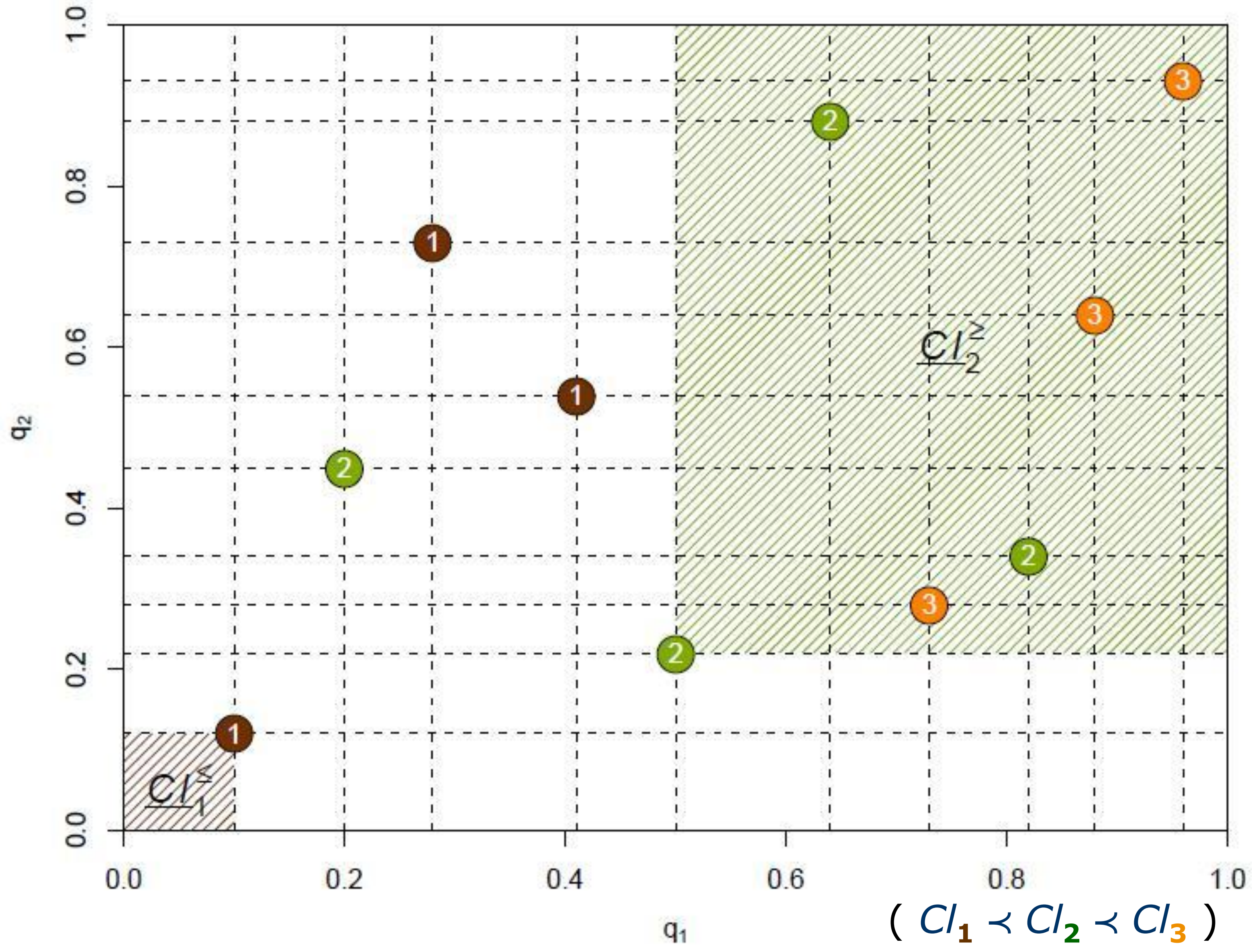
Dominance cones wrt object x – example ($Cl_1 \prec Cl_2 \prec Cl_3$)



Dominance cones wrt object x – example ($Cl_1 \prec Cl_2 \prec Cl_3$)

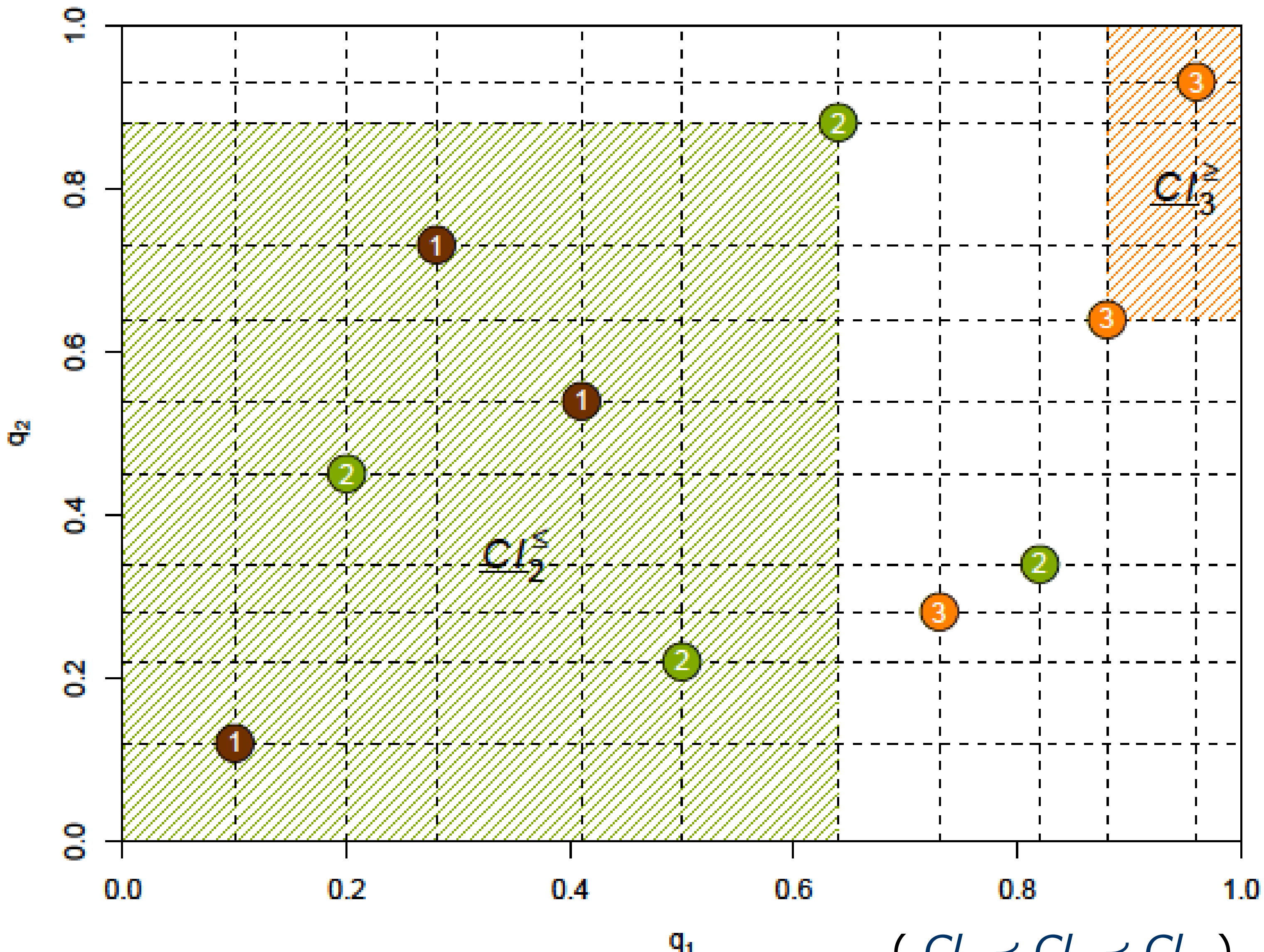


Lower approximations of „at most Cl_1" and „at least Cl_2"



($Cl_1 \prec Cl_2 \prec Cl_3$)

Lower approximations of „at most Cl_2" and „at least Cl_3"

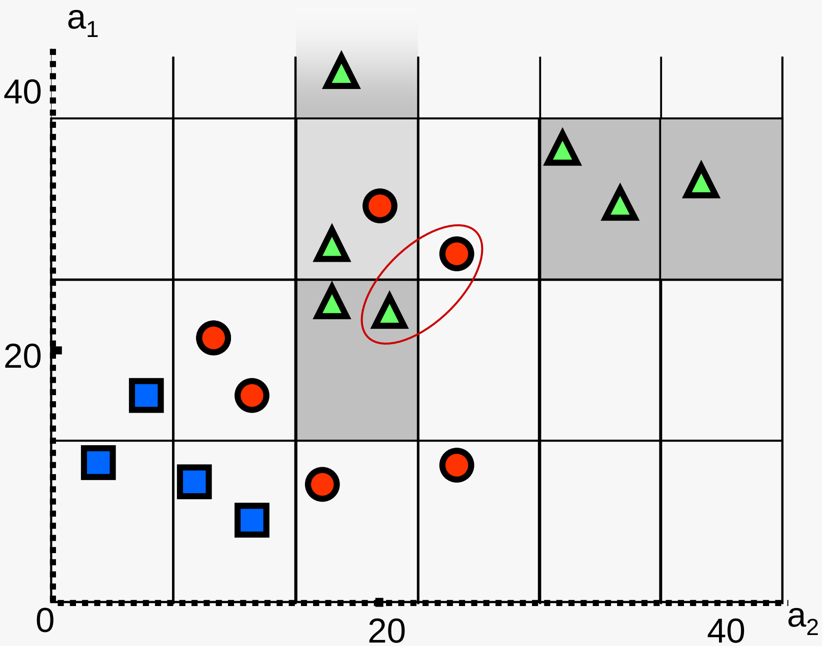


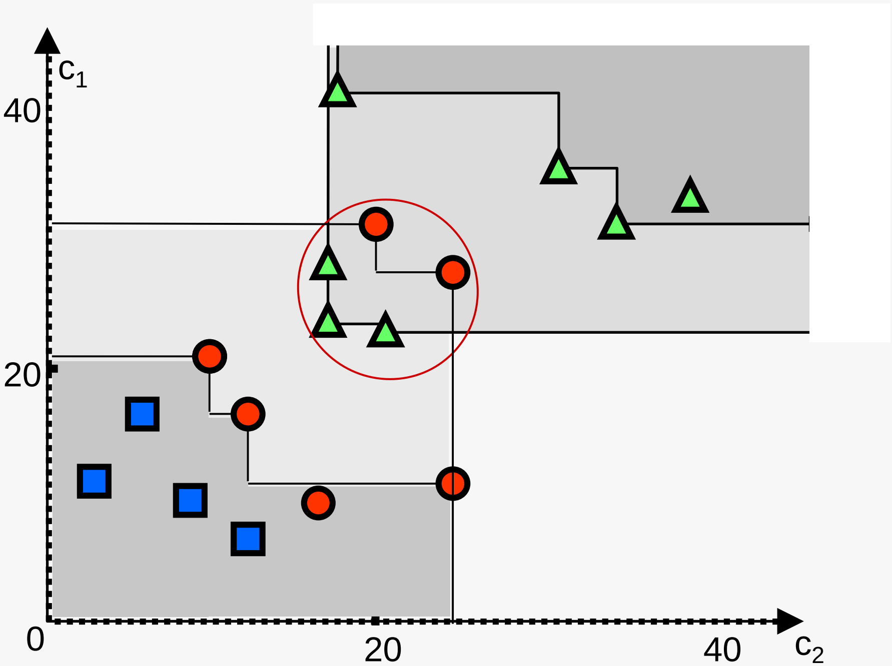
($Cl_1 \prec Cl_2 \prec Cl_3$)

Dominance-based Rough Set Approach vs. Classical RSA

Comparison of CRSA and DRSA

Classes: △ ≻ ● ≻ ■





$$\underline{P}(X)=\{x\in U: I_P(x)\subseteq X\}$$

$$\underline{P}\left(Cl_t^{\geq}\right)=\left\{x\in U:\ D_P^{+}(x)\subseteq Cl_t^{\geq}\right\}$$

$$\overline{P}(X)=\bigcup_{x\in X} I_P(x)$$

$$\overline{P}\left(Cl_t^{\geq}\right)=\bigcup_{x\in Cl_t^{\geq}} D_P^{+}(x)$$

DRSA for multiple criteria classification

- Example of preference information about students:

Student	Mathematics (**M**)	Physics (**Ph**)	Literature (**L**)	Overall class
S1	**good**	**medium**	**bad**	**bad**
S2	**medium**	**medium**	**bad**	**medium**
S3	**medium**	**medium**	**medium**	**medium**
S4	**good**	**good**	**medium**	**good**
S5	**good**	**medium**	**good**	**good**
S6	**good**	**good**	**good**	**good**
S7	**bad**	**bad**	**bad**	**bad**
S8	**bad**	**bad**	**medium**	**bad**

- Examples of classification of S1 and S2 are inconsistent

  Quality of approximation by {**M**,**Ph**,**L**} = 6/8 = 0.75

DRSA for multiple criteria classification

- If we eliminate Literature, then more inconsistencies appear:

Student	Mathematics (**M**)	Physics (**Ph**)	Literature (**L**)	Overall class
S1	**good**	**medium**	**bad**	**bad**
S2	**medium**	**medium**	**bad**	**medium**
S3	**medium**	**medium**	**medium**	**medium**
S4	**good**	**good**	**medium**	**good**
S5	**good**	**medium**	**good**	**good**
S6	**good**	**good**	**good**	**good**
S7	**bad**	**bad**	**bad**	**bad**
S8	**bad**	**bad**	**medium**	**bad**

- Examples of classification of S1, S2, S3 and S5 are inconsistent

DRSA for multiple criteria classification

- Elimination of Mathematics does not increase inconsistencies:

Student	Mathematics (**M**)	Physics (**Ph**)	Literature (**L**)	Overall class
S1	**good**	**medium**	**bad**	**bad**
S2	**medium**	**medium**	**bad**	**medium**
S3	**medium**	**medium**	**medium**	**medium**
S4	**good**	**good**	**medium**	**good**
S5	**good**	**medium**	**good**	**good**
S6	**good**	**good**	**good**	**good**
S7	**bad**	**bad**	**bad**	**bad**
S8	**bad**	**bad**	**medium**	**bad**

- Subset of criteria {Ph,L} is a reduct of {M,Ph,L}

DRSA for multiple criteria classification

- Elimination of Physics also does not increase inconsistencies:

Student	Mathematics (**M**)	Physics (**Ph**)	Literature (**L**)	Overall class
S1	**good**	**medium**	**bad**	**bad**
S2	**medium**	**medium**	**bad**	**medium**
S3	**medium**	**medium**	**medium**	**medium**
S4	**good**	**good**	**medium**	**good**
S5	**good**	**medium**	**good**	**good**
S6	**good**	**good**	**good**	**good**
S7	**bad**	**bad**	**bad**	**bad**
S8	**bad**	**bad**	**medium**	**bad**

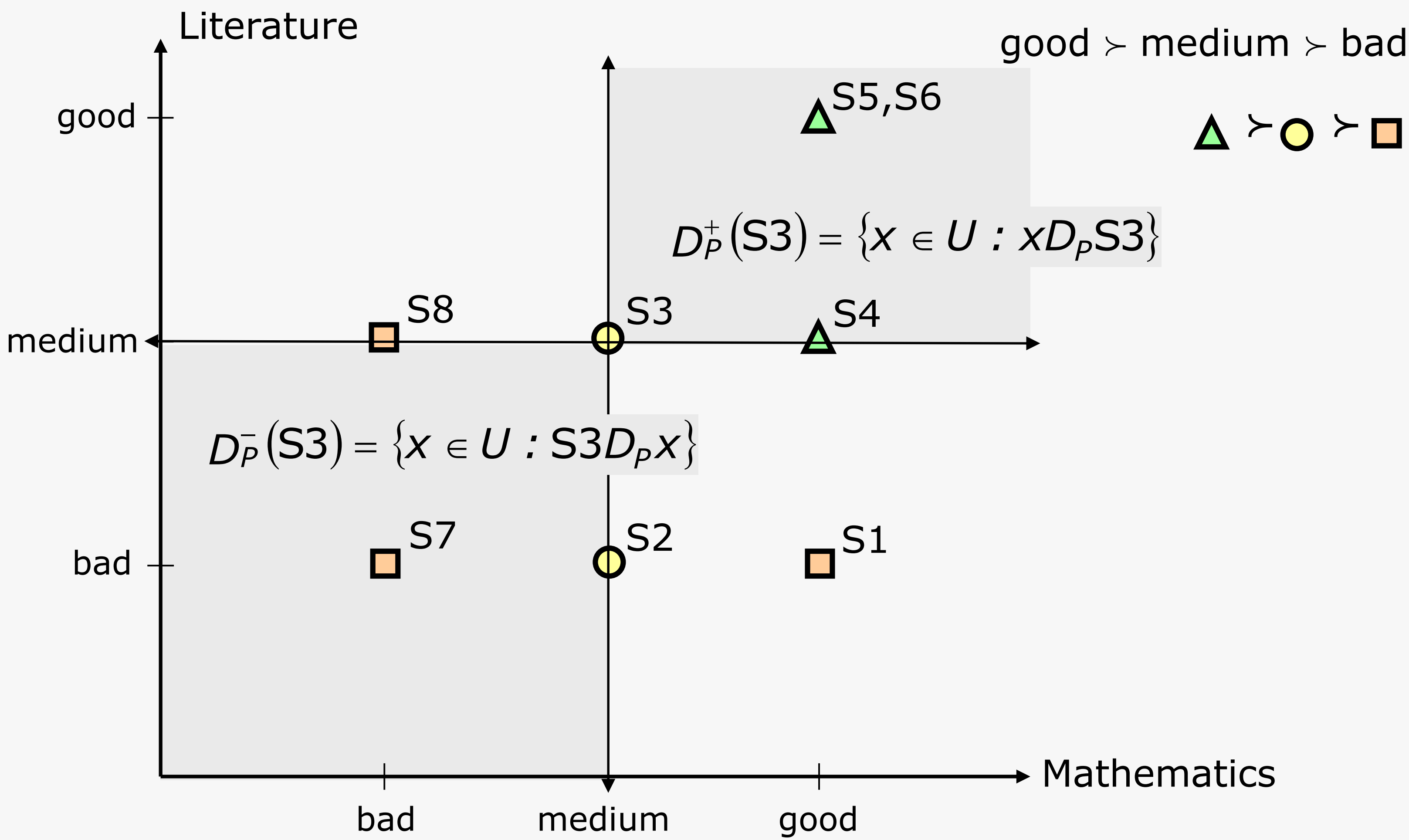
- Subset of criteria {M,L} is a reduct of {M,Ph,L}
- Intersection of reducts {M,L} and {Ph,L} gives the core {L}

DRSA for multiple criteria classification

- Let us represent the students in condition space {M,L} :

DRSA for multiple criteria classification

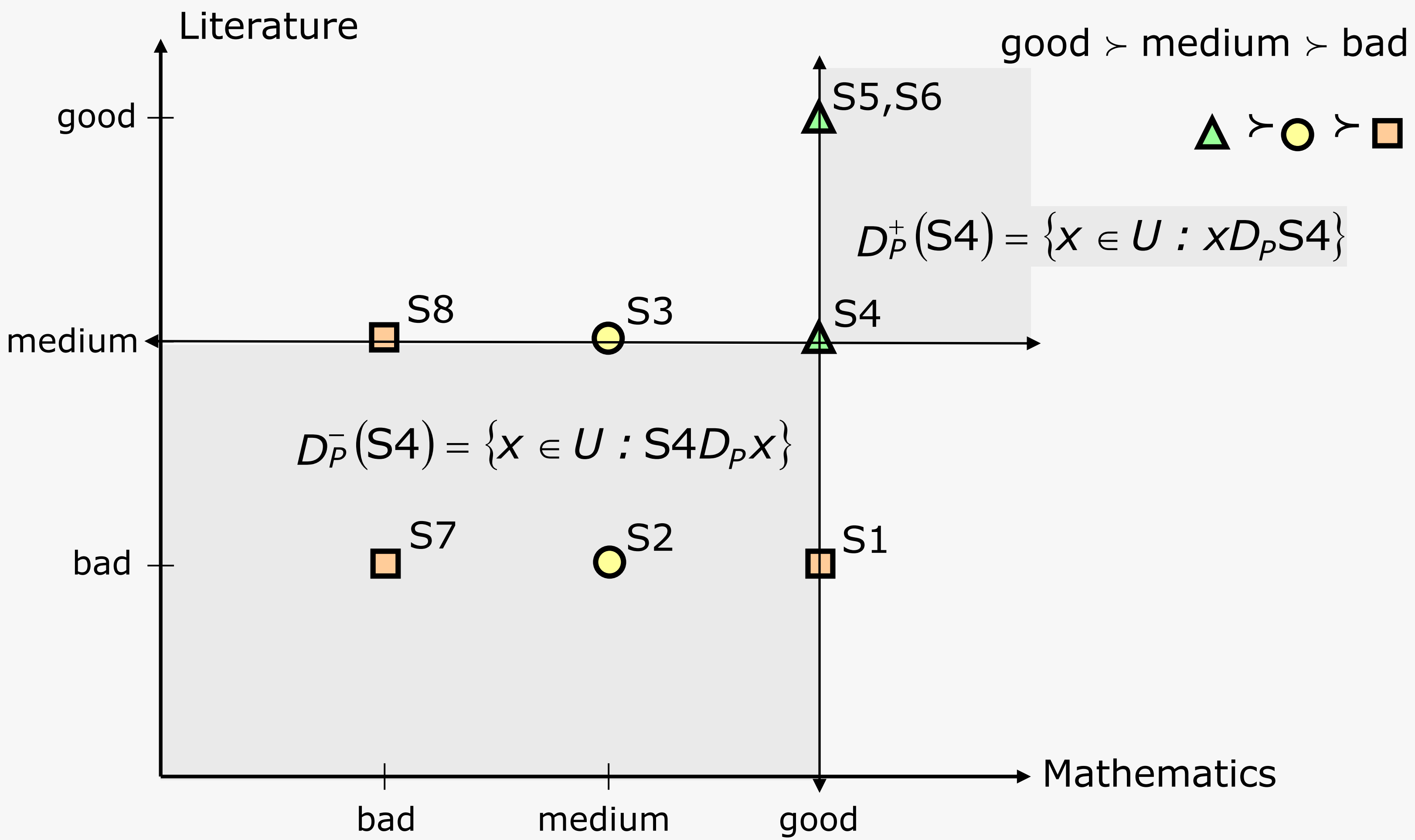
- Dominance cones in condition space {M,L} :



- P={M,L}

DRSA for multiple criteria classification

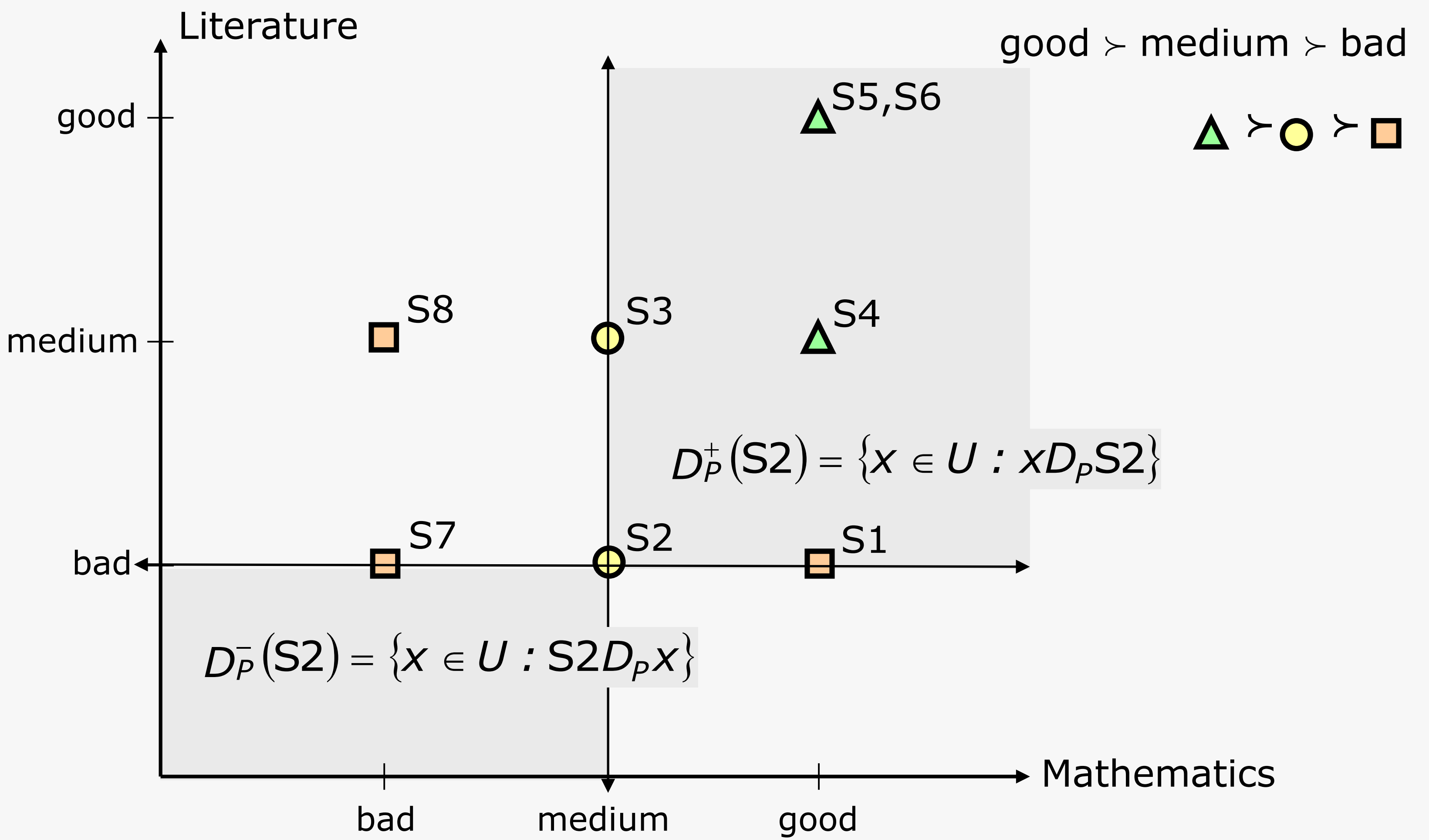
- Dominance cones in condition space {M,L} :



- P={M,L}

DRSA for multiple criteria classification

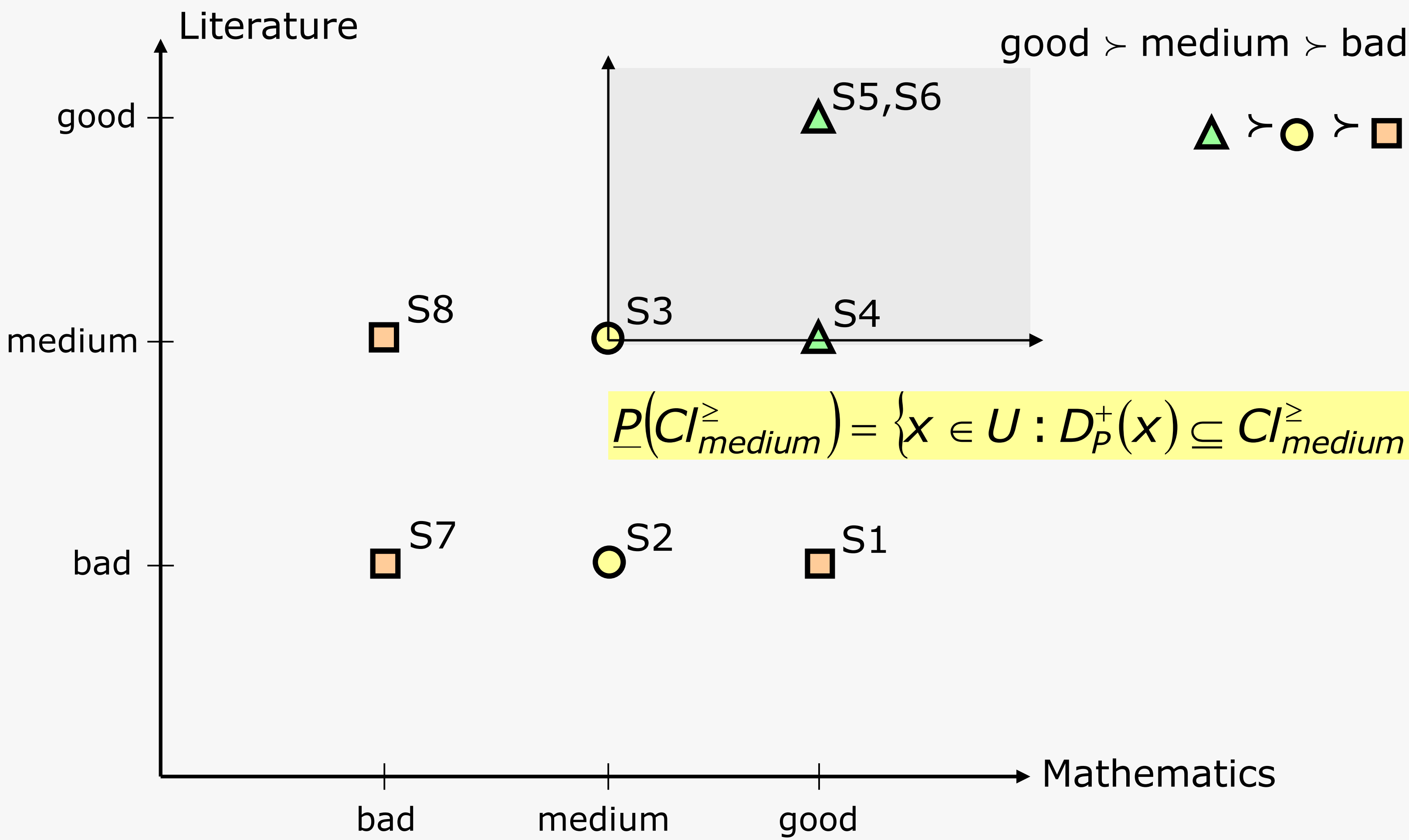
- Dominance cones in condition space {M,L} :



- P={M,L}

DRSA for multiple criteria classification

- Lower approximation of at least medium students:

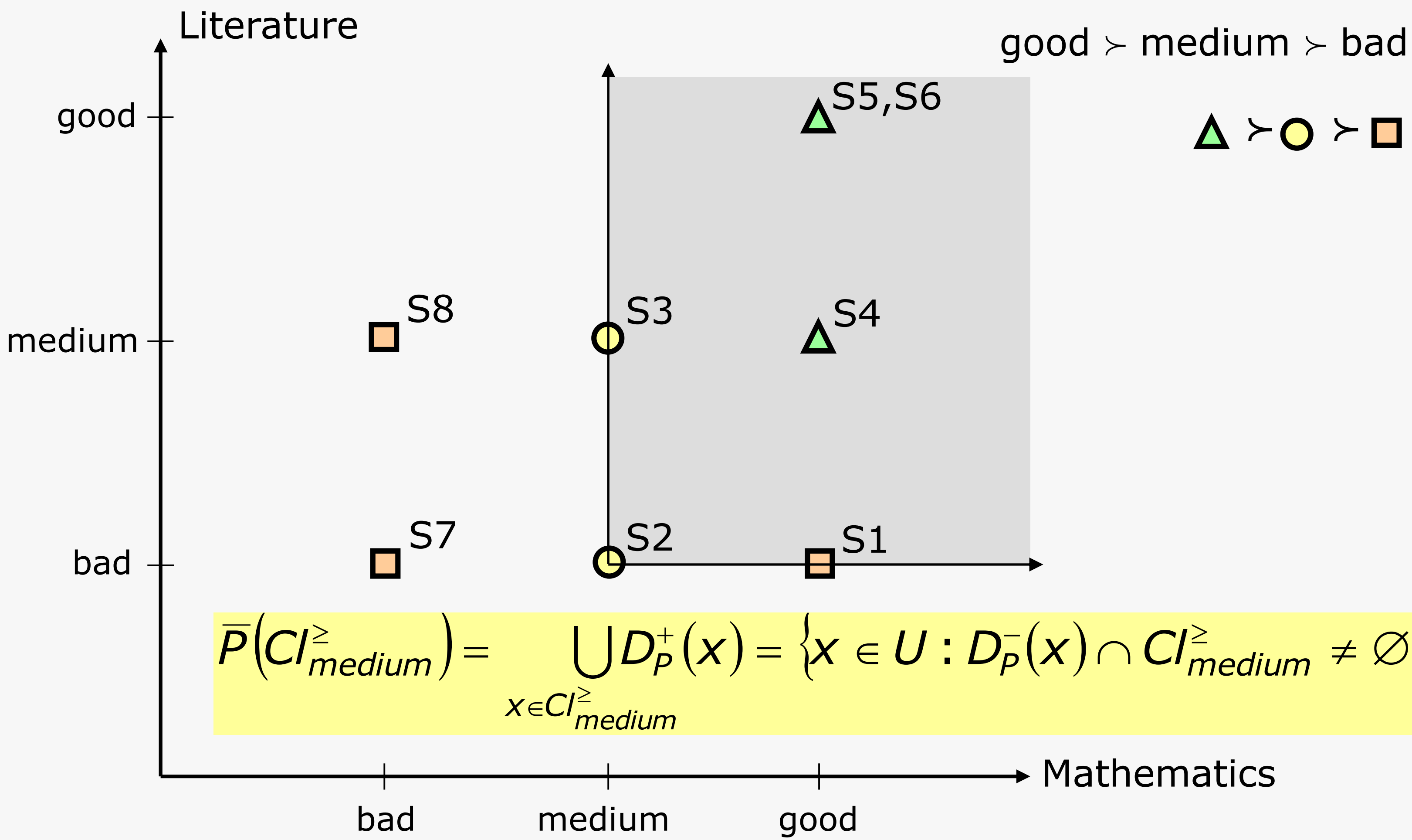


$$\underline{P}\left(Cl^{\geq}_{medium}\right) = \left\{x \in U : D^{+}_{P}(x) \subseteq Cl^{\geq}_{medium}\right\}$$

- P={M,L}

DRSA for multiple criteria classification

- Upper approximation of <u>at least</u> medium students:

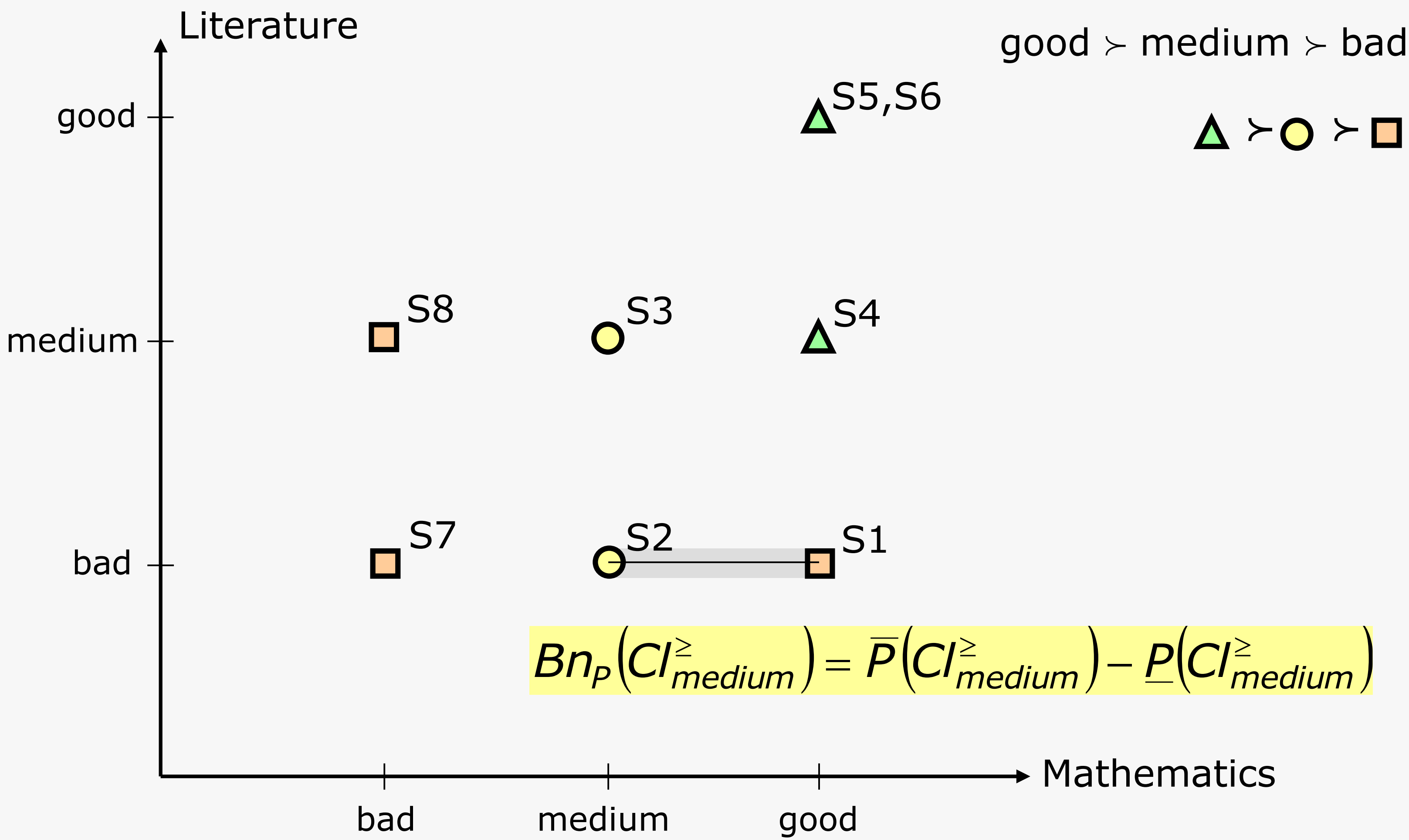


$$\overline{P}\left(Cl^{\geq}_{medium}\right) = \bigcup_{x \in Cl^{\geq}_{medium}} D^{+}_{P}(x) = \left\{x \in U : D^{-}_{P}(x) \cap Cl^{\geq}_{medium} \neq \varnothing\right\}$$

- P={M,L}

DRSA for multiple criteria classification

- Boundary region of <u>at least</u> medium students:

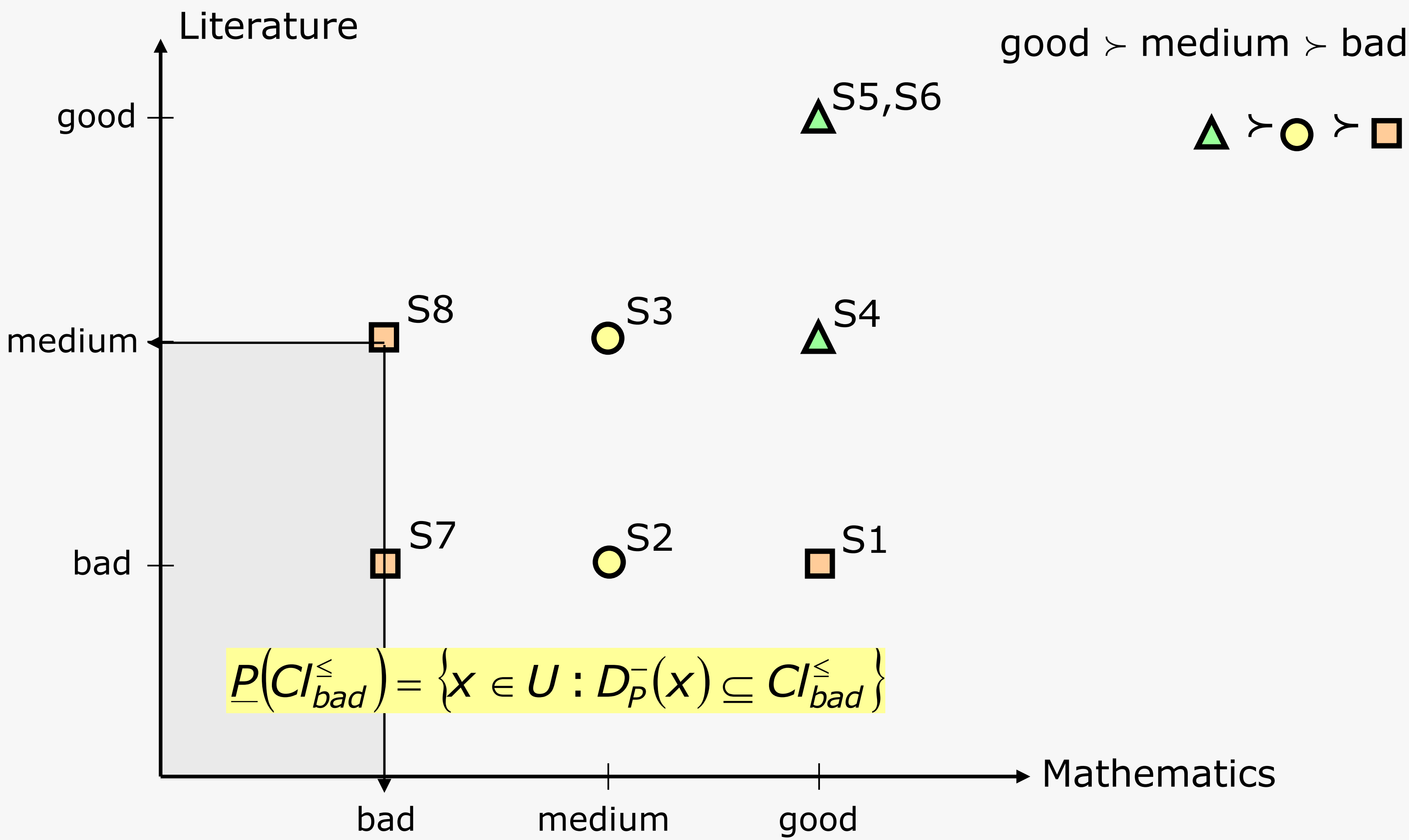


$$Bn_P\left(Cl^{\geq}_{medium}\right) = \overline{P}\left(Cl^{\geq}_{medium}\right) - \underline{P}\left(Cl^{\geq}_{medium}\right)$$

- P={M,L}

DRSA for multiple criteria classification

- Lower approximation of <u>at most</u> bad students:

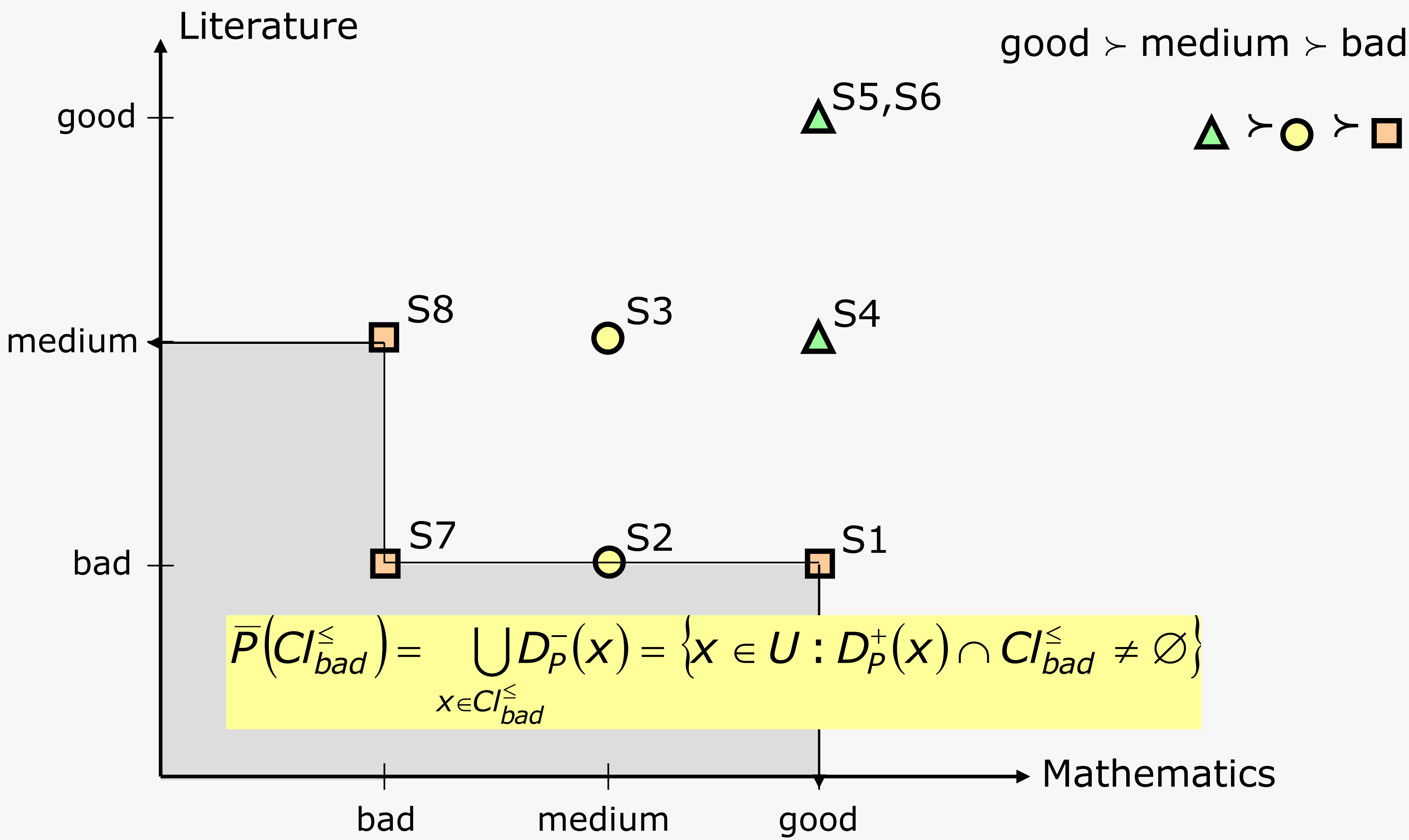


$$\underline{P}\left(Cl_{bad}^{\leq}\right)=\left\{x\in U: D_{P}^{-}(x)\subseteq Cl_{bad}^{\leq}\right\}$$

- P={M,L}

DRSA for multiple criteria classification

- Upper approximation of at most bad students:

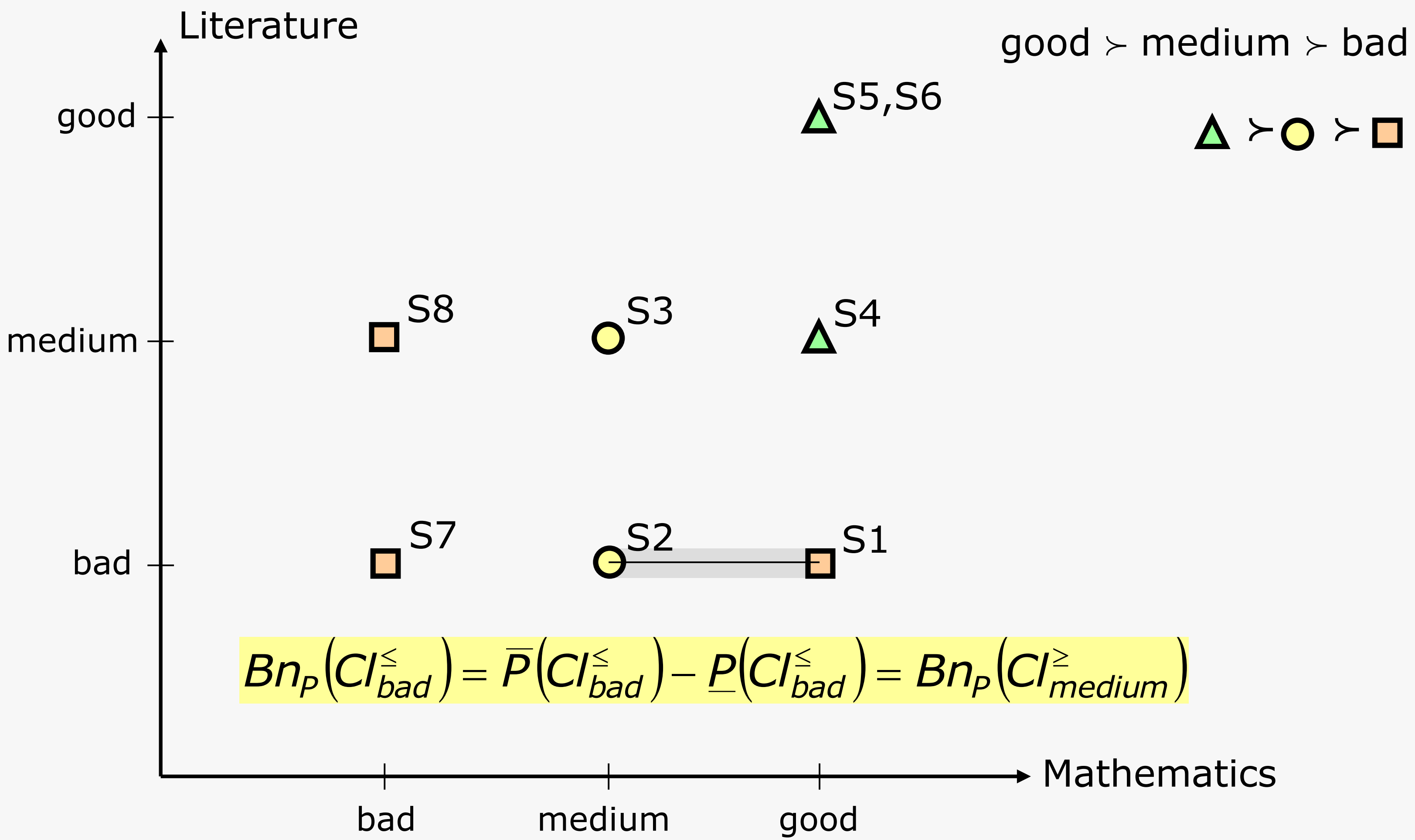


$$\overline{P}\left(Cl_{bad}^{\leq}\right)=\bigcup_{x\in Cl_{bad}^{\leq}} D_P^{-}(x)=\left\{x\in U: D_P^{+}(x)\cap Cl_{bad}^{\leq}\neq\emptyset\right\}$$

- P={M,L}

DRSA for multiple criteria classification

- Boundary region of at most bad students:



$$Bn_P\left(Cl_{bad}^{\leq}\right) = \overline{P}\left(Cl_{bad}^{\leq}\right) - \underline{P}\left(Cl_{bad}^{\leq}\right) = Bn_P\left(Cl_{medium}^{\geq}\right)$$

- P={M,L}

DRSA – properties

- Basic properties of rough approximations

$$\underline{P}(Cl_t^{\geq}) \subseteq Cl_t^{\geq} \subseteq \overline{P}(Cl_t^{\geq}) \qquad \underline{P}(Cl_t^{\leq}) \subseteq Cl_t^{\leq} \subseteq \overline{P}(Cl_t^{\leq})$$

$$\underline{P}(Cl_t^{\geq}) = U - \overline{P}(Cl_{t-1}^{\leq}),\ \text{ for } t=2,\ldots,m$$

- Identity of boundaries $Bn_P(Cl_t^{\geq}) = Bn_P(Cl_{t-1}^{\leq})$, for t=2,…,m

- Quality of approximation of classification $\boldsymbol{Cl}=\{Cl_t,\ t=1,\ldots m\}$ by set $P\subseteq C$

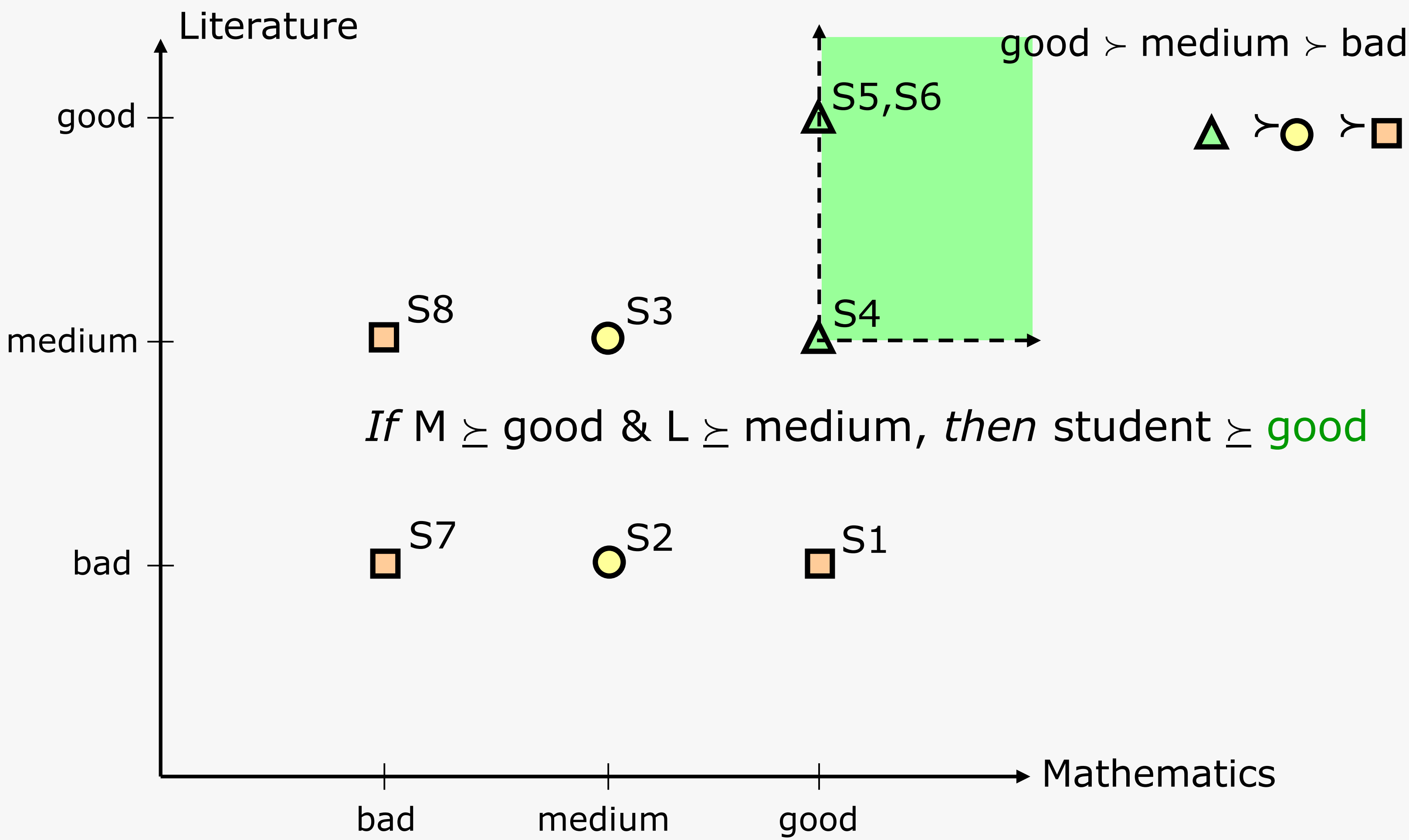
$$\gamma_P(\boldsymbol{Cl}) = \frac{\left|U - \bigcup_{t\in\{2,\ldots,m\}} Bn_P(Cl_t^{\geq})\right|}{|U|}$$

- ***Cl***-reducts and ***Cl***-core of $P\subseteq C$

$$CORE_{\boldsymbol{Cl}}(P) = \bigcap RED_{\boldsymbol{Cl}}(P)$$

DRSA for multiple criteria classification

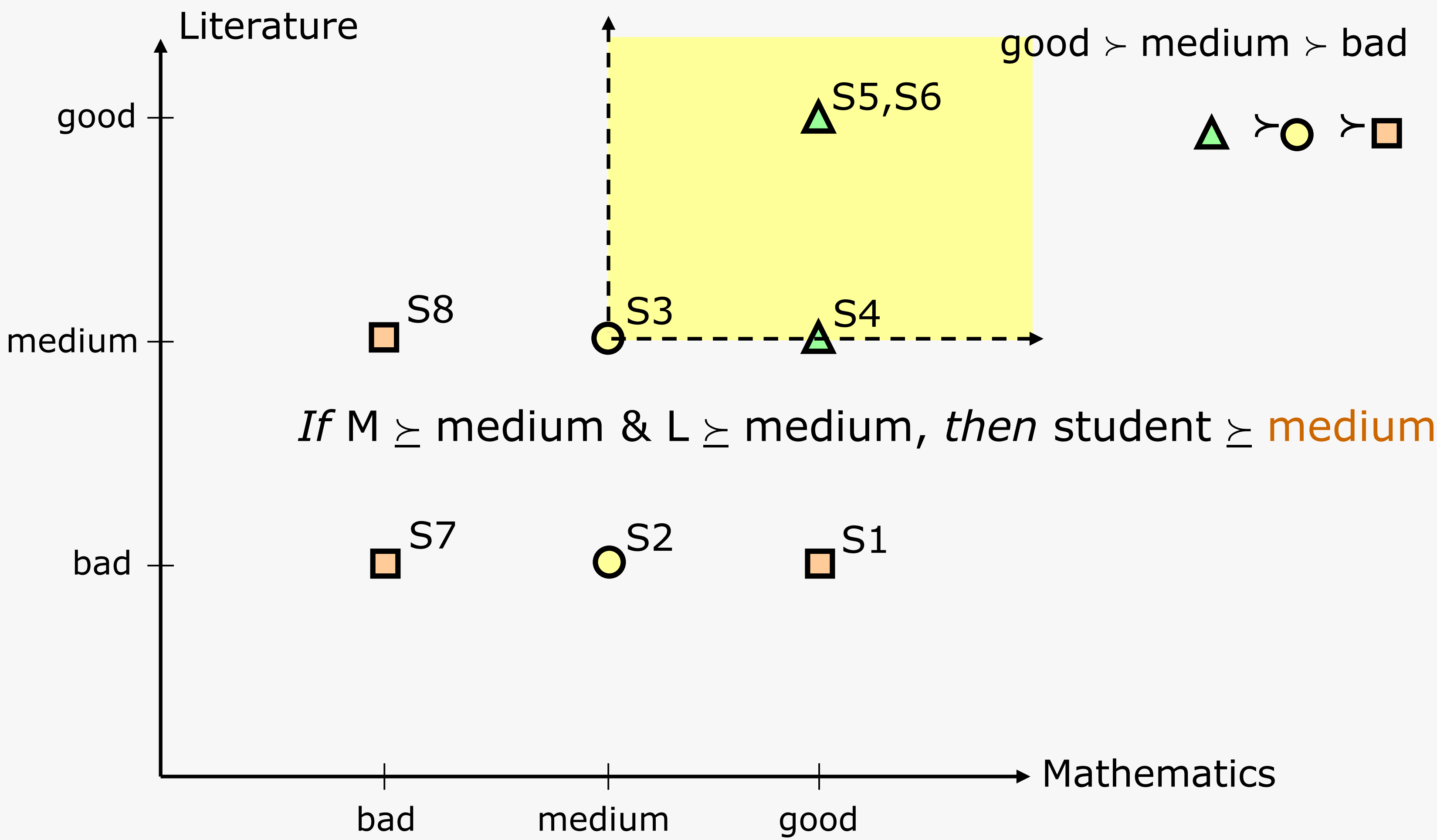
- Decision rules in terms of {M,L} :



- $D_{\geq}$ - certain rule

DRSA for multiple criteria classification

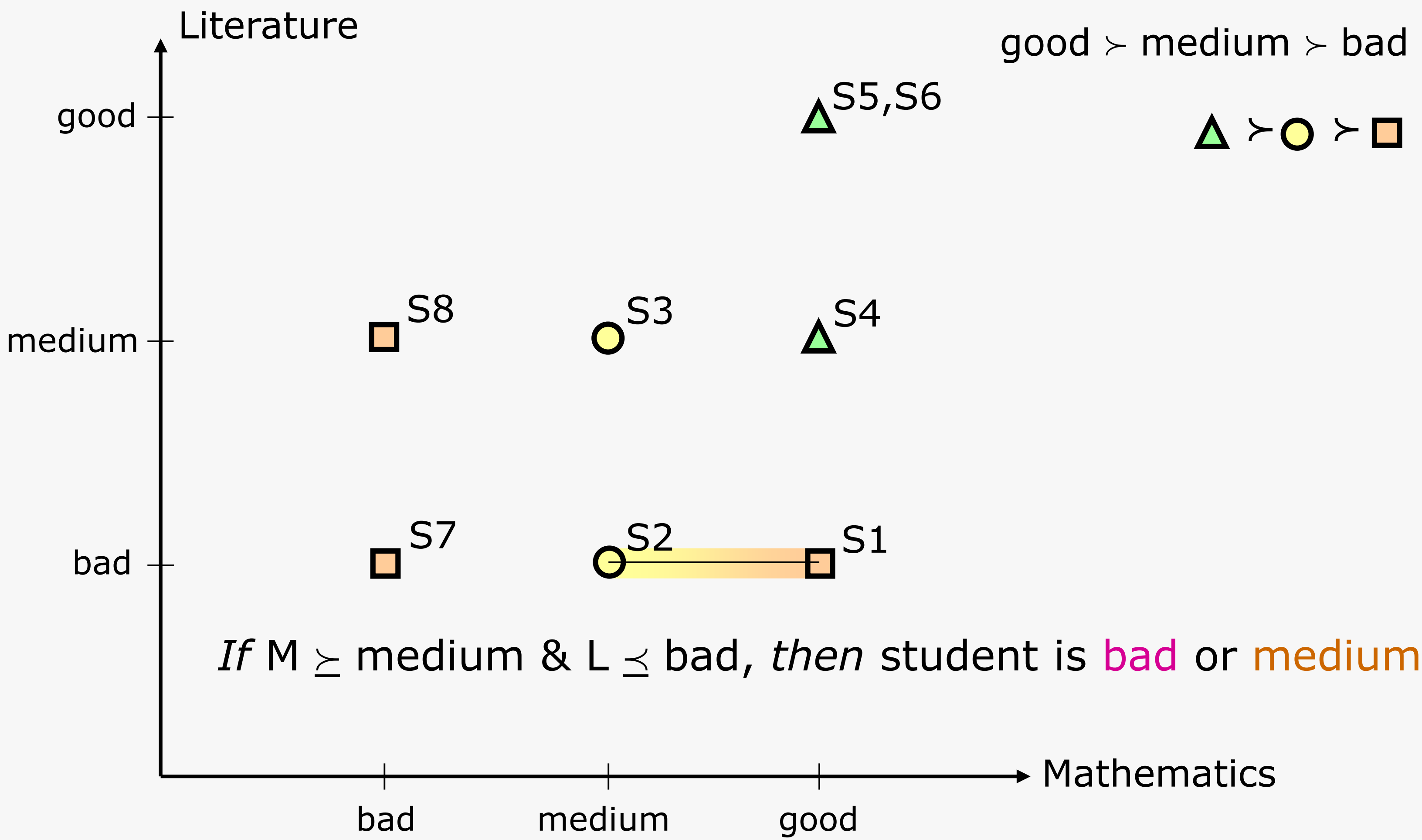
- Decision rules in terms of {M,L} :



- $D_{\geq}$ - certain rule

DRSA for multiple criteria classification

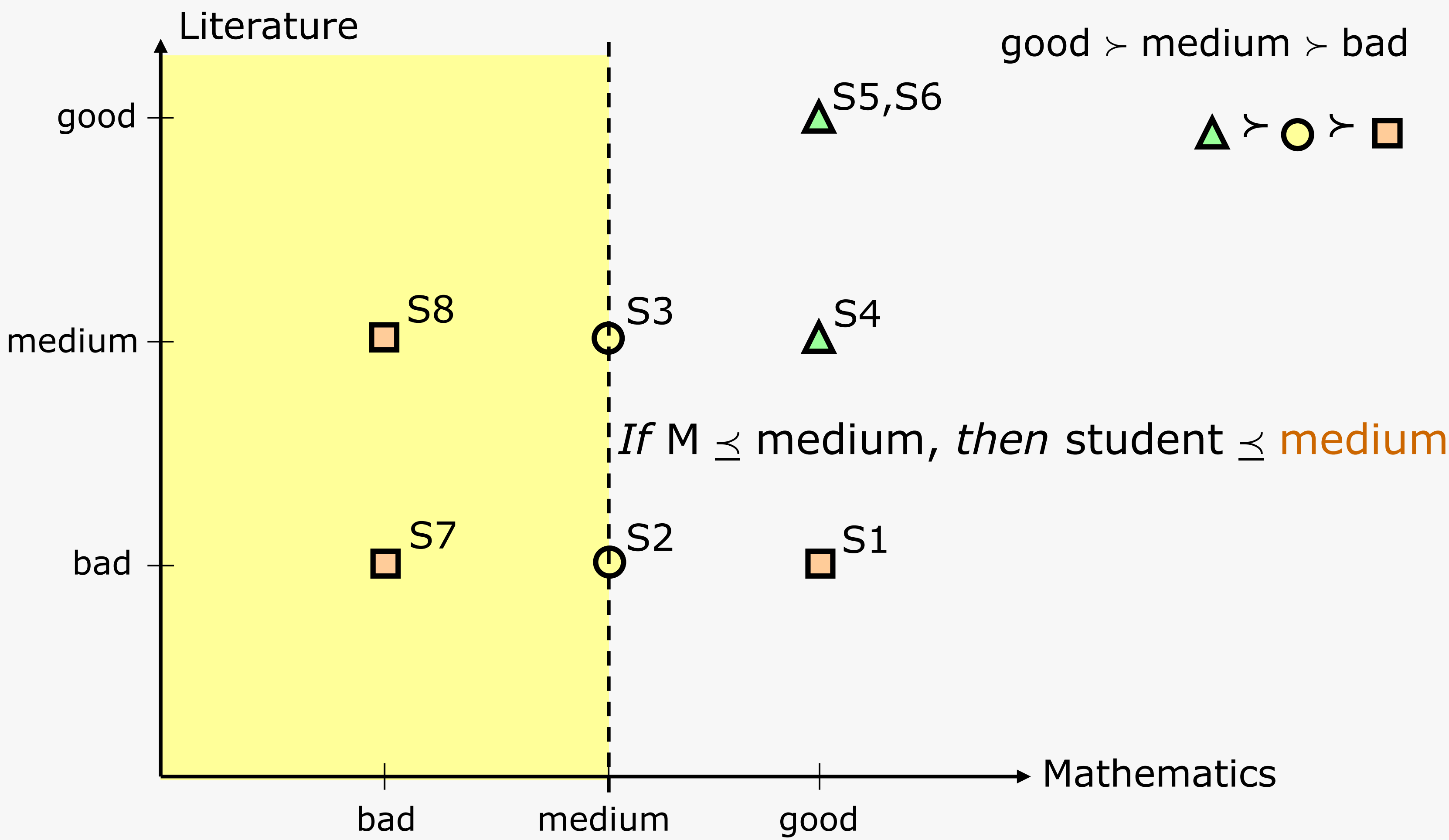
- Decision rules in terms of {M,L} :



- $D_{\geq\leq}$ - approximate rule

DRSA for multiple criteria classification

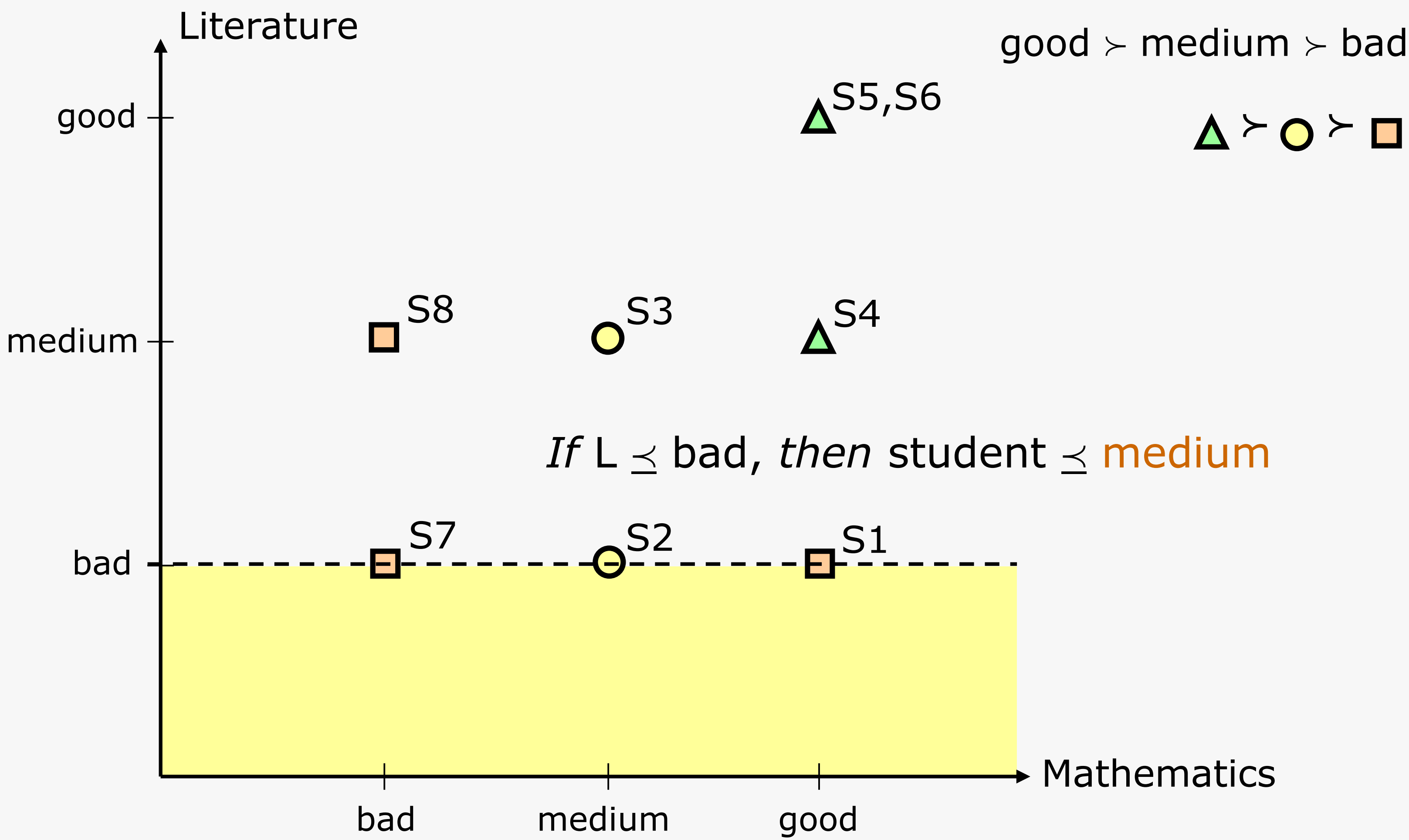
- Decision rules in terms of {M,L} :



- $D_{\leq}$ - certain rule

DRSA for multiple criteria classification

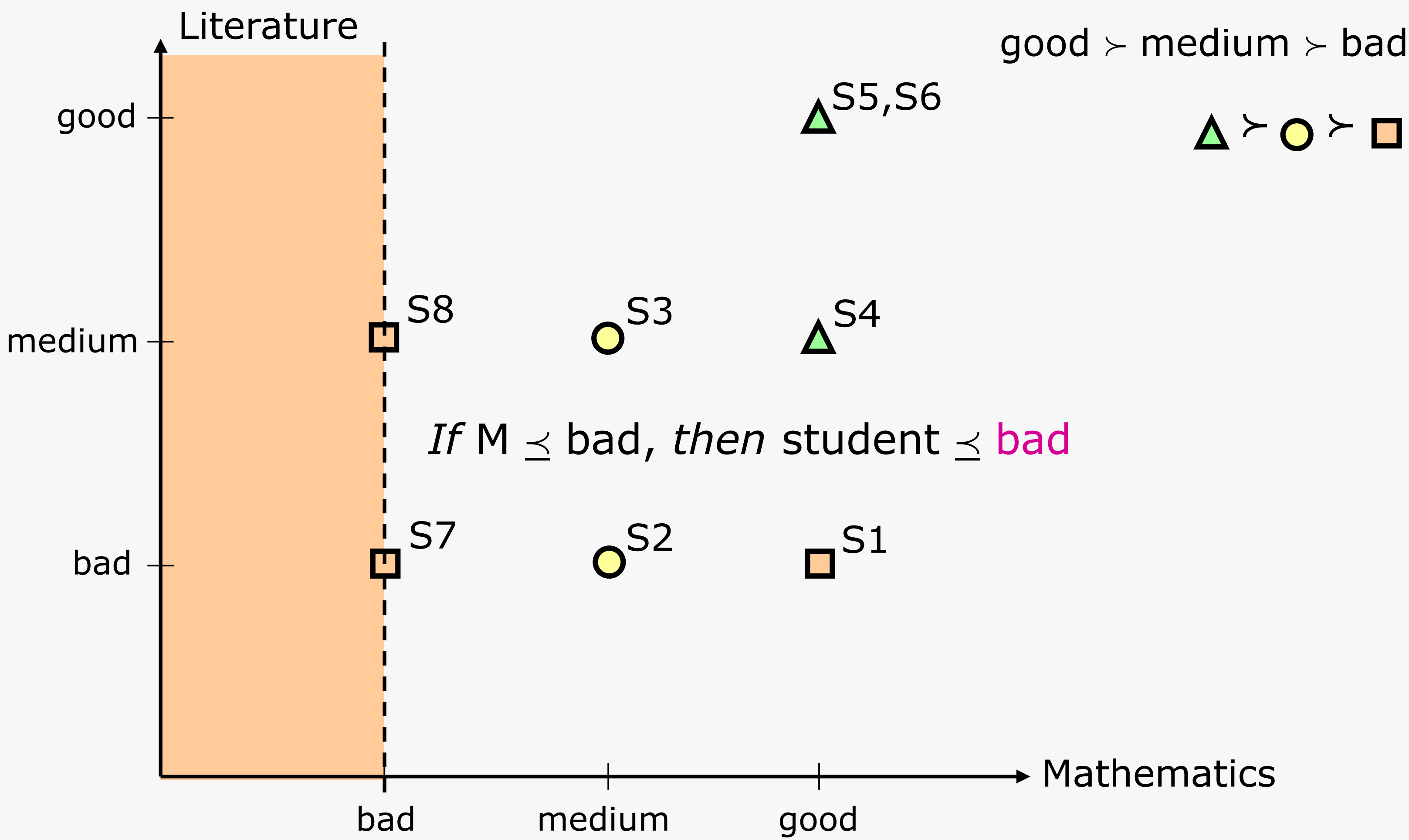
- Decision rules in terms of {M,L} :



- $D_{\leq}$ - certain rule

DRSA for multiple criteria classification

- Decision rules in terms of {M,L} :



- $D_{\leq}$ - certain rule

DRSA for multiple criteria classification

- Set of decision rules in terms of {M, L} representing preferences:

If M $\succeq$ good & L $\succeq$ medium, *then* student $\succeq$ good {S4,S5,S6}

If M $\succeq$ medium & L $\succeq$ medium, *then* student $\succeq$ medium {S3,S4,S5,S6}

If M $\succeq$ medium & L $\preceq$ bad, *then* student is bad or medium {S1,S2}

If M $\preceq$ medium, *then* student $\preceq$ medium {S2,S3,S7,S8}

If L $\preceq$ bad, *then* student $\preceq$ medium {S1,S2,S7}

If M $\preceq$ bad, *then* student $\preceq$ bad {S7,S8}

DRSA for multiple criteria classification

- Set of decision rules in terms of {M,Ph,L} representing preferences:

If M $\succeq$ good & L $\succeq$ medium, *then* student $\succeq$ good {S4,S5,S6}

If M $\succeq$ medium & L $\succeq$ medium, *then* student $\succeq$ medium {S3,S4,S5,S6}

If M $\succeq$ medium & L $\preceq$ bad, *then* student is bad or medium {S1,S2}

If Ph $\preceq$ medium & L $\preceq$ medium *then* student $\preceq$ medium {S1,S2,S3,S7,S8}

If M $\preceq$ bad, *then* student $\preceq$ bad {S7,S8}

- The preference model involving all three criteria is more concise

Using DRSA rules as a decision model

- New student to be evaluated

Student	Mathematics	Physics	Literature
S9	medium	medium	good

- Set of activated decision rules:

If M $\succeq$ medium & L $\succeq$ medium, *then* student $\succeq$ medium {S3,S4,S5,S6}

If M $\preceq$ medium, *then* student $\preceq$ medium {S2,S3,S7,S8}

- Set of non-activated decision rules:

If M $\succeq$ good & L $\succeq$ medium, *then* student $\succeq$ good {S4,S5,S6}

If M $\succeq$ medium & L $\preceq$ bad, *then* student is bad or medium {S1,S2}

If M $\preceq$ bad, *then* student $\preceq$ bad {S7,S8}

DRSA for multiple criteria classification

- Importance and interaction among criteria

- Quality of approximation of classification $\gamma_P(\boldsymbol{Cl})$ $(P \subseteq C)$ is a fuzzy measure with the property of Choquet capacity
  $(\gamma_\emptyset(\boldsymbol{Cl})=0,\ \gamma_C(\boldsymbol{Cl})=r$ and $\gamma_R(\boldsymbol{Cl}) \leq \gamma_P(\boldsymbol{Cl}) \leq r$ for any $R \subseteq P \subseteq C)$

- Such measure can be used to calculate Shapley value or Benzhaf index, i.e., an average „contribution" of criterion q in all coalitions of criteria, $q \in \{1,\ldots,n\}$

- Fuzzy measure theory permits, moreover, to calculate interaction indices for pairs (or larger subsets) of criteria (Murofushi & Soneda, Grabisch or Roubens),
  i.e., an average „added value" resulting from putting together q and q' in all coalitions of criteria, $q,q' \in \{1,\ldots,n\}$

DRSA for multiple criteria classification

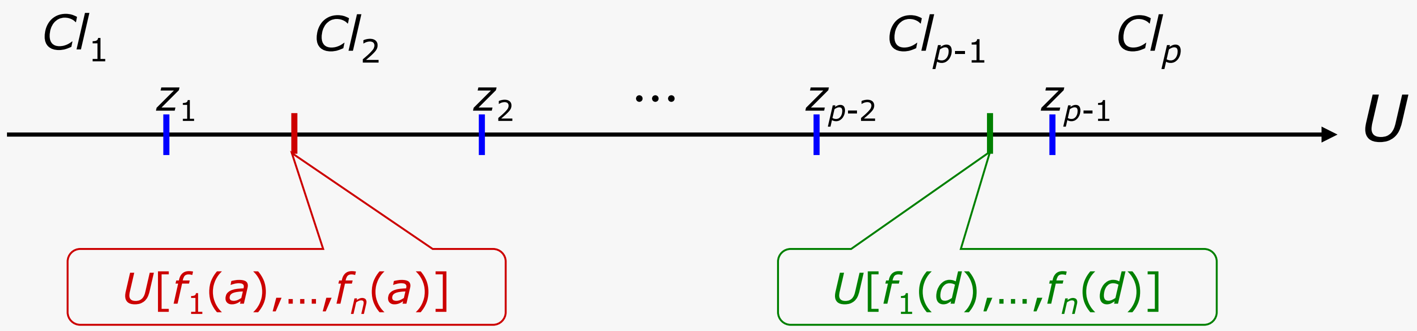
- Quality of approximation of classification of students

$$\gamma_C(\boldsymbol{Cl}) = [8-|\{S1,S2\}|]/8 = 0.75$$

Set of criteria P	Ambiguous objects	Non-ambiguous objects	Quality of classification	Shapley value
{Mathematics}	S1,S2,S3,S4,S5,S6	S7,S8	0.25	0.167
{Physics}	S1,S2,S3,S5	S4,S6,S7,S8	0.5	0.292
{Literature}	S1,S2,S3,S4,S7,S8	S5,S6	0.25	0.292
{Mathematics, Physics}	S1,S2,S3,S5	S4,S6,S7,S8	0.5	-0.375
{Mathematics, Literature}	S1,S2	S3,S4,S5,S6,S7,S8	0.75	0.125
{Physics, Literature}	S1,S2	S3,S4,S5,S6,S7,S8	0.75	-0.125
{Mathematics, Physics, Literature}	S1,S2	S3,S4,S5,S6,S7,S8	0.75	-0.125

Comparison of decision rule preference model and utility function

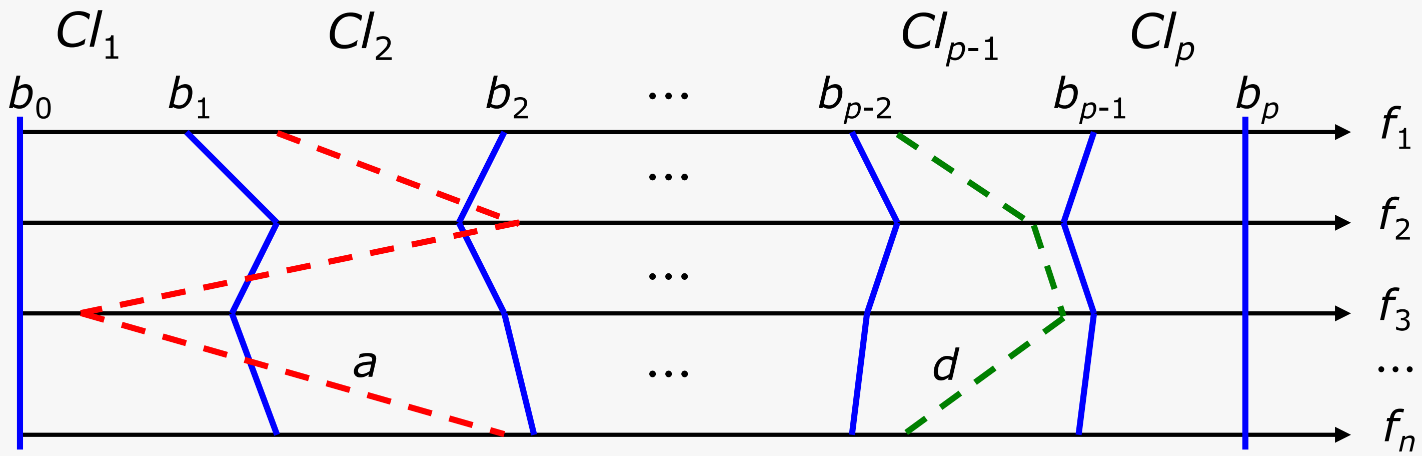
- Value-driven methods
- The preference model is a utility function U and a set of thresholds z_t, t=1,…,p-1, on U, separating the decision classes Cl_t, t=0,1,…,p



- A value of utility function U is calculated for each action $a \in A$
- e.g. $a \rightarrow Cl_2$, $d \rightarrow Cl_{p-1}$

Comparison of decision rule preference model and outranking relation

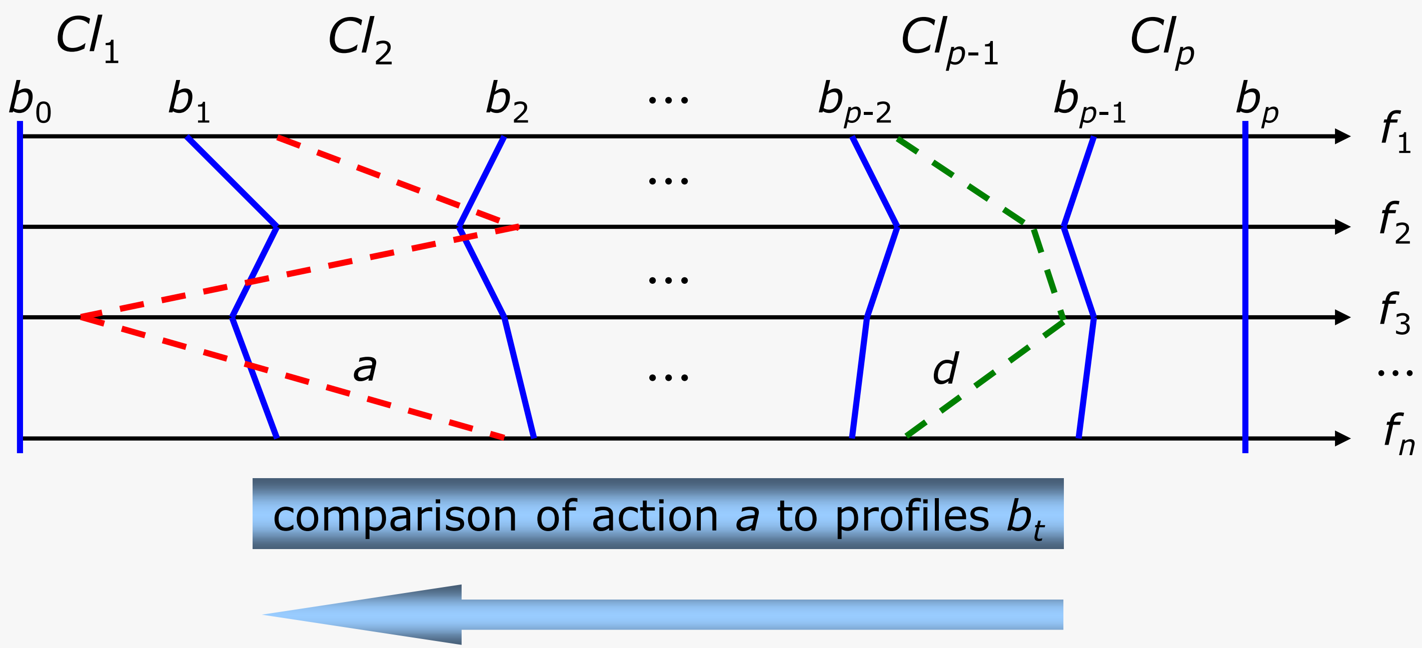
- ELECTRE TRI
- Decision classes Cl_t are caracterized by limit profiles b_t, t=0,1,…,p



- The preference model, i.e. outranking relation S, is constructed for each couple (a, b_t), for every $a \in A$ and b_t, t=0,1,…,p

Comparison of decision rule preference model and outranking relation

- ELECTRE TRI
- Decision classes Cl_t are caracterized by limit profiles b_t, t=0,1,...,p

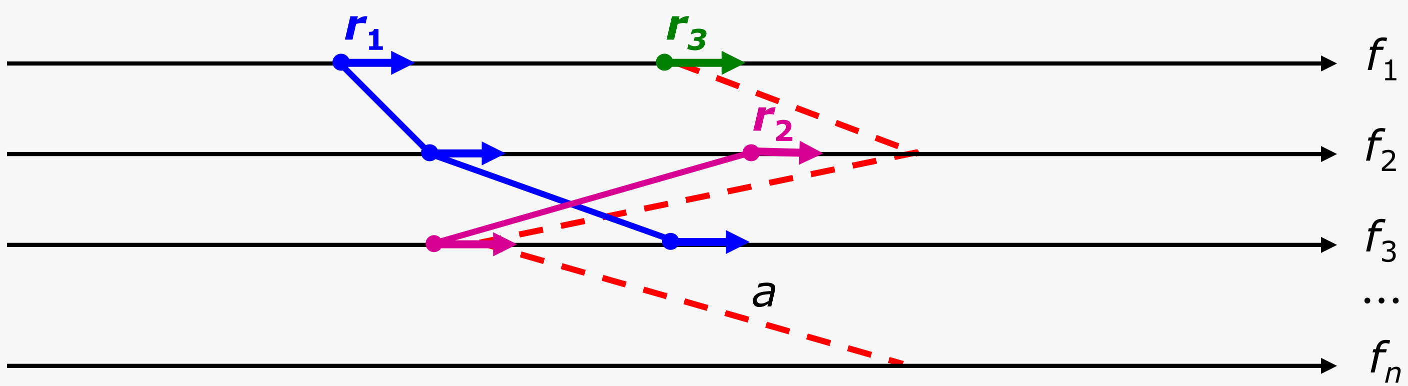


- Compare action a successively to each profile b_t, t=p-1,...,1,0;
  if b_t is the first profile such that aSb_t, then $a \rightarrow Cl_{t+1}$
- e.g. $a \rightarrow Cl_1$, $d \rightarrow Cl_{p-1}$

Comparison of decision rule preference model and outranking relation

- Rule-based classification
- The preference model is a set of decision rules for unions $Cl_t^{\geq}$, $t=2,\ldots,p$

e.g. for $Cl_2^{\geq}$



- A decision rule compares an action profile to a partial profile using a dominance relation
- e.g. $a \rightarrow Cl_2^{\geq}$, because profile of a dominates partial profiles of $\boldsymbol{r_2}$ and $\boldsymbol{r_3}$

Important feature of DRSA

- DRSA exploits ordinal information only, and decision rules do not convert ordinal information into numeric one

- *„Si l'ordre apparaît quelque part dans la qualité, pourquoi chercherions-nous à passer par l'intermédiaire du numbre?"* (G.Bachelard 1934)

  (*„If an order appears somewhere in quality, why should we like to interprete this order through numerical values?"*)

- Pareto-dominance can be replaced by Lorenz-dominance, making decision rules more equitable and risk-averse

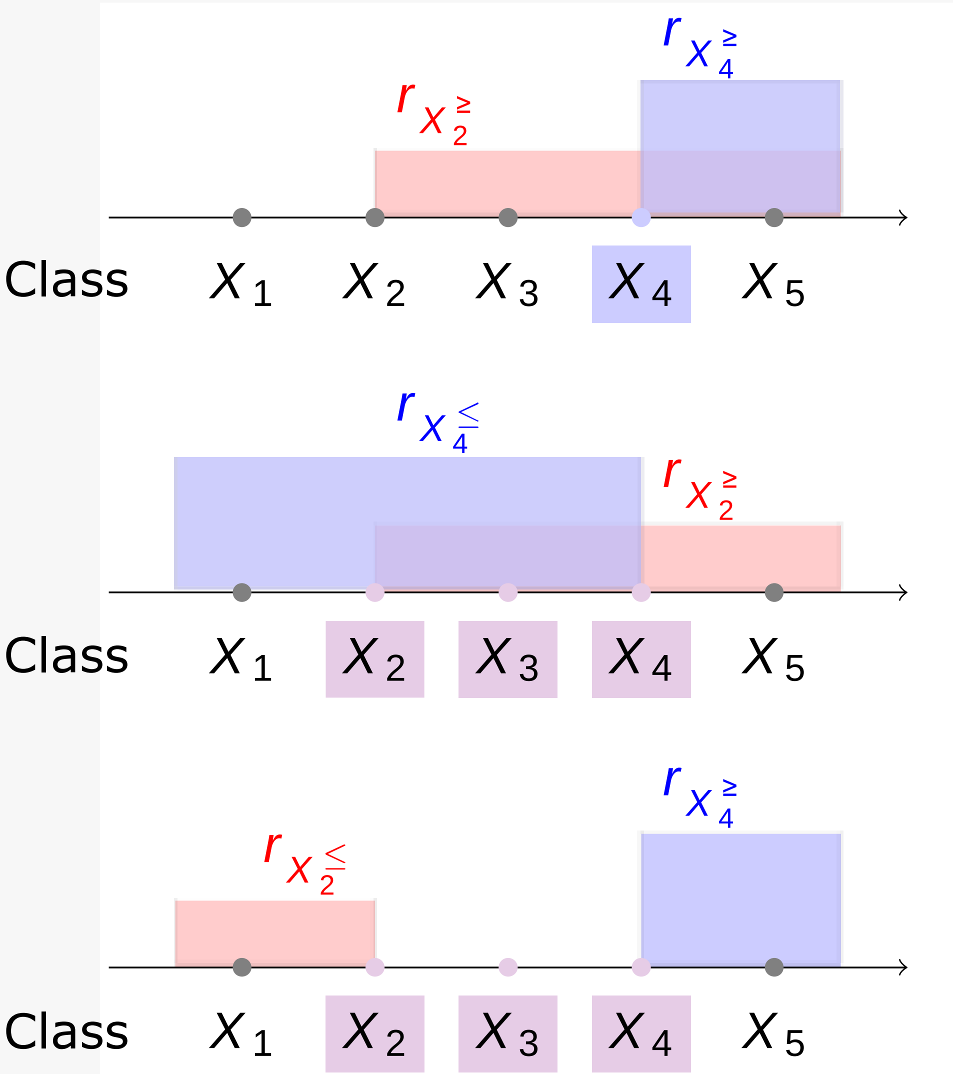
Preference modelling by dominance-based decision rules

- Dominance-based *„if…, then…”* decision rules are the only aggregation operators that:
  - give account of most complex interactions among criteria,
  - are non-compensatory,
  - accept ordinal evaluation scales and do not convert ordinal evalautions into cardinal ones,
- Rules identify values that drive DM's decisions – each rule is a scenario of a causal relationship between evaluations on a subset of criteria and a comprehensive judgment

R.Słowiński, S.Greco, B.Matarazzo: Axiomatization of utility, outranking and decision-rule preference models for multiple-criteria classification problems under partial inconsistency with the dominance principle, *Control and Cybernetics*, 31 (2002) no.4, 1005-1035

Classification with monotonic decision rules

Application of decision rules to multiple criteria classification



Only „at least" or „at most" rules

Overlapping „at least" and „at most" rules

Disjoint „at least" and „at most" rules

Application of decision rules to multiple criteria classification

- Let $\varphi_1 \rightarrow \psi_1$,…, $\varphi_k \rightarrow \psi_k$, be the rules matching object x

  $\left(\psi_j = Cl_s^{\geq} \text{ or } \psi_j = Cl_q^{\leq},\ s,q \in \{1,\dots,m\},\ j = 1,\dots,k\right)$

- $\boxed{R_t(x)=\{j:\ Cl_t \in \psi_j,\ j=1,\dots,k\},}$ $\boxed{R_{\neg t}(x)=\{j:\ Cl_t \notin \psi_j,\ j=1,\dots,k\}}$

  $\|\varphi_j\|, \|\psi_j\|$ are sets of objects with property φ_j, ψ_j, respectively, $j=1,\dots,k$

- For classified object x, the score is calculated for each candidate class Cl_t, $t=1,\dots,m$

$$\boxed{score(Cl_t, x) = score^{+}(Cl_t, x) - score^{-}(Cl_t, x)}$$

$$score^{+}(Cl_t, x) = \frac{\left|\bigcup_{j \in R_t(x)} \left(\|\varphi_j\| \cap Cl_t\right)\right|^2}{\left|\bigcup_{j \in R_t(x)} \|\varphi_j\|\right| \times |Cl_t|} \qquad score^{-}(Cl_t, x) = \frac{\left|\bigcup_{j \in \neg R_t(x)} \left(\|\varphi_j\| \cap \|\psi_j\|\right)\right|^2}{\left|\bigcup_{j \in \neg R_t(x)} \|\varphi_j\|\right| \times \left|\bigcup_{j \in \neg R_t(x)} \|\psi_j\|\right|}$$

Application of decision rules to multiple criteria classification

- *score*$^+$(Cl_t , x) can be interpreted as confidence $\times$ coverage

$$score^+\left(Cl_t,x\right) = Pr\left(\left\{\varphi_j\colon j \in R_t\left(x\right)\right\} \mid Cl_t\right) \times Pr\left(Cl_t \mid \left\{\varphi_j\colon j \in R_t\left(x\right)\right\}\right)$$

- *score*$^-$(Cl_t , x) can be interpreted as

$$score^-\left(Cl_t,x\right) = Pr\left(\left\{\varphi_j\colon j \in R_{\neg t}\left(x\right)\right\} \mid \neg Cl_t\right) \times Pr\left(\neg Cl_t \mid \left\{\varphi_j\colon j \in R_{\neg t}\left(x\right)\right\}\right)$$

Recommendation: $x \rightarrow Cl_t$

where $Cl_t = \underset{t\in\{1,\dots,m\}}{arg\,max}\left(score\left(Cl_t,x\right)\right)$

J.Błaszczyński, S.Greco, R.Słowiński: Multi-criteria classification – a new scheme for application of dominance-based decision rules. *European J. Operational Research*, 181 (2007) 1030-1044

J.Błaszczyński, R.Słowiński, M.Szeląg: Sequential covering rule induction algorithm for variable consistency rough set approaches. *Information Sciences*, 181 (2011) 987-1002

Computational experiment – data sets

Id	Data set	# Objects	# Attributes	# Classes
1	balance	625	4	3
2	breast-c	286	7	2
3	breast-w	699	9	2
4	car	1296	6	4
5	cpu	209	6	4
6	bank-g	1411	16	2
7	fame	1328	10	5
8	denbosch	119	8	2
9	ERA	1000	4	9
10	ESL	488	4	9
11	housing	506	13	4
12	LEV	1000	4	5
13	SWD	1000	10	4
14	windsor	546	10	4

The directions of ordering in the domains of the attributes are known.

Predictive accuracy is measured on the basis of averaged stratified 10-fold cross validation estimates.

Classification accuracy results: ϵ-VC-DomLEM gets minimal MAE

- Averaged stratified 10-fold cross validation estimates

rank

data set	ϵ-VC-DomLEM	μ-VC-DomLEM	Naive Bayes	SMO	Ripper	J48	OLM	OSDL
balance	0.1621 (2) ±0.001996	0.1659 (3) ±0.002719	0.1104 (1) ±0.002613	0.1723 (4) ±0.003017	0.2917 (5) ±0.01088	0.3088 (6) ±0.02174	0.6384 (7) ±0.01713	0.7003 (8) ±0.004588
breast-cancer	0.2331 (1) ±0.003297	0.2436 (3) ±0.007185	0.2564 (4) ±0.005943	0.3217 (7) ±0.01244	0.2960 (5) ±0.01154	0.2424 (2) ±0.003297	0.324 (8) ±0.01835	0.3065 (6) ±0.001648
breast-w	0.03720 (2) ±0.002023	0.04578 (6) ±0.003504	0.03958 (3) ±0.0006744	0.03243 (1) ±0.0006744	0.04483 (5) ±0.004721	0.05532 (7) ±0.00751	0.1764 (8) ±0.00552	0.04149 (4) ±0.001168
car	0.03421 (1) ±0.0007275	0.03524 (2) ±0.0009624	0.1757 (7) ±0.002025	0.08668 (4) ±0.002025	0.2029 (8) ±0.01302	0.1168 (6) ±0.003108	0.09156 (5) ±0.005358	0.04141 (3) ±0.0009624
cpu	0.08293 (1) ±0.01479	0.0925 (2) ±0.01579	0.1707 (5) ±0.009832	0.4386 (8) ±0.01579	0.1611 (4) ±0.01372	0.1196 (3) ±0.01790	0.3461 (7) ±0.02744	0.3158 (6) ±0.01034
bank-g	0.04536 (1) ±0.001531	0.04867 (2) ±0.000884	0.1146 (6) ±0.01371	0.1280 (7) ±0.001205	0.0489 (3) ±0.00352	0.0515 (4) ±0.005251	0.05528 (5) ±0.001736	0.1545 (8) ±0
fame	0.3406 (1.5) ±0.001878	0.3469 (3) ±0.004	0.4829 (6) ±0.002906	0.3406 (1.5) ±0.001775	0.3991 (5) ±0.003195	0.3863 (4) ±0.005253	1.577 (7) ±0.03791	1.592 (8) ±0.007555
denbosch	0.1232 (1) ±0.01048	0.1289 (2.5) ±0.01428	0.1289 (2.5) ±0.01428	0.2129 (7) ±0.003961	0.1737 (6) ±0.02598	0.1653 (5) ±0.01048	0.2633 (8) ±0.02206	0.1541 (4) ±0.003961
ERA	1.307 (2) ±0.002055	1.364 (7) ±0.006018	1.325 (5) ±0.003771	1.318 (3) ±0.007257	1.681 (8) ±0.01558	1.326 (6) ±0.006018	1.321 (4) ±0.01027	1.280 (1) ±0.00704
ESL	0.3702 (3) ±0.01352	0.4146 (5) ±0.005112	0.3456 (2) ±0.003864	0.4262 (6) ±0.01004	0.4296 (7) ±0.01608	0.3736 (4) ±0.01089	0.474 (8) ±0.01114	0.3422 (1) ±0.005019
housing	0.3235 (2) ±0.01133	0.3083 (1) ±0.00559	0.5033 (7) ±0.006521	0.3551 (3) ±0.005187	0.3676 (4) ±0.007395	0.3676 (5) ±0.01556	0.3867 (6) ±0.01050	1.078 (8) ±0.00796
LEV	0.4813 (6) ±0.004028	0.5187 (7) ±0.002867	0.475 (5) ±0.004320	0.4457 (4) ±0.003399	0.4277 (3) ±0.00838	0.426 (2) ±0.01476	0.615 (8) ±0.0099	0.4033 (1) ±0.003091
SWD	0.454 (4) ±0.004320	0.4857 (7) ±0.005249	0.475 (6) ±0.004320	0.4503 (2) ±0.002867	0.452 (3) ±0.006481	0.4603 (5) ±0.004497	0.5707 (8) ±0.007717	0.433 (1) ±0.002160
windsor	0.5024 (1) ±0.006226	0.5201 (3) ±0.003956	0.5488 (4) ±0.005662	0.5891 (6) ±0.02101	0.6825 (8) ±0.03332	0.652 (7) ±0.03721	0.5757 (5) ±0.006044	0.5153 (2) ±0.006044
average rank	2.04	3.82	4.54	4.54	5.29	4.71	6.71	4.36

- **SMO** - Sequential Minimal Optimization – implementation of SVM in Java (WEKA)
  Ripper - Repeated Incremental Pruning to Produce Error Reduction - version of IREP
  J48 – implementation of C4.5 in Java (WEKA), **OLM** - Ordinal Learning Method
  OSDL - Ordinal Stochastic Dominance Learner

Expressiveness of models – „black-box”

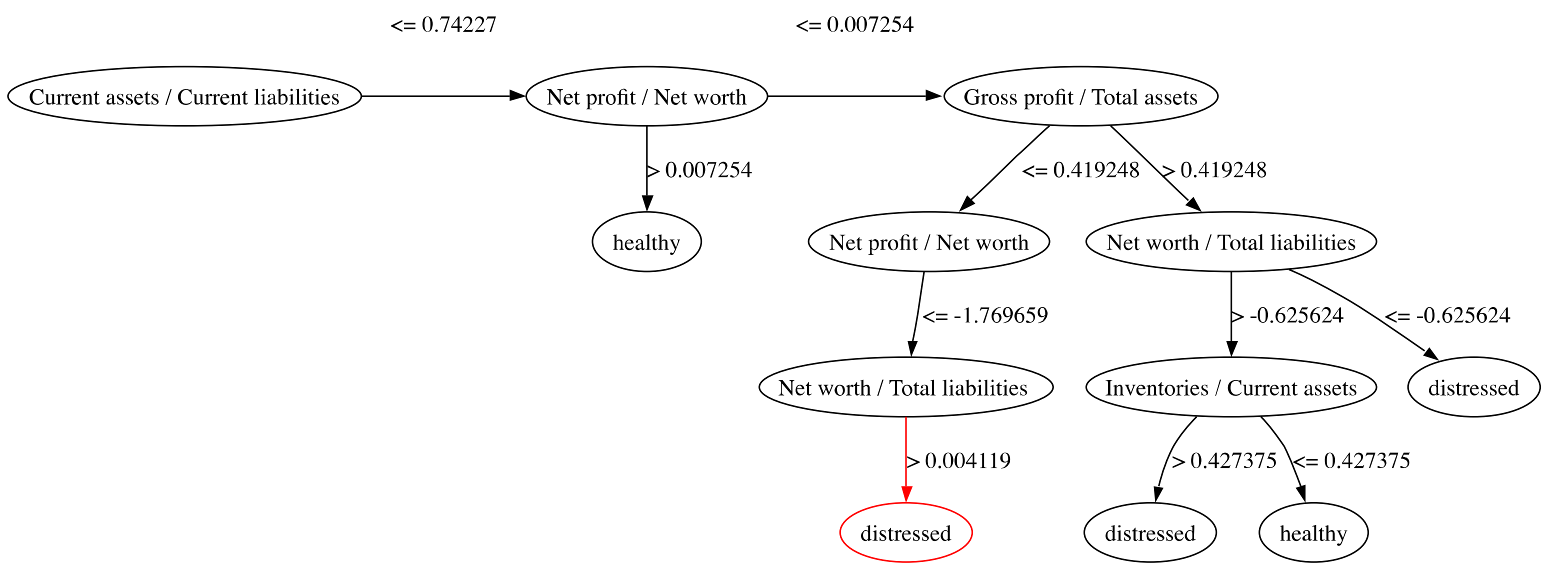
Black-box Classifiers

Classifiers that only give suggestion:

Naïve Bayes, SVM, OLM, OSDL, . . .

Decision Trees

Model can be counterintuitive (orders of preference are neglected).



Expressiveness of models – „glass-box”

Object to classify

Quick ratio	Solvency ratio	Interest cover	EBIT margin	EBITDA margin
1.74	59.72	10.4	43.59	64.7

The matching rules

if $(Solvency\ ratio\ \geq 33.0)$ and $(EBITDA\ margin \geq 48.07)$ then $(STATUS \leq Secure)$ with ϵ-consistency 0.99

...

Objects supporting the matching rules

Quick ratio	Solvency ratio	Interest cover	EBIT margin	EBITDA margin	STATUS
0.61	33.0	14.69	29.98	48.07	Secure
0.52	35.3	13.21	33.33	49.01	Secure
0.55	34.2	14.03	25.14	50.25	Secure
0.67	33.3	15.21	30.13	51.00	Secure

...

Illustrative examples

Example – Prime d'Excellence Scientifique (PES) *with jMAF*

- Multiple criteria classification of candidates for PES award:
  1. Comprehensive assessment (Global)
  2. Publications (Avis 1)
  3. Supervision of PhD students (Avis 2)
  4. Influence (Avis 3)
  5. Administrative responsibility (Avis 4)

Example – Prime d'Excellence Scientifique (PES) *with jMAF*

Attributes: 5 Examples: 118

No	Global (+)	Avis_1 (+)	Avis_2 (+)	Avis_3 (+)	Avis_4 (+)	PRIME (+)
37	B	A	B	C	B	0
38	B	A	B	B	B	1
39	B	A	A	B	B	1
40	B	A	B	C	B	0
41	B	A	B	B	B	1
42	B	B	B	B	B	0
43	B	A	B	C	B	0
44	B	A	B	B	C	0
45	B	B	B	B	C	0
46	B	A	A	C	B	1
47	B	B	A	B	B	0
48	B	A	A	B	A	1
49	B	B	B	B	B	0
50	B	B	C	C	C	0
51	B	A	B	A	B	1
52	B	A	C	B	C	0

Example – Prime d'Excellence Scientifique (PES) *with jMAF*

Quality of approximation: 0.975

Union name	Accuracy	Cardina...
At most 0	0.962	79
Lower		77
Upper		80
Boundary		3
Example_23		
Example_31		
Example_47		
At least 1	0.927	39
Lower		38
Upper		41
Boundary		3

Name	Cardinality	Content
Core	4	Avis_1, Avis_2, Avis_3, Avis_4
Reducts	1	
Reduct 1	4	Avis_1, Avis_2, Avis_3, Avis_4

Example – Prime d'Excellence Scientifique (PES) *with jMAF*

Number of rules: 7

ID	DECISION PART 1	<=	CONDITION 1		CONDITION 2		CONDITION 3
1	(PRIME >= 1)	<=	(Avis_2 >= B)	&	(Avis_4 >= A)		
2	(PRIME >= 1)	<=	(Avis_1 >= A)	&	(Avis_2 >= A)		
3	(PRIME >= 1)	<=	(Avis_1 >= A)	&	(Avis_3 >= B)	&	(Avis_4 >= B)
4	(PRIME <= 0)	<=	(Global <= B)	&	(Avis_4 <= C)		
5	(PRIME <= 0)	<=	(Avis_1 <= B)	&	(Avis_3 <= C)		
6	(PRIME <= 0)	<=	(Avis_2 <= B)	&	(Avis_3 <= C)	&	(Avis_4 <= B)
7	(PRIME <= 0)	<=	(Avis_1 <= B)	&	(Avis_2 <= B)	&	(Avis_4 <= B)

Console | Reducts of PES_RS_var5.isf | Monotonic Unions | Statistics of PES_RS_var5.rules

Rule type: **CERTAIN** Usage type: **AT LEAST** Characteristic class: 1

Support:	28
SupportingExamples:	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 22, 26, 30, 38, 39, 41, 48, 51, 65
Strength:	0.237
Confidence:	1
CoverageFactor:	0.718
Coverage:	28
CoveredExamples:	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 22, 26, 30, 38, 39, 41, 48, 51, 65
NegativeCoverage:	0
InconsistencyMeasure:	0
f-ConfirmationMeasure:	1
A-ConfirmationMeasure:	0.63
Z-ConfirmationMeasure:	1

Example – Prime d'Excellence Scientifique (PES) *with jMAF*

ID	DECISION PART 1	<=	CONDITION 1		CONDITION 2		CONDITION 3
8	(PRIME >= 1)	<=	(Avis_2 >= B)	&	(Avis_4 >= A)		
9	(PRIME >= 1)	<=	(Avis_2 >= A)	&	(Avis_3 >= B)		
10	(PRIME >= 1)	<=	(Avis_1 >= A)	&	(Avis_2 >= A)		
11	(PRIME >= 1)	<=	(Avis_1 >= A)	&	(Avis_3 >= B)	&	(Avis_4 >= B)
12	(PRIME <= 0)	<=	(Global <= B)	&	(Avis_4 <= C)		
13	(PRIME <= 0)	<=	(Global <= C)	&	(Avis_3 <= C)		
14	(PRIME <= 0)	<=	(Avis_1 <= B)	&	(Avis_4 <= B)		
15	(PRIME <= 0)	<=	(Avis_2 <= B)	&	(Avis_3 <= C)	&	(Avis_4 <= B)

Console | Reducts of PES_RS_var5.isf | Monotonic Unions | Statistics of PES_RS_var5.rules

Rule type: **POSSIBLE** Usage type: **AT LEAST** Characteristic class: 1

Support:	24
SupportingExamples:	1, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 23, 28, 30, 33, 39, 48, 81
Strength:	0.203
Confidence:	0.923
CoverageFactor:	0.615
Coverage:	26
CoveredExamples:	1, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 23, 28, 30, 31, 33, 39, 47, 48, 81
NegativeCoverage:	2
NegativeCoveredExamples:	31, 47
InconsistencyMeasure:	0.025
f-ConfirmationMeasure:	0.921
A-ConfirmationMeasure:	0.507
Z-ConfirmationMeasure:	0.885

Illustrative example – Thierry's choice [Bouyssou et al. 2006]

- Objects: 14 cars; Criteria: Crit_1,…,Crit_5

Car	↓ Cost	↓ Accel	↓ Pick-up	↑ Brakes	↑ Road-h
1. Fiat Tipo	18 342	30.7	37.2	2.33	3.00
2. Alfa 33	15 335	30.2	41.6	2.00	2.50
3. Nissan Sunny	16 973	29.0	34.9	2.66	2.50
4. Mazda 323	15 460	30.4	35.8	1.66	1.50
5. Mitsubishi Colt	15 131	29.7	35.6	1.66	1.75
6. Toyota Corolla	13 841	30.8	36.5	1.33	2.00
7. Honda Civic	18 971	28.0	35.6	2.33	2.00
8. Opel Astra	18 319	28.9	35.5	1.66	2.00
9. Ford Escort	19 800	29.4	34.7	2.00	1.75
10. Renault 19	16 966	30.0	37.7	2.33	23.25
11. Peugeot 309 16V	17 537	28.3	34.8	2.33	2.75
12. Peugeot 309	15 980	29.6	35.3	2.33	2.75
13. Mitsubishi Galant	17 219	30.2	36.9	1.66	1.25
14. Renault 21	21 334	28.9	36.7	2.00	2.25

Illustrative example – Thierry's choice by DRSA (inductive learning of rules)

- 8 reference objects assigned by the DM to ordered classes: Good ≻ Bad

No	name	price...	accel...	pick_up...	brakes...	road_h...	class ...
1	2	15335.000	30.200	41.600	2.000	2.500	Bad
2	7	18971.000	28.000	35.600	2.330	2.000	Good
3	9	19800.000	29.400	34.700	2.000	1.750	Bad
4	10	16966.000	30.000	37.700	2.330	3.250	Good
5	11	17537.000	28.300	34.800	2.330	2.750	Good
6	12	15980.000	29.600	35.300	2.330	2.750	Good
7	13	17219.000	30.200	36.900	1.660	1.250	Bad
8	14	21334.000	28.900	36.700	2.000	2.250	Bad

Console | Reducts of ThierrysChoice_8bin.isf

Name	Cardinality	Content
Core	0	
Reducts	5	
Reduct 1	2	pick_up, road_h
Reduct 2	2	price, pick_up
Reduct 3	1	brakes
Reduct 4	2	accel, road_h
Reduct 5	2	price, accel

Illustrative example – Thierry's choice by DRSA (inductive learning of rules)

- All rules induced from binary classification of 8 reference objects

Number	Condition	Decision	Stren...	Relative Strength
1.	(price >= 19800)	class at most Bad	2	50,00 %
2.	(accel >= 30,2)	class at most Bad	2	50,00 %
3.	(pick_up >= 41,6)	class at most Bad	1	25,00 %
4.	(brakes <= 2)	class at most Bad	4	100,00 %
10.	(brakes >= 2,33)	class at least Good	4	100,00 %
11.	(road_h >= 2,75)	class at least Good	3	75,00 %
12.	(price <= 16966) & (accel <= 30)	class at least Good	2	50,00 %
13.	(price <= 18971) & (pick_up <= 35,6)	class at least Good	3	75,00 %
14.	(price <= 16966) & (pick_up <= 37,7)	class at least Good	2	50,00 %
15.	(pick_up <= 35,6) & (road_h >= 2)	class at least Good	3	75,00 %

Minimal set of rules covering all examples:

{4, 10}

Supporting Examples:

	name	price	accel	pick_up	brakes	road_h	class
1.	2	15335	30,2	41,6	2	2,5	Bad
3.	9	19800	29,4	34,7	2	1,75	Bad
7.	13	17219	30,2	36,9	1,66	1,25	Bad
8.	14	21334	28,9	36,7	2	2,25	Bad

Illustrative example – Thierry’s choice by DRSA (inductive learning of rules)

- **Re**classification of 8 reference objects by rules with relative strength ≥75%

	Orig. Decision	Dec. Type	Decision	Used Rules	name	price	accel	pick_up	brakes	road_h
1.	Bad	OK	Bad	1	2	15335	30,2	41,6	2	2,5
2.	Good	OK	Good	3	7	18971	28	35,6	2,33	2
3.	Bad	OK	Bad	2	9	19800	29,4	34,7	2	1,75
4.	Good	OK	Good	2	10	16966	30	37,7	2,33	3,25
5.	Good	OK	Good	4	11	17537	28,3	34,8	2,33	2,75
6.	Good	OK	Good	4	12	15980	29,6	35,3	2,33	2,75
7.	Bad	OK	Bad	2	13	17219	30,2	36,9	1,66	1,25
8.	Bad	OK	Bad	2	14	21334	28,9	36,7	2	2,25

Used Rules:

Number	Condition	Decision	Support	Relative Strength
10.	(brakes >= 2,33)	class at least Good	4	1,0000
13.	(price <= 18971) & (pick_up <= 35,6)	class at least Good	3	0,7500
15.	(pick_up <= 35,6) & (road_h >= 2)	class at least Good	3	0,7500

Attractiveness measures of monotonic decision rules

Why the rule attractiveness measures are important?

The number of rules
induced from datasets is usually quite large

- overwhelming for human comprehension
- many rules are irrelevant, weak or obvious (low practical value)

rule evaluation – attractiveness (interestingness) measures
(e.g. support, confidence, measures of Bayesian confirmation)

- each measure was proposed to capture different characteristics of rules
- the number of proposed measures is very large
- to make a proper choice of an attractiveness measure one has to know its properties

Rule induction

- Discovering rules from data is the domain of inductive reasoning (IR)
- IR uses data about a sample of larger reality to start inference
- $S=\langle U, A\rangle$ – *data table*, where U and A are finite, non-empty sets
  U – *universe*; A – set of *attributes*
- $S=\langle U, C, D\rangle$ – *decision table*, where C – set of *condition attributes*,
  D – set of *decision attributes*, $C\cap D=\varnothing$
- *Rule* induced from S is a *consequence relation*:

  E → H *read as* ***if E, then H***

  where E is condition (evidence or premise)

  and H is conclusion (hypothesis or decision)

  formula built from attribute-value pairs (q,v)

Decision rules

- $\|E\|_S$ is the set of all objects from U, having property E in S
- $\|H\|_S$ is the set of all objects from U, having property H in S
- In the *Rough Set approach*, $\|H\|_S$ is:
  - C-lower approximation, or
  - C-upper approximation, or
  - C-boundary of a union of classes H in S,

  giving thus a *certain*, or *possible*, or *approximate* rule $\boldsymbol{E \rightarrow H}$, resp.
- Basic quantitative characteristics of rules

Measures characterizing decision rules in system $S=\langle U, C, D\rangle$

- *Support* of decision rule $E\rightarrow H$ in S:

$$sup_S(E,H) = card(\|E \wedge H\|_S)$$

- *Strength* of decision rule $E\rightarrow H$ in S:

$$str_S(E,H) = \frac{card(\|E \wedge H\|_S)}{card(U)}$$

- *Confidence factor* for decision rule $E\rightarrow H$ in S (Łukasiewicz, 1913): (called also *certainty*)

$$cer_S(E,H) = \frac{card(\|E \wedge H\|_S)}{card(\|E\|_S)}$$

- *Coverage factor* for decision rule $E\rightarrow H$ in S:

$$cov_S(E,H) = \frac{card(\|E \wedge H\|_S)}{card(\|H\|_S)}$$

Measures characterizing decision rules in system $S=\langle U, C, D\rangle$

- Certainty and coverage factors refer to *Bayes' theorem*

$$cer_S(E,H) = Pr(H|E) = \frac{Pr(H \wedge E)}{Pr(E)}, \quad cov_S(E,H) = Pr(E|H) = \frac{Pr(E \wedge H)}{Pr(H)}$$

- Given a decision table S, the probability (*frequency*) is calculated as:

$$Pr(E) = \frac{card(\|E\|_S)}{card(U)}, \quad Pr(H) = \frac{card(\|H\|_S)}{card(U)}, \quad Pr(E \wedge H) = \frac{card(\|E \wedge H\|_S)}{card(U)}$$

- In fact, *without referring to prior and posterior probability*:

$$cer_S(E,H) \times card(\|E\|_S) = cov_S(E,H) \times card(\|H\|_S)$$

- What is the certainty factor for ***E→H*** is the coverage factor for ***H→E***
- This underlines a directional character of the statement *if E, then H*
  (e.g. „if *x* is a square, then *x* is a rectangle")

Notation

- Notation corresponding to a 2x2 contingency table of rule's premise (E) and conclusion (H)

  $a=sup(H,E)$ is the number of objects in U satisfying both the premise E and the conclusion H of a rule $\boldsymbol{E \rightarrow H}$,

  $b=sup(H, \neg E)$,

  $c=sup(\neg H, E)$,

  $d=sup(\neg H, \neg E)$,

  $a+c=sup(E)$,

  $a+b=sup(H)$,…

	H	$\neg H$	Σ
E	a	c	$a+c$
$\neg E$	b	d	$b+d$
Σ	$a+b$	$c+d$	$a+b+c+d=n$

- a, b, c and d can also be regarded as frequencies that can be used to estimate probabilities, e.g. :

  $Pr(E)=(a+c)/\mathrm{n}$, $Pr(H)=(a+b)/\mathrm{n}$, $Pr(H|E)=a/(a+c)$, $Pr(E|H)=a/(a+b)$

Property of confirmation

- Generally, measures possessing the property of confirmation (confirmation measures) are expected to obtain:
  - values **>0** when the premise of a rule confirms the conclusion
  - values **= 0** when the rule’s premise and conclusion are neutral to each other
  - values **< 0** when the premise disconfirms the conclusion

- What does „premise confirms conclusion” mean?
- How to quantify such confirmation?

Property of confirmation

- Four definitions are possible:
  - Bayesian confirmation: $Pr(H|E)>Pr(H)$
  - strong Bayesian confirmation: $Pr(H|E)>Pr(H|\neg E)$
  - likelihoodist confirmation: $Pr(E|H)>Pr(E)$
  - strong likelihoodist confirmation: $Pr(E|H)>Pr(E|\neg H)$
- An attractiveness measure $c(H,E)$, has the property of Bayesian confirmation if is satisfies the following condition:

$$c(H,E)\begin{cases} >0 & \text{if } Pr(H|E)>Pr(H) \\ =0 & \text{if } Pr(H|E)=Pr(H) \\ <0 & \text{if } Pr(H|E)<Pr(H) \end{cases}$$

Property of confirmation

- Bayesian approach is related to the idea that E confirms H,
  if H is more frequent with E rather than with $\neg E$
  (perspective of rule's conclusion)

- Bayesian confirmation: $Pr(H|E)>Pr(H)$
  - H is satisfied more often when E is satisfied
    [then, this frequency is $Pr(H|E)$], rather than generically [$Pr(H)$]
    Assumption: $Pr(E)\neq 0$

- strong Bayesian confirmation: $Pr(H|E)>Pr(H|\neg E)$
  - H is satisfied more often, when E is satisfied,
    rather than when $\neg E$ is satisfied
    Assumption: $Pr(E)\neq 0$, $Pr(\neg E)\neq 0$

Property of confirmation

- Likelihoodist approach is based on the idea that E confirms H, if E is more frequent with H rather than with $\neg H$ (perspective of rule's premise)
  - likelihoodist confirmation: $Pr(E|H)>Pr(E)$
  - strong likelihoodist confirmation: $Pr(E|H)>Pr(E|\neg H)$

Logical equivalance of four definitions of confirmation

- Bayesian confirmation: $a/(a+c) > (a+b)/n$
- strong Bayesian confirmation: $a/(a+c) > b/(b+d)$
- likelihoodist confirmation: $a/(a+b) > (a+c)/n$
- strong likelihoodist confirmation: $a/(a+b) > c/(c+d)$
- Obviously, the above definitions differ
  - What is the relationship between them?
  - Do they „switch" (between +, zero and —) at the same time?
- All four definitions boil down to one general, always-defined formulation:

$$c(H,E)\begin{cases} > 0 & \text{if } ad\text{-}bc > 0 \\ = 0 & \text{if } ad\text{-}bc = 0 \\ < 0 & \text{if } ad\text{-}bc < 0 \end{cases}$$

Advantage: *ad-bc* is never undefined, no denominator

Popular measures of Bayesian confirmation

There are many alternative, non-equivalent measures of Bayesian confirmation

$$D(H,E)=\frac{a}{a+c}-\frac{a+b}{a+b+c+d}$$ (Carnap 1950/1962)

$$S(H,E)=\frac{a}{a+c}-\frac{b}{b+d}$$ (Christensen 1999)

$$M(H,E)=\frac{a}{a+b}-\frac{a+c}{a+b+c+d}$$ (Mortimer 1988)

$$N(H,E)=\frac{a}{a+b}-\frac{c}{c+d}$$ (Nozick 1981)

$$C(H,E)=\frac{a}{a+b+c+d}-\frac{(a+c)(a+b)}{(a+b+c+d)^2}$$ (Carnap 1950/1962)

$$R(H,E)=\frac{a(a+b+c+d)}{(a+c)(a+b)}-1$$ (Finch 1960)

$$G(H,E)=1-\frac{c(a+b+c+d)}{(a+c)(c+d)}$$ (Rips 2001)

$$F(H,E)=\frac{ad-bc}{ad+bc+2ac}$$ (Kemeny and Oppenheim 1952)

Popular measures of Bayesian confirmation

- To avoid that some measures are undefined, e.g., for

$$D(H,E) = Pr(H \mid E) - Pr(H) = \frac{a}{a+c} - \frac{a+b}{n} = \frac{ad-bc}{n(a+c)}$$

when $a+c=0$, we impose that all measures take value 0 for ad-$bc = 0$

Monotonicity property of the confirmation measures

- Desirable property of c(E,H) = $f(a,b,c,d)$: monotonicity (M)*

f should be non-decreasing with respect to a and d and non-increasing with respect to b and c

$a=sup_S(E,H)$, $b=sup_S(\neg E,H)$, $c=sup_S(E,\neg H)$, $d=sup_S(\neg E,\neg H)$

- Interpretation of (M): ($E \rightarrow H \equiv$ *if x is a raven, then x is black*)

  a) the more black ravens we observe, the more credible becomes $E \rightarrow H$

  b) the more black non-ravens we observe, the less credible becomes $E \rightarrow H$

  c) the more non-black ravens we observe, the less credible becomes $E \rightarrow H$

  d) the more non-black non-ravens we observe, the more credible becomes $E \rightarrow H$

- Example of c(E,H) with property (M) (Kemeny & Oppenheim 1952, Good 1984, Heckerman 1988, Pearl 1988, Fitelson 2001)

$$\boxed{F(H,E) = \frac{ad - bc}{ad + bc + 2ac}}$$

*S.Greco, Z.Pawlak, R.Słowiński: Can Bayesian confirmation measures be useful for rough set decision rules? *Engineering Applications of Artificial Intelligence*, 17 (2004) no.4, 345-361

Confidence vs. confirmation F

- Consider the possible result of rolling a die: 1,2,3,4,5,6, and let the conclusion H="the result is 6"
  - E_1="the result is divisible by 3" $conf(H, E_1)=1/2$, $F(H, E_1)=2/3$
  - E_2="the result is divisible by 2" $conf(H, E_2)=1/3$, $F(H, E_2)=3/7$
  - E_3="the result is divisible by 1" $conf(H, E_3)=1/6$, $F(H, E_3)=0$

- The value of F has a more meaningful interpretation than *conf*
- In particular, in case of $E_3 \rightarrow H \equiv$ „in any case, the result is 6”, the „any case” does not add any information which could confirm that the result is 6, thus $F(H, E_3)=0$

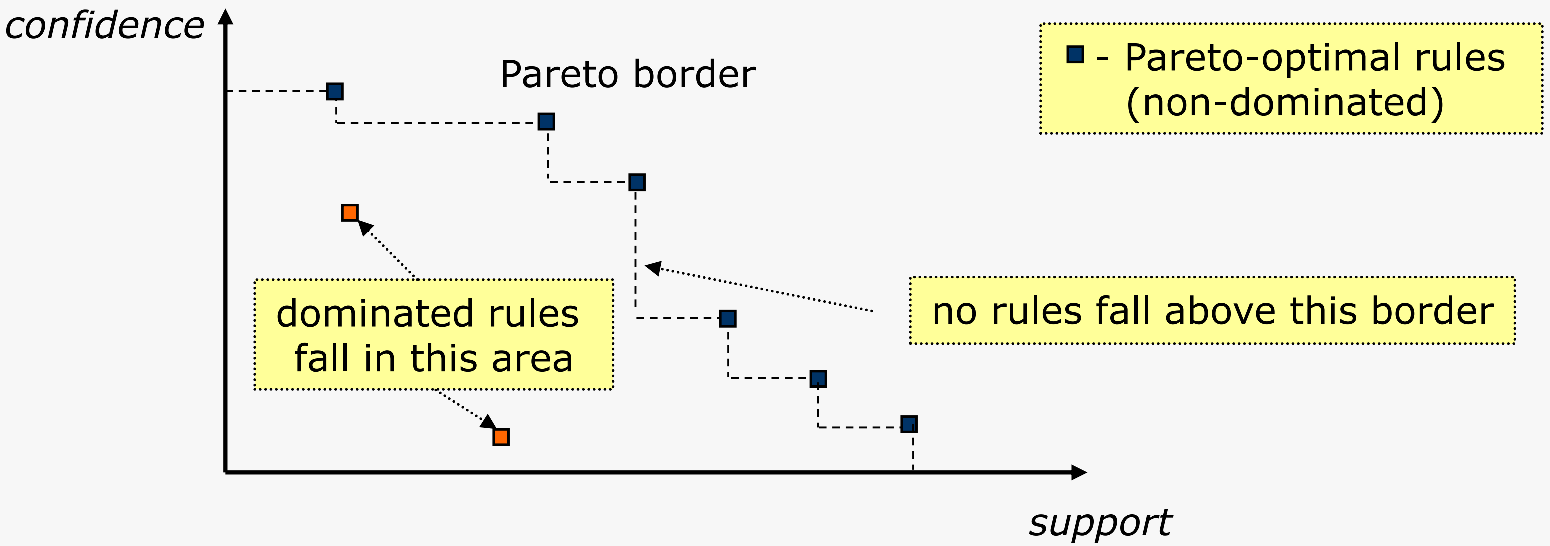
Confidence vs. confirmation F

- Consider the possible result of rolling a die: 1,2,3,4,5,6, and let the premise be kept fixed at E="the result is divisible by 2"
  - H_1 ="the result is 6" $conf(H_1, E)=1/3$, $F(H_1, E)=3/7$
  - H_2 ="the result is *not* 6" $conf(H_2, E)=2/3$, $F(H_2, E)=-3/7$

- $E \rightarrow H_2$ has greater confidence than $E \rightarrow H_1$
- However, $E \rightarrow H_2$ is less interesting than $E \rightarrow H_1$ because E reduces the probability of conclusion H_2 from 5/6=$sup(H_2)$ to 2/3=$conf(H_2, E)$, while it augments the probability of conclusion H_1 from 1/6=$sup(H_1)$ to 1/3= $conf(H_1, E)$
- In consequence, premise E disconfirms conclusion H_2, which is expressed by a negative value of $F(H_2, E)=-3/7$, and it confirms conclusion H_1, which is expressed by a positive value of $F(H_1, E)=3/7$

Support-confidence Pareto border

- Support-confidence Pareto border is the set of non-dominated, Pareto-optimal rules with respect to both *rule support* and *confidence*

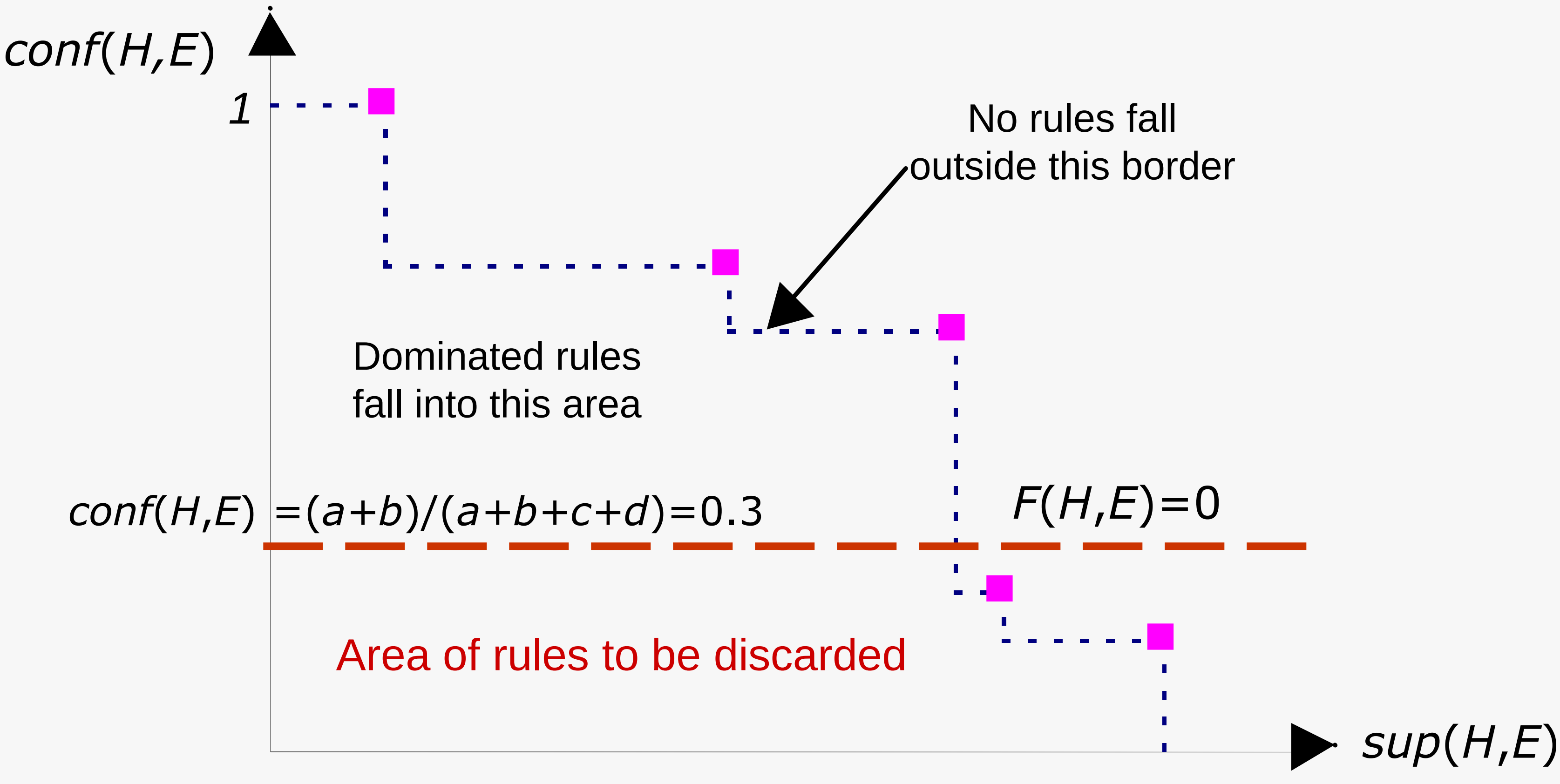


- Mining the border identifies rules optimal with respect to measures such as: *lift*, *gain*, *conviction*, *piatetsky-shapiro*,…

(Bayardo and Agrawal 1999)

Support-confidence vs. support-confirmation Pareto border

The set of rules located on the **support-confidence** Pareto border is exactly the same as on the **support-*F*** Pareto border (Greco, Brzezińska, Słowiński 2007)



The **support-*F*** Pareto border **is more meaningful** than the support-confidence Pareto border

Computational experiment: general info about the dataset

- Dataset ***"CENSUS"*** by B. Becker & R. Kohavi 1996
- **32 561 instances**
- 9 nominal attributes
  - workclass: Private, Local-gov, etc.;
  - education: Bachelors, Some-college, etc.;
  - marital-status: Married, Divorced, Never-married, et.;
  - occupation: Tech-support, Craft-repair, etc.;
  - relationship: Wife, Own-child, Husband, etc.;
  - race: White, Asian-Pac-Islander, etc.;
  - sex: Female, Male;
  - native-country: United-States, Cambodia, England, etc.;
  - salary: >50K, <=50K
- throughout the experiment, $sup(E \rightarrow H)$ denotes relative rule support [0,1]

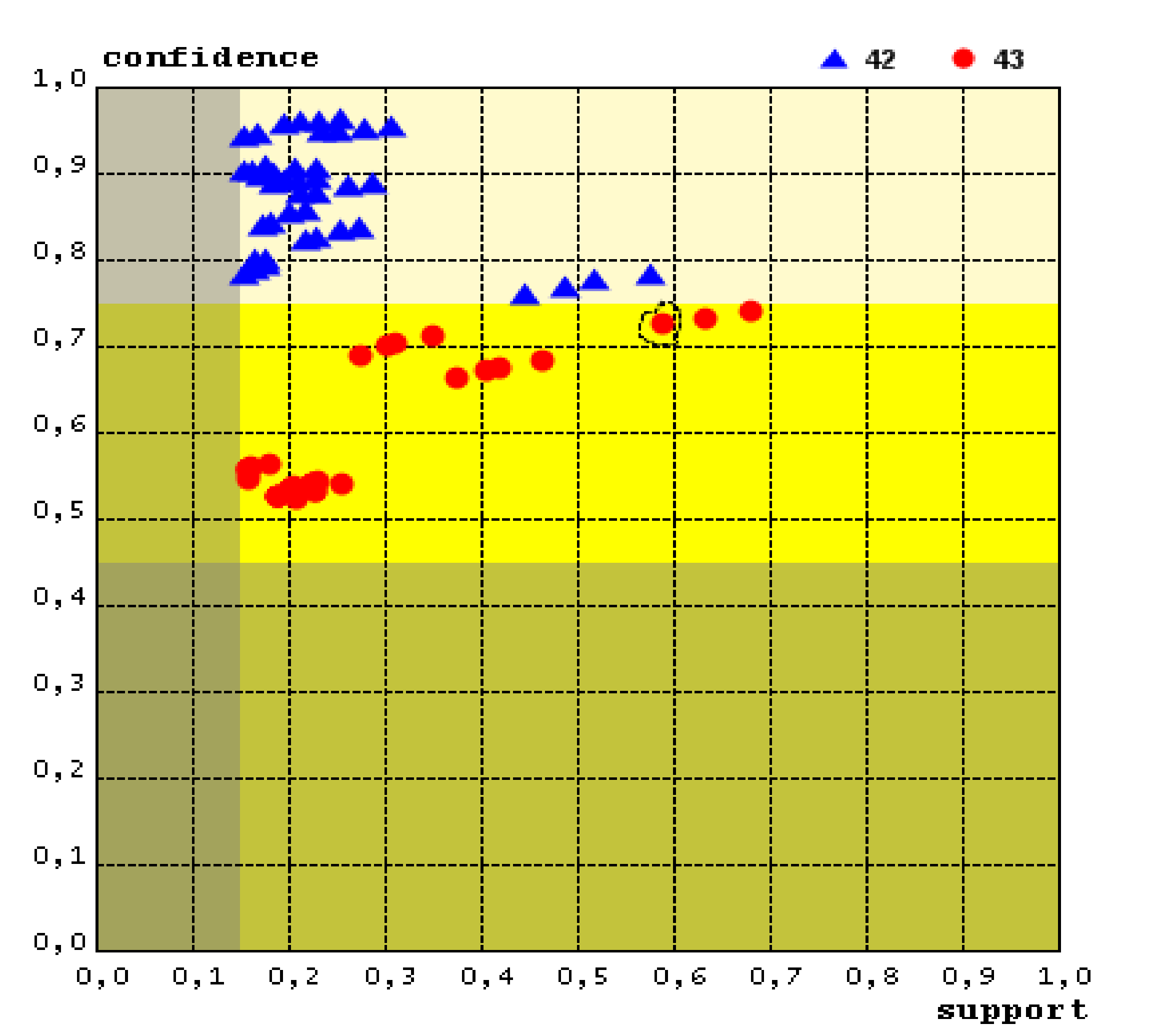
Support-confidence vs. support-confirmation Pareto border

- Example of „CENSUS” dataset:
  - 9 attributes
  - 32.561 instances (objects)

Association rules

premise	conclusion	support	confid.	confirmation s	confirmation f
race is White	native-country is United-States	0,80	0,93	0,16	0,15
native-country is United-States	race is White	0,80	0,88	0,24	0,09
class is <=50K	native-country is United-States	0,68	0,91	-0,03	-0,04
native-country is United-States	class is <=50K	0,68	0,75	-0,06	-0,01
native-country is United-States	workclass is Private	0,67	0,73	-0,08	-0,02
workclass is Private	native-country is United-States	0,67	0,90	-0,03	-0,05
race is White	workclass is Private	0,63	0,74	-0,01	0,00
workclass is Private	race is White	0,63	0,86	0,00	0,00
race is White	class is <=50K	0,63	0,74	-0,11	-0,04
class is <=50K	race is White	0,63	0,84	-0,07	-0,07
native-country is United-States	sex is Male	0,62	0,68	0,00	0,00
sex is Male	native-country is United-States	0,62	0,91	0,00	0,00
race is White	sex is Male	0,60	0,70	0,14	0,05
sex is Male	race is White	0,60	0,89	0,08	0,11
workclass is Private	native-country is United-States and race is White	0,59	0,80	-0,03	-0,02
native-country is United-States and workclass is Private	race is White	0,59	0,88	0,06	0,09
race is White and workclass is Private	native-country is United-States	0,59	0,93	0,04	0,10

Support-confidence vs. support-confirmation Pareto border



„CENSUS” dataset

association rules

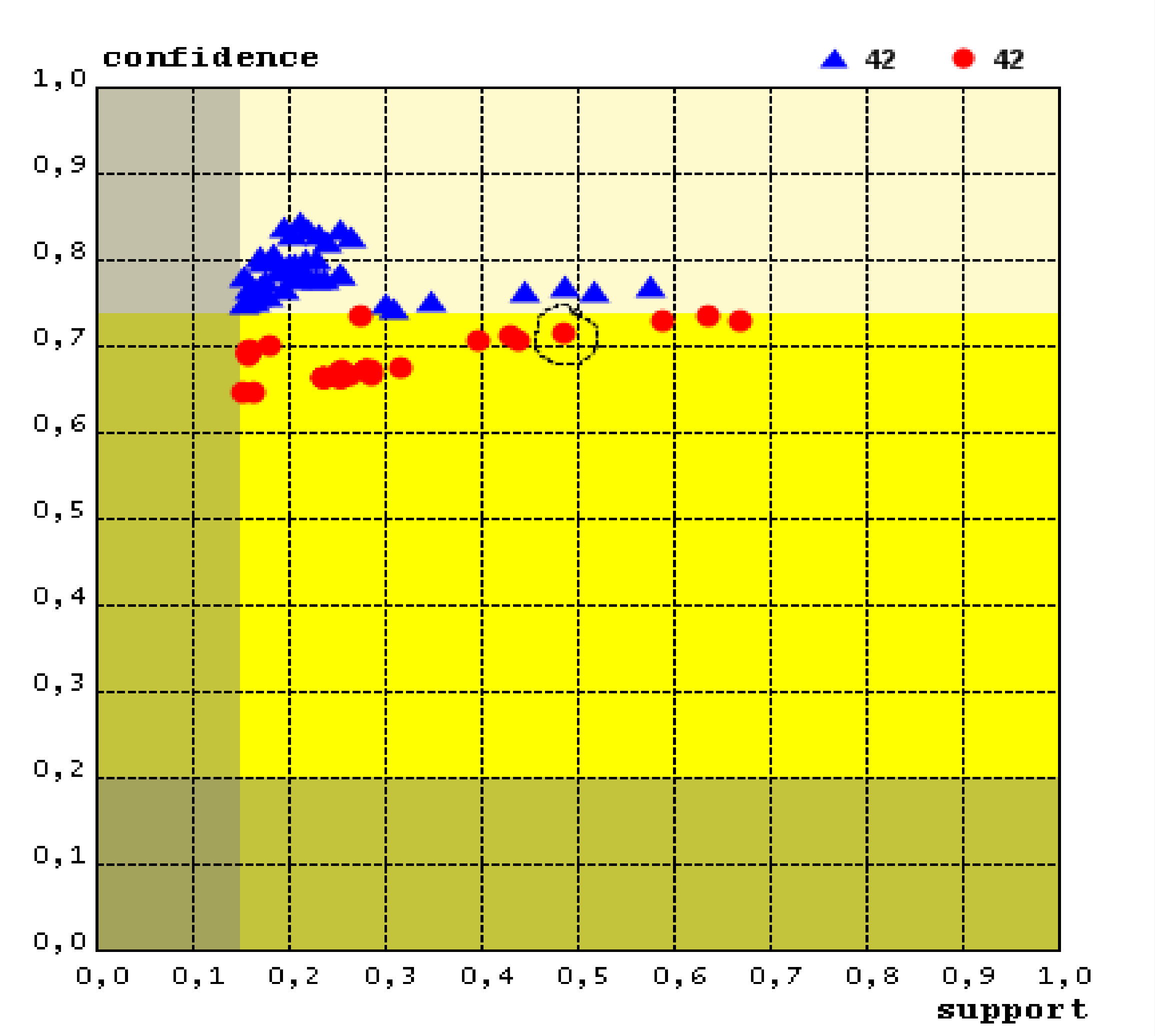
supp ≥ 15%

conf ≥ 45%

● **confirmation<=0**

premise	conclusion	supp	conf	s	f
native-country is United-States and race is White	class is <=50K	0,59	0,73	-0,11	-0,05

Support-confidence vs. support-confirmation Pareto border



„CENSUS" dataset

association rules

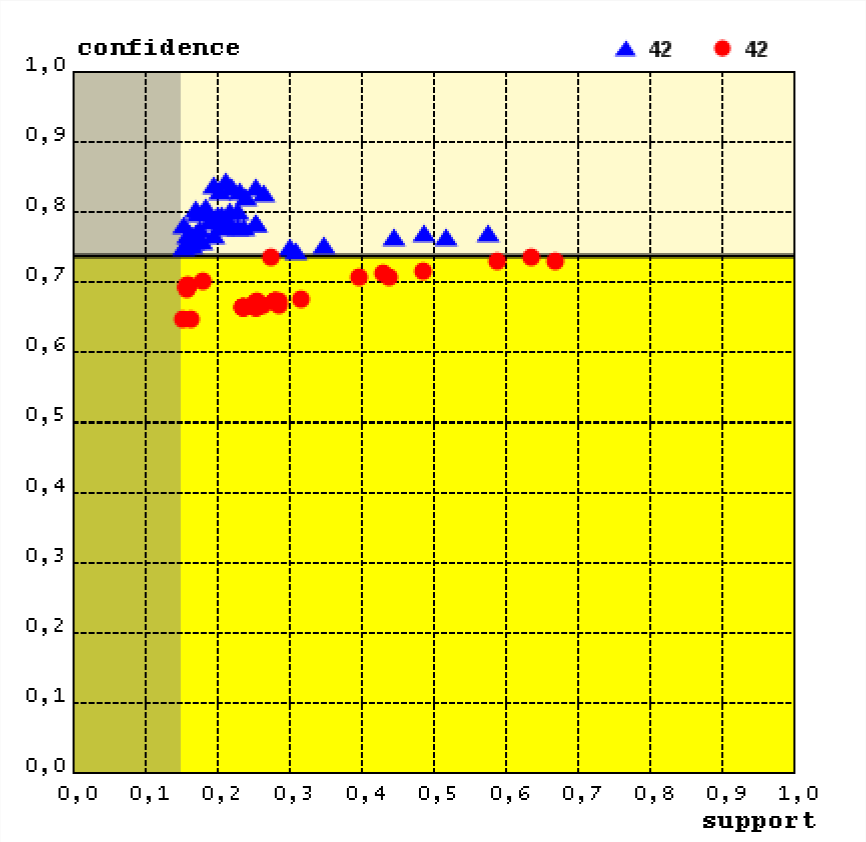
supp ≥ 15%

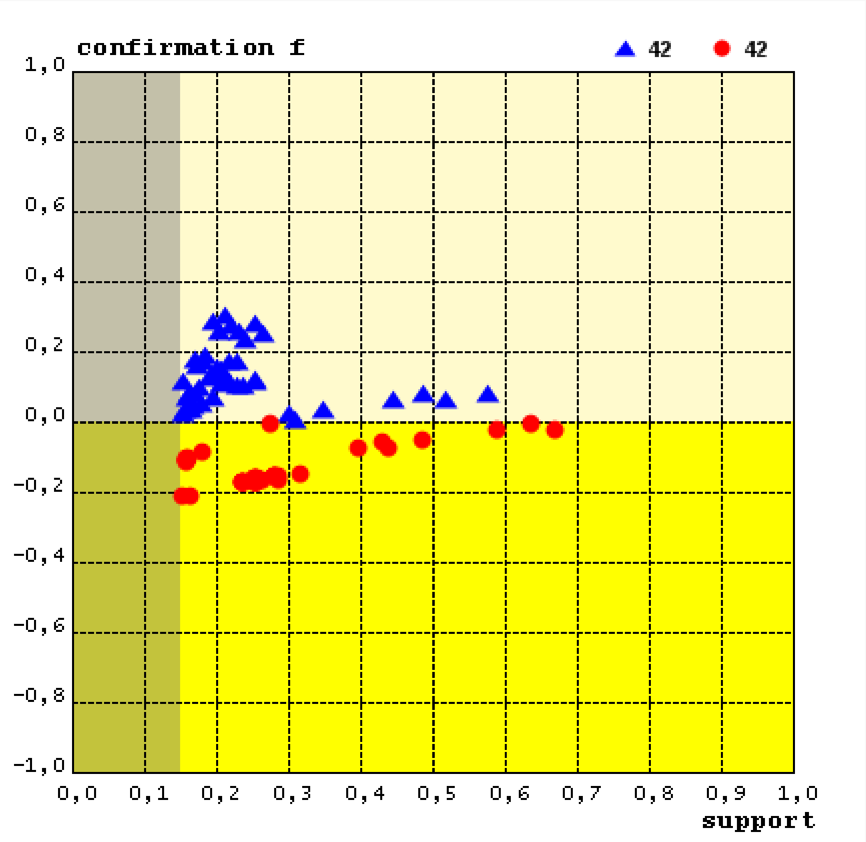
conf ≥ 20%

- **confirmation<=0**

premise	conclusion	supp	conf	s	f
sex is Male	workclass is Private	0,49	0,72	-0,06	-0,05

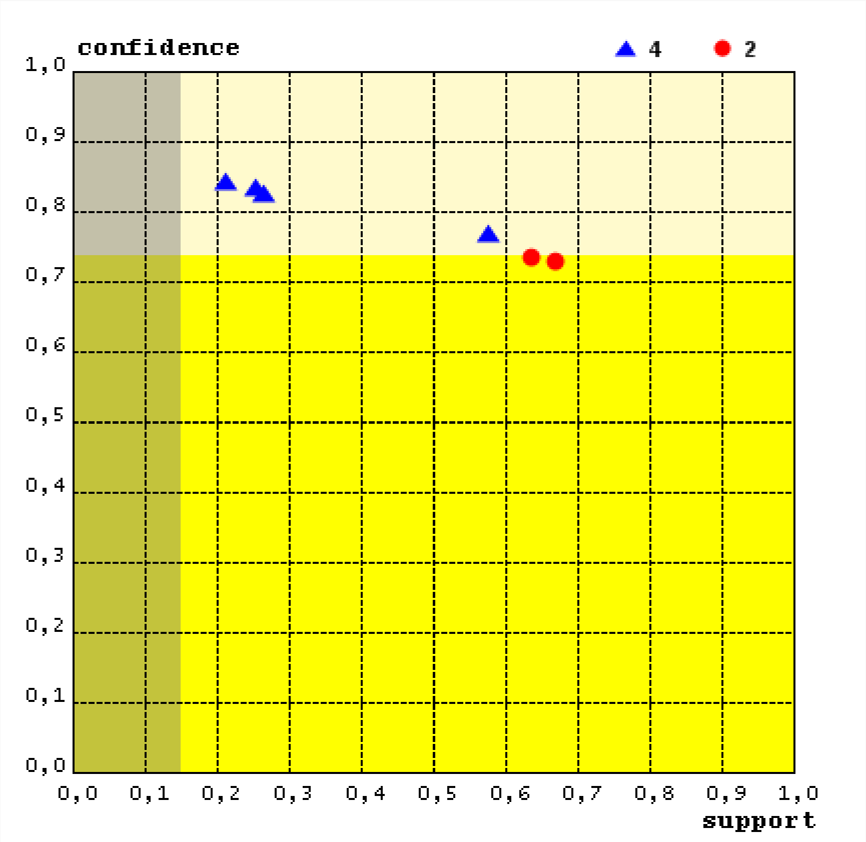
Support-confidence vs. support-confirmation Pareto border

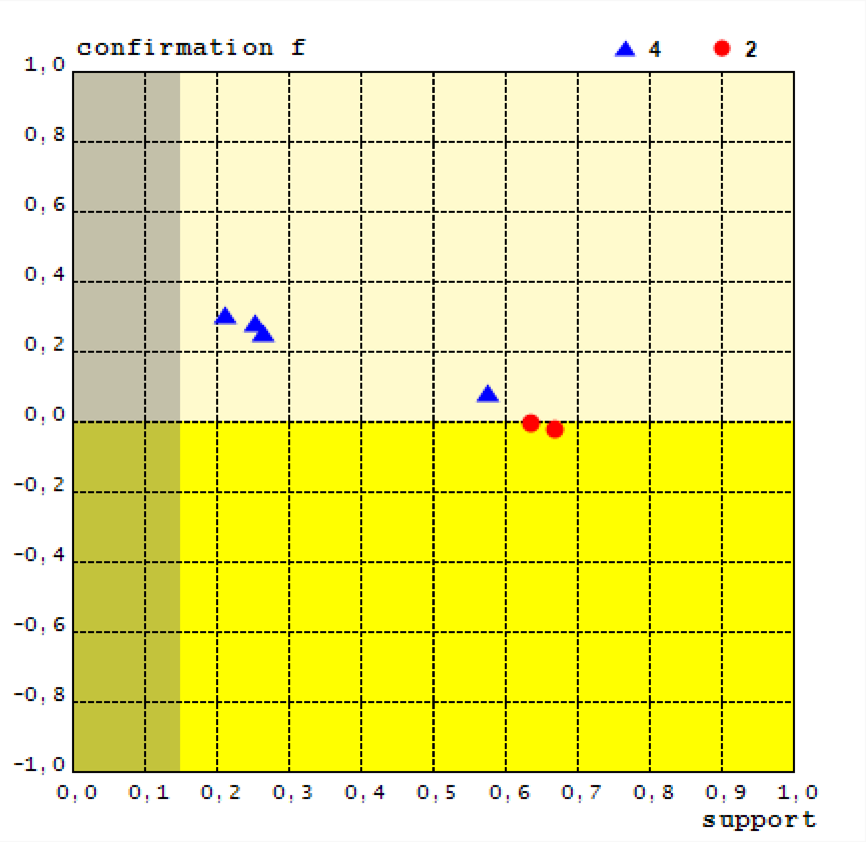




- ● indicates rules with negative confirmation
- the decision class constitutes over 70% of the whole dataset
- rules with high confidence can be disconfirming
- even some rules from the Pareto border need to be discarded

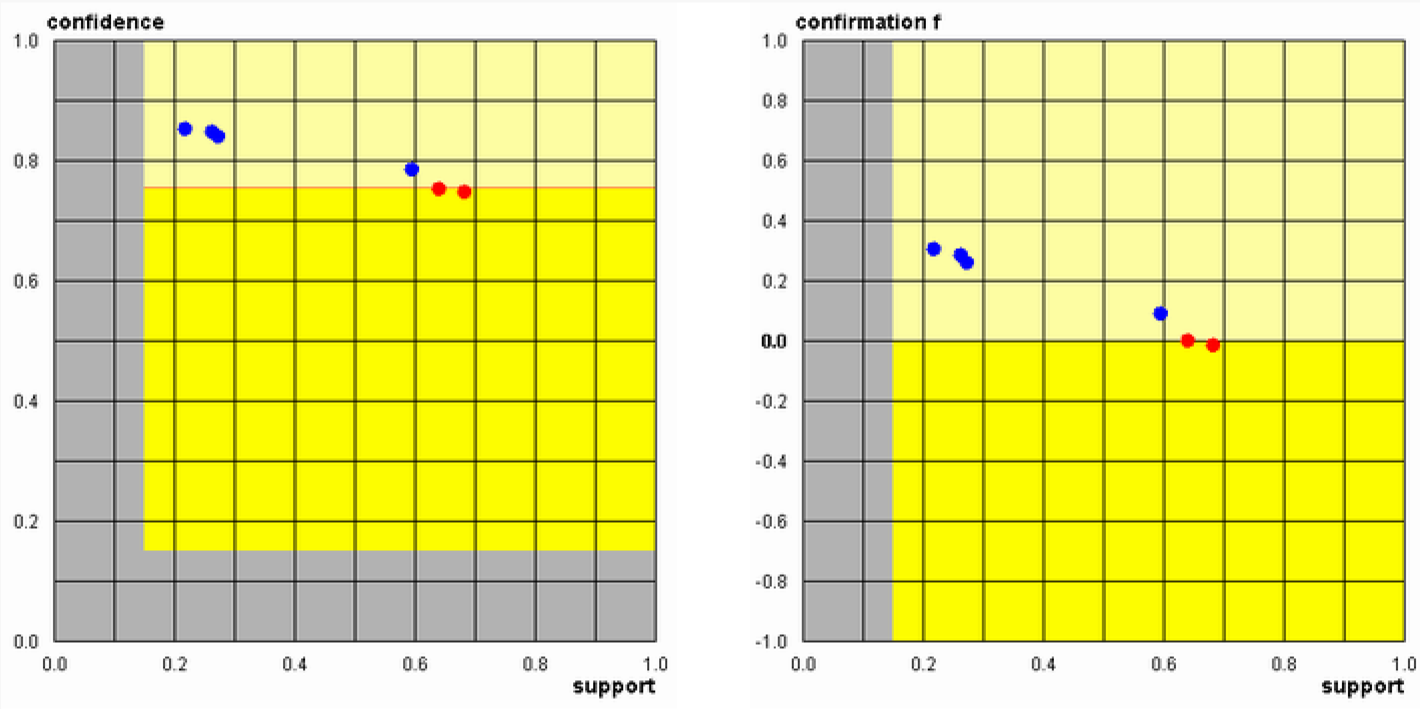
Support-confidence vs. support-confirmation Pareto border





- ● indicates rules with negative confirmation
- both Pareto borders contain the same rules

Support-confidence vs. support-confirmation Pareto border



premise	conclusion	supp	conf	s	f	a-supp
marital-status is Never-married **and race is** White **and class is** =50K	**workclass is** Private	0.22	0.85	0.13	0.30	0.04
marital-status is Never-married **and class is** =50K	**workclass is** Private	0.26	0.85	0.13	0.28	0.05
marital-status is Never-married	**workclass is** Private	0.27	0.84	0.13	0.26	0.05
race is White	**workclass is** Private	0.64	0.75	-0.01	-0.00	0.21
native-country is United-States	**workclass is** Private	0.68	0.75	-0.07	-0.02	0.23
class is =50K	**workclass is** Private	0.60	0.78	0.13	0.09	0.16

Z-measure

- It can be observed that:

$$D_{\text{norm}} = S_{\text{norm}} = M_{\text{norm}} = N_{\text{norm}} = C_{\text{norm}} = R_{\text{norm}} = G_{\text{norm}}$$

- Crupi et al. (2007) have therefore proposed to call them all by one name: ***Z*-measure**

$$Z(H,E) = \begin{cases} \dfrac{ad - bc}{(a+c)(c+d)} = G, & \text{in case of confirmation} \\[2em] \dfrac{ad - bc}{(a+c)(a+b)} = R, & \text{in case of disconfirmation} \end{cases}$$

A-measure

- In particular, we propose the likelihoodist counterpart of the approach of Crupi et al. that transforms all of the considered measures into measure *A*:

$$A(H,E)=\begin{cases}\dfrac{ad-bc}{(a+b)(b+d)} & \text{in case of confirmation}\\[2em] \dfrac{ad-bc}{(b+d)(c+d)} & \text{in case of disconfirmation}\end{cases}$$

S.Greco, R.Słowiński, I.Szczęch: Properties of rule interestingness measures and alternative approaches to normalization of measures. *Information Sciences*, 216 (2012) 1-16

Complementarity of measures *Z and A*

- Measures Z and A can be regarded as complementary since:
  - measure Z comes from Bayesian inspiration,
    while measure A comes from likelihoodist inspiration
  - measure Z can be expressed in terms of $Pr(H|E)$ and $Pr(H)$,
    while measure A in terms of $Pr(H|\neg E)$ and $Pr(H)$

Support – *Anti-support* Pareto border

- Theorem*:*

  For a set of rules with the same conclusion *H*,

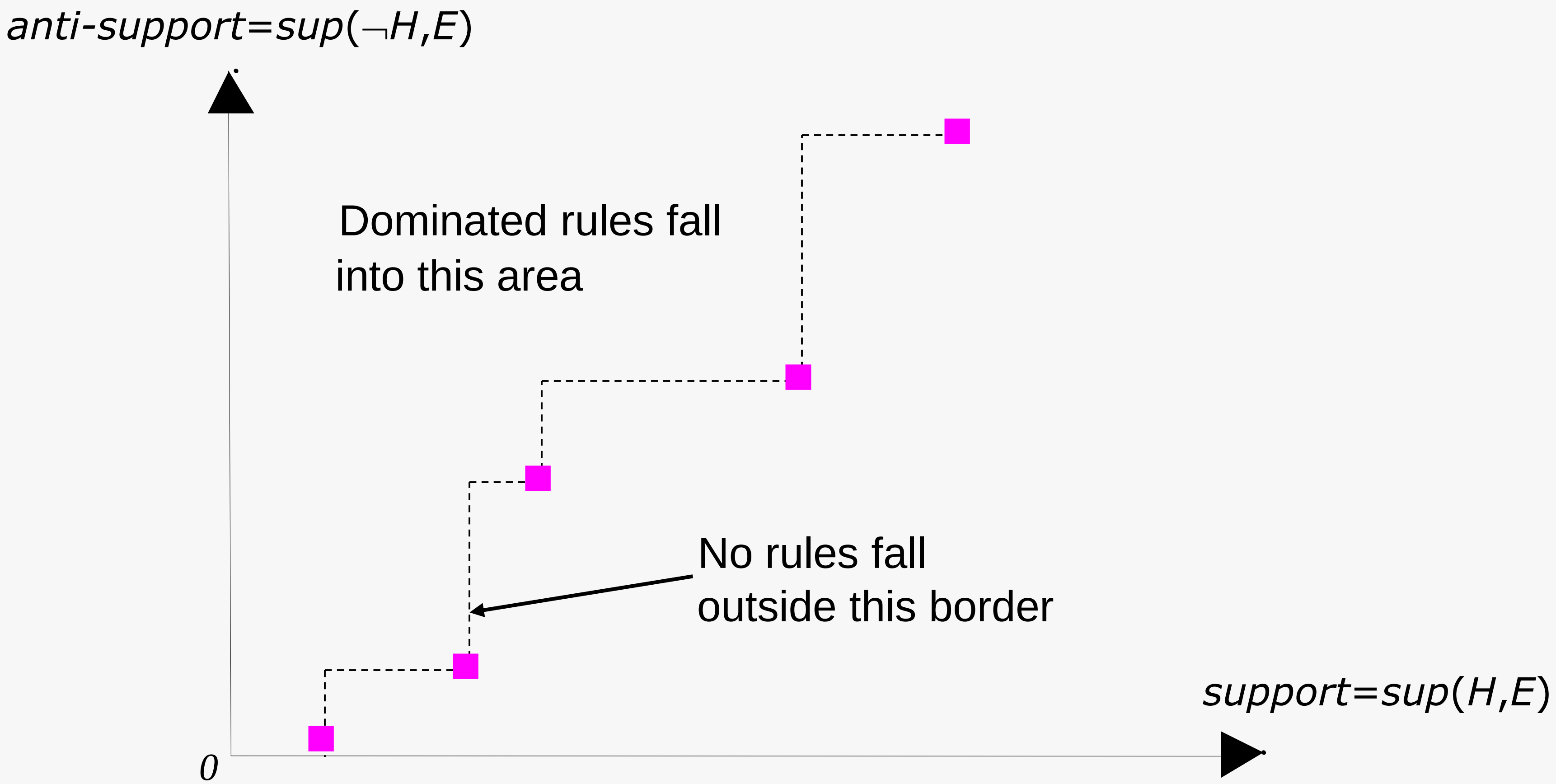
  due to (anti) monotonic dependencies between
  measures of *support* and *anti-support* on one hand,
  and any attractiveness measure with property M on the other hand,

  the best rules according to any measure with the property M
  must reside on the ***support* – *anti-support* Pareto optimal border**

- The support – anti-support Pareto border is a set of non-dominated rules with respect to *support* and *anti-support*

S.Greco, R.Słowiński, I.Szczęch: Measures of rule interestingness in four perspectives of confirmation. *Information Sciences*, 346–347 (2016) 216–235.

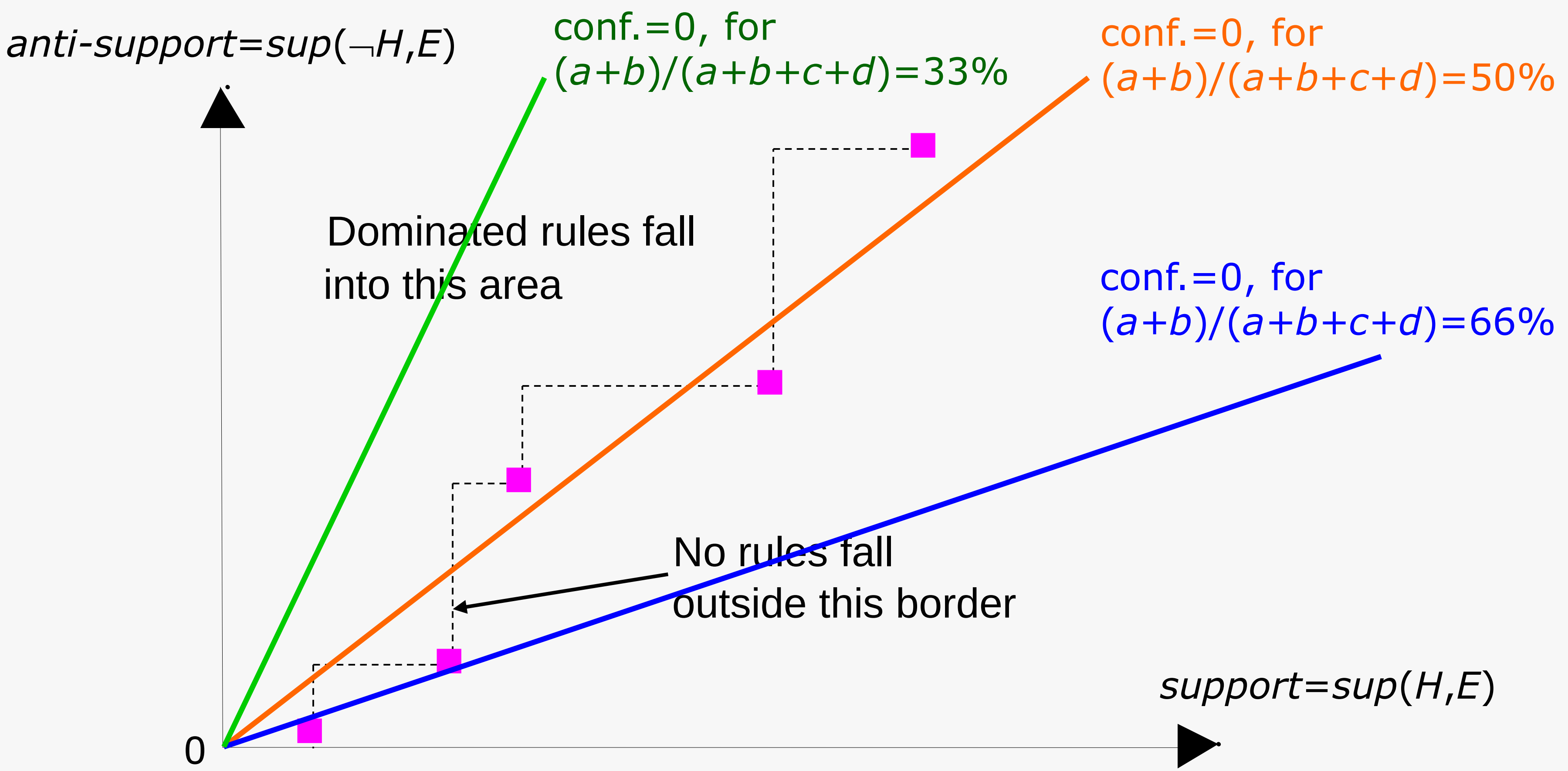
Support – Anti-support Pareto border



The best rules according to any measure with the property M must reside on the *support – anti-support* Pareto border

Measures Z, A and $c_{1\text{-}4}$ all satisfy property M

Support – Anti-support Pareto border

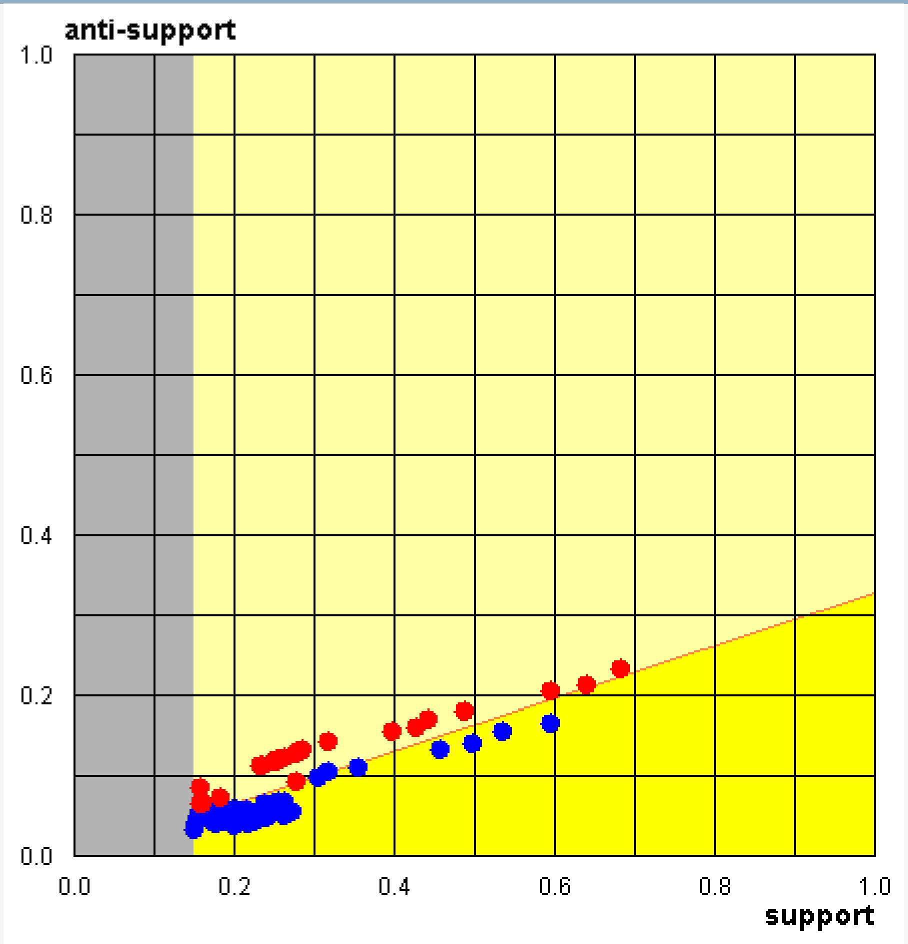


Rules lying above a linear function (line of conf.=0):

$$sup(\neg H,E) = sup(H,E)[(a+b+c+d)/(a+b)-1]$$

have a negative value of any confirmation measure

Support – anti-support (workclass=Private)

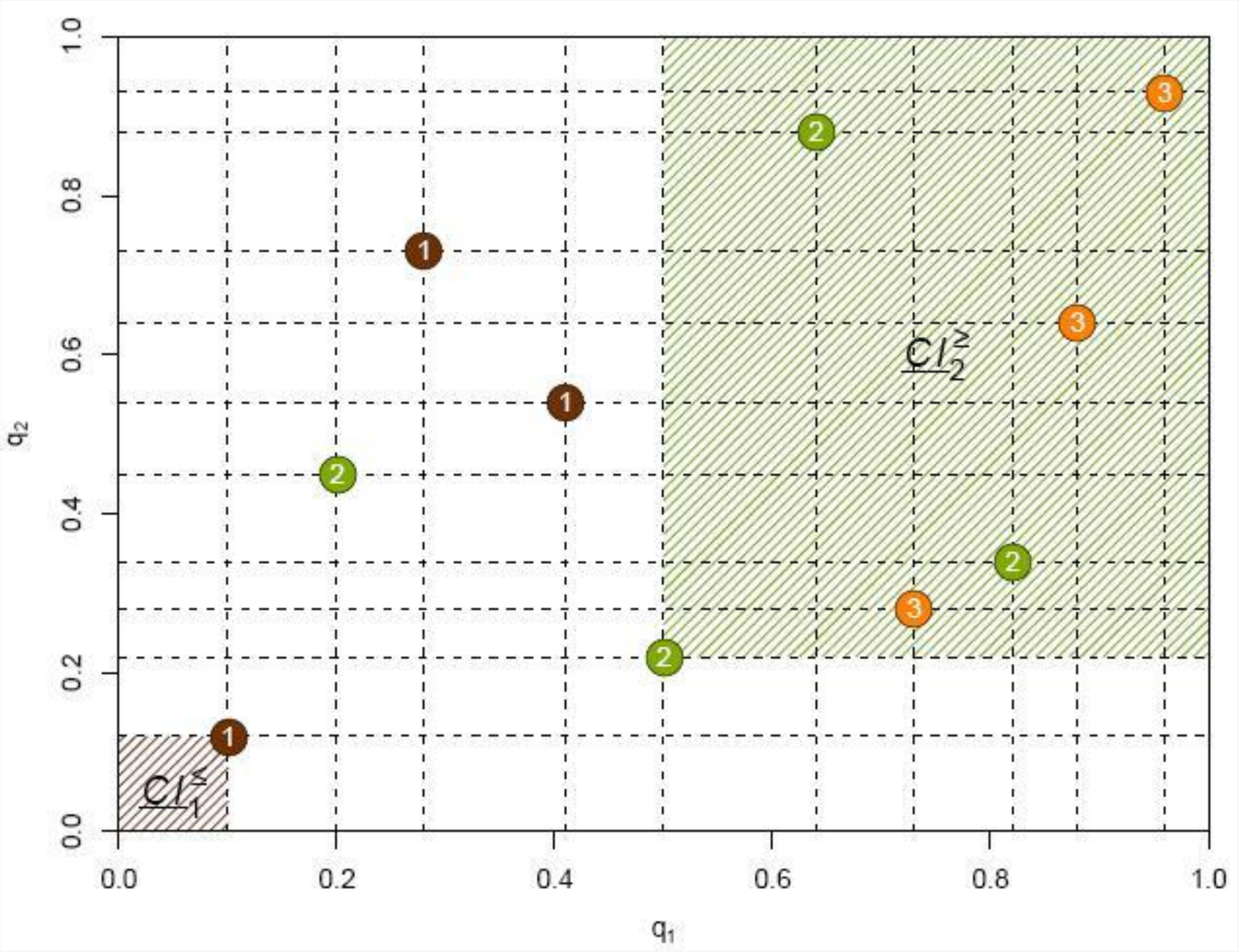


- ● indicates rules with negative confirmation
- even some rules from the Pareto border need to be discarded

Variable-consistency DRSA

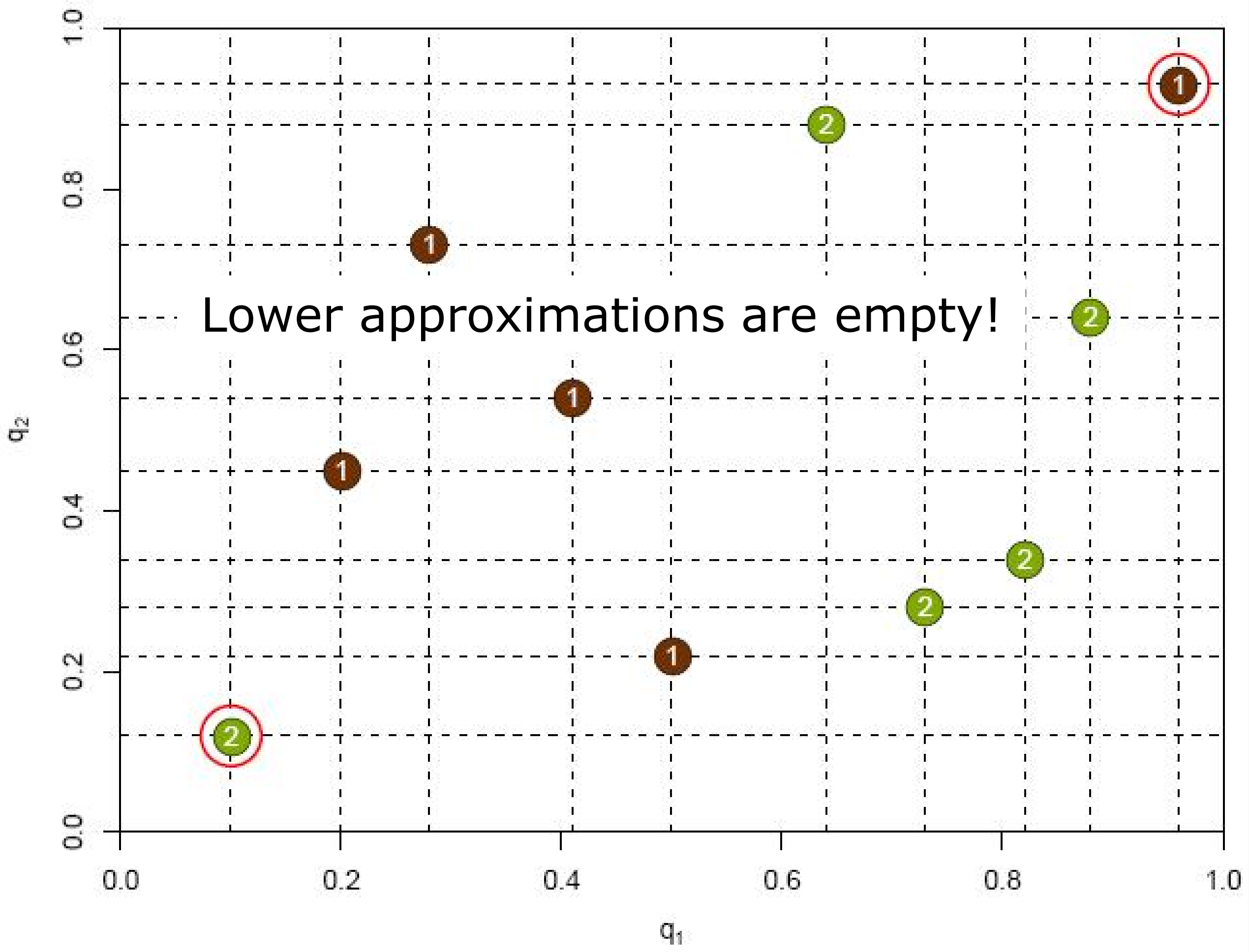
How to deal with „malicious" inconsistency in data?

- „Orthodox" lower approximations:



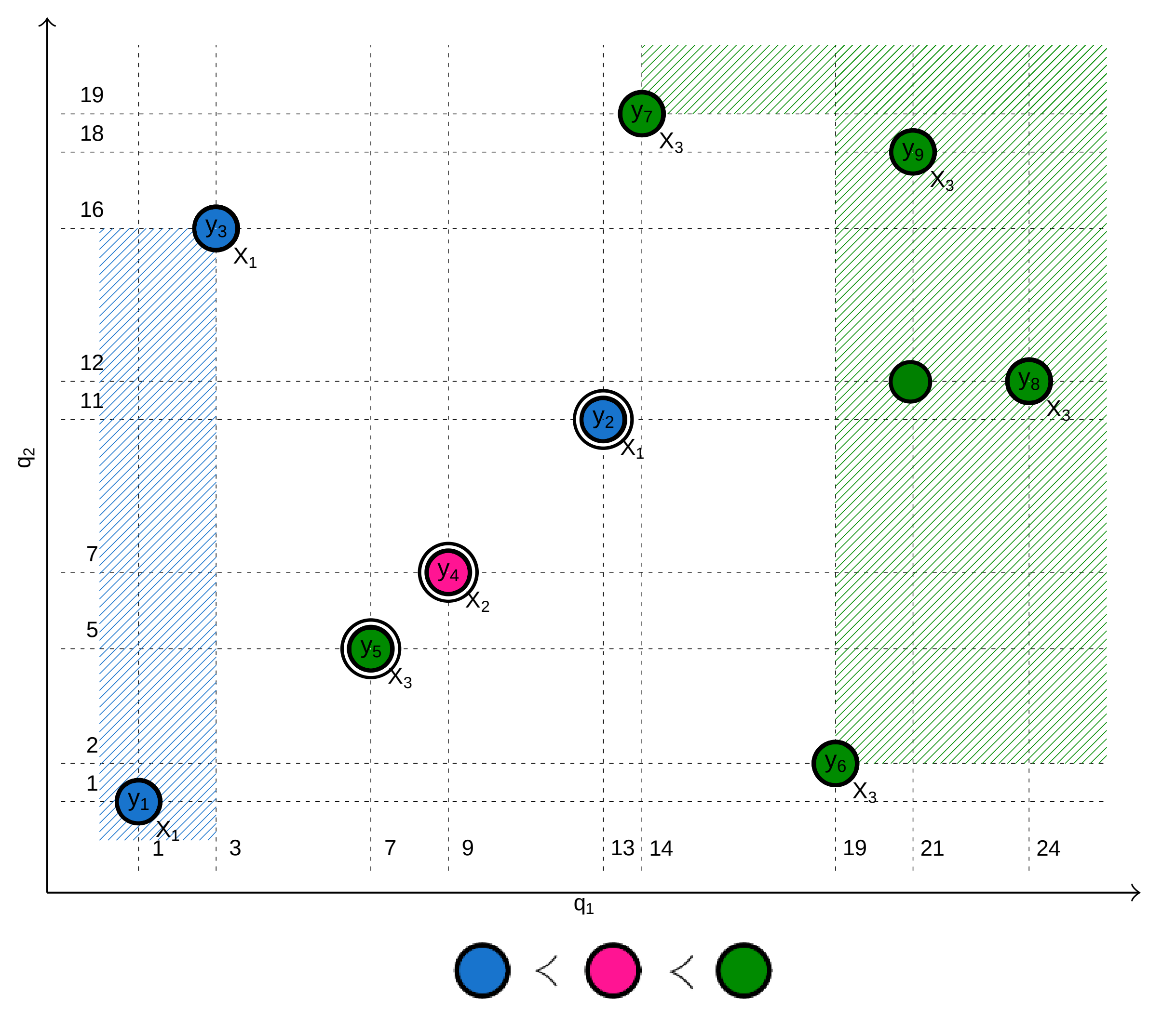
How to deal with „malicious" inconsistency in data?

- „Orthodox" lower approximations:

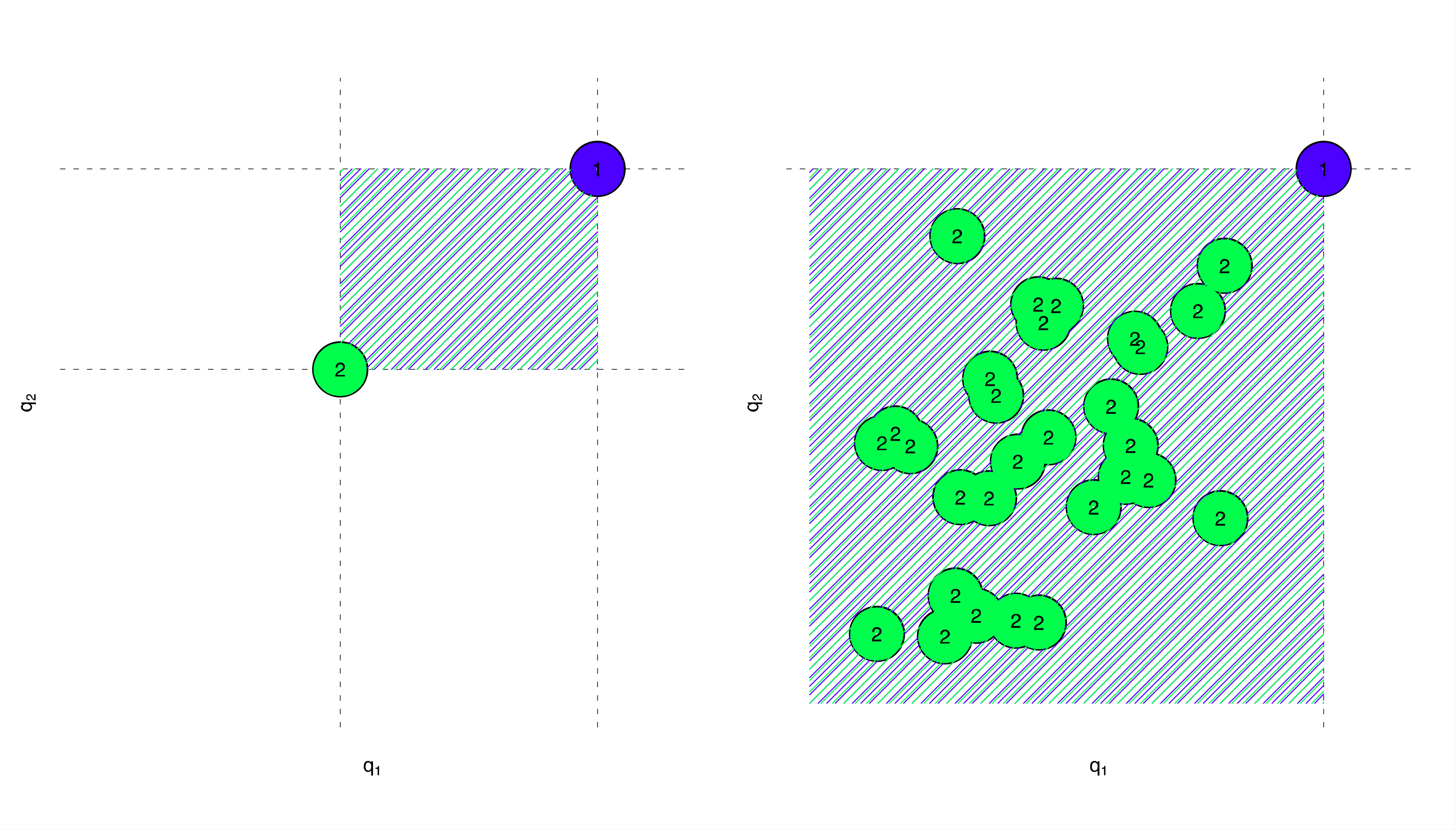


- One „malicious" object is enough to empty the lower approximation

Another example of inconsistent data



How to quantify inconsistency?



Consistency measures in VC-DRSA

- Consistency measures $f(x)$ and $g(x)$ are used to control consistency of object x included in the extended P-lower approximation
- gain-type consistency measures:

$$\underline{P}^{\alpha_{Cl_t^{\geq}}}\left(Cl_t^{\geq}\right) = \left\{x \in Cl_t^{\geq}:\ f_{Cl_t^{\geq}}^{P}\left(x\right) \geq \alpha_{Cl_t^{\geq}}\right\}$$

$$\underline{P}^{\alpha_{Cl_t^{\leq}}}\left(Cl_t^{\leq}\right) = \left\{x \in Cl_t^{\leq}:\ f_{Cl_t^{\leq}}^{P}\left(x\right) \geq \alpha_{Cl_t^{\leq}}\right\}$$

- cost-type consistency measures:

$$\underline{P}^{\beta_{Cl_t^{\geq}}}\left(Cl_t^{\geq}\right) = \left\{x \in Cl_t^{\geq}:\ g_{Cl_t^{\geq}}^{P}\left(x\right) \leq \beta_{Cl_t^{\geq}}\right\}$$

$$\underline{P}^{\beta_{Cl_t^{\leq}}}\left(Cl_t^{\leq}\right) = \left\{x \in Cl_t^{\leq}:\ g_{Cl_t^{\leq}}^{P}\left(x\right) \leq \beta_{Cl_t^{\leq}}\right\}$$

- Consistency measures are also used to control the consistency of induced decision rules.

Monotonicity properties of consistency measures

- Consistency measure $f(x)$ (or $g(x)$) is monotonic iff it does not decrease (or does not increase) when:
  - (m1) the set of attributes is growing,
  - (m2) the set of objects is growing,
  - (m3) the union of ordered classes is growing,
  - (m4) x improves its evaluation, so that it dominates more objects.

J.Błaszczyński, S.Greco, R.Słowiński, M.Szeląg: Monotonic variable consistency rough set approaches. *Int. J. of Approximate Reasoning*, 50 (2009) no.7, 979–999

S.Greco, B.Matarazzo, R.Słowiński: Parameterized rough set model using rough membership and Bayesian confirmation measures. *Int. J. of Approximate Reasoning*, 49 (2008) 285-300

Consistency measures

- Gain-type consistency measure:

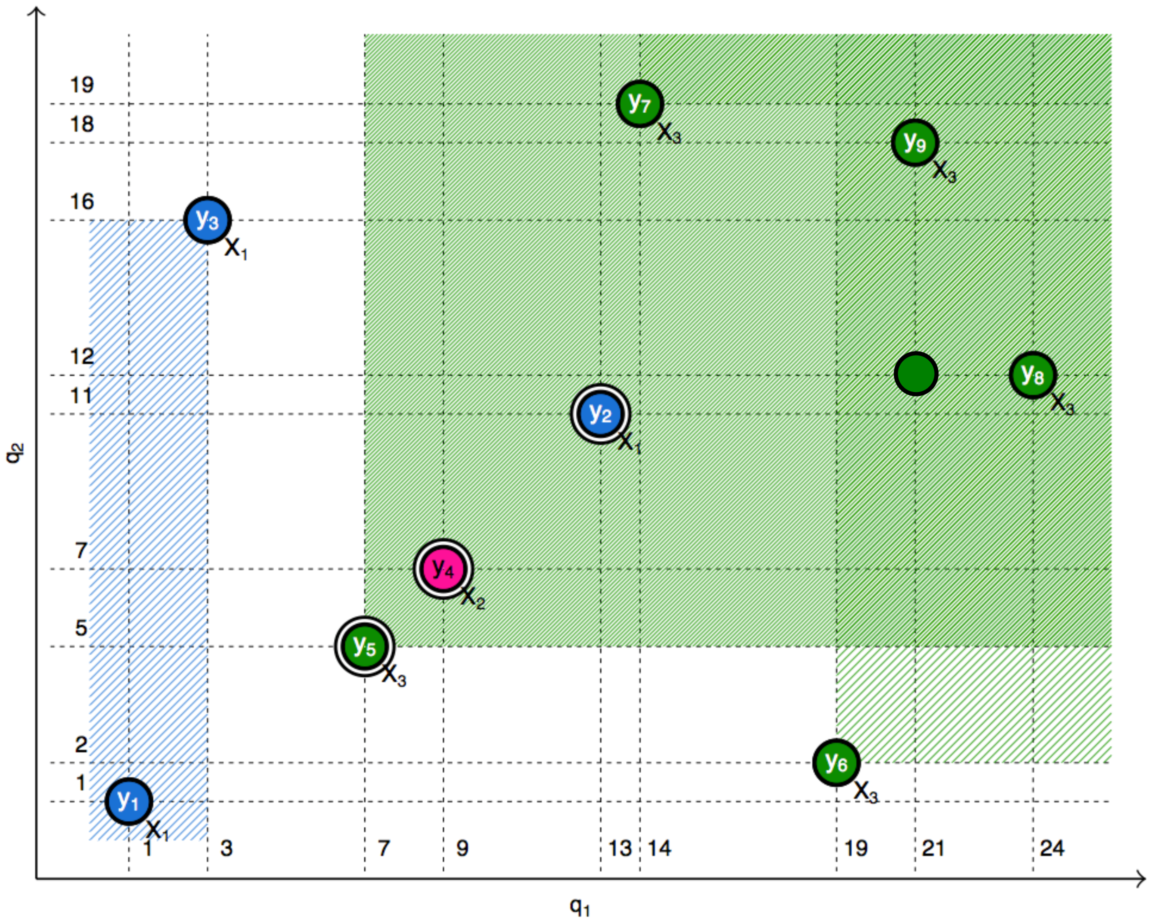
  rough membership, μ-consistency measure

$$\mu_{Cl_t^{\geq}}^{P}\left(x\right)=\frac{\left|D_P^{+}\left(x\right)\cap Cl_t^{\geq}\right|}{\left|D_P^{+}\left(x\right)\right|}\qquad\qquad \mu_{Cl_t^{\leq}}^{P}\left(x\right)=\frac{\left|D_P^{-}\left(x\right)\cap Cl_t^{\leq}\right|}{\left|D_P^{-}\left(x\right)\right|}$$

- It can be interpreted as an estimate of conditional probability:

$$Pr\left(y\in Cl_t^{\geq}\mid y\in D_P^{+}\left(x\right)\right)\qquad\qquad Pr\left(y\in Cl_t^{\leq}\mid y\in D_P^{-}\left(x\right)\right)$$

Coming back to our example…



$$\mu^{P}_{X_3^{\geq}}(y_5) = \frac{5}{7}$$

Consistency measures

- Cost-type consistency measure:
  ε-consistency measure

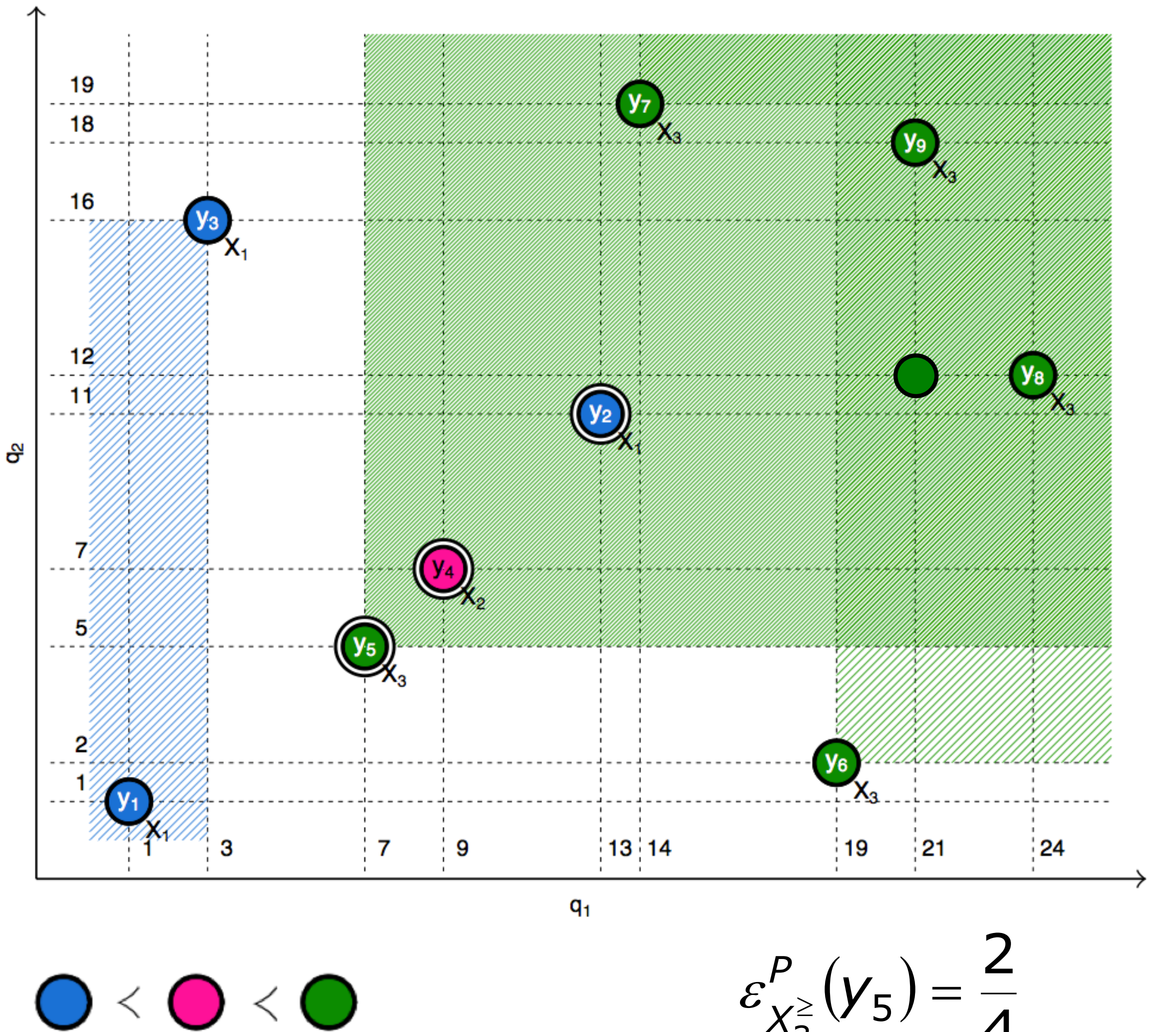
$$\varepsilon^{P}_{Cl_t^{\geq}}(x) = \frac{\left|D_P^{+}(x) \cap \neg Cl_t^{\geq}\right|}{\left|\neg Cl_t^{\geq}\right|} \qquad \varepsilon^{P}_{Cl_t^{\leq}}(x) = \frac{\left|D_P^{-}(x) \cap \neg Cl_t^{\leq}\right|}{\left|\neg Cl_t^{\leq}\right|}$$

- It can be interpreted as an estimate of conditional probability:

$$Pr\left(y \in D_P^{+}(x) \mid y \in \neg Cl_t^{\geq}\right) \qquad Pr\left(y \in D_P^{-}(x) \mid y \in \neg Cl_t^{\leq}\right)$$

- The intuition behind ε-consistency measure : it says how far the implications $y \in D_P^{+}(x) \Rightarrow y \in Cl_t^{\geq}$, $y \in D_P^{-}(x) \Rightarrow y \in Cl_t^{\leq}$ are not supported by the data

Coming back to our example…



$$\varepsilon^{P}_{X_3^{\geq}}(y_5) = \frac{2}{4}$$

Monotonicity properties of consistency measures

consistency measure	(m1)	(m2)	(m3)	(m4)
μ (rough)	no	yes	yes	no
μ'	no	yes	yes	yes
B (Bayesian)	no	no	no	no
β	no	yes	yes	yes
ε	yes	yes	no	yes
ε^*	yes	yes	yes	yes
ε'	yes	yes	yes	yes
$\bar{\mu}$	yes	yes	yes	yes

Induction of monotonic decision rules

- Rules are induced by VC-DomLEM algorithm
  - VC-DomLEM is a sequential covering algorithm that induces strong rules characterised by required level of consistency
  - Selection of the elementary conditions is based on two factors:
    - consistency of the constructed rule
    - support of the constructed rule
  - The result is a minimal set of rules that covers all objects from the *P*-lower approximations
  - Computational complexity of VC-DomLEM: $O(n^2m^2)$ where n=number of objects; m=number of attributes

J.Błaszczyński, R.Słowiński, M.Szeląg: Sequential covering rule induction algorithm for variable consistency rough set approaches. *Information Sciences,* 181 (2011) 987-1002

Stochastic DRSA

Probabilistic model for DRSA

- To each object $x_i \in U$, we assign a probability that x_i belongs to „class” at least t:

$$\Pr(y_i \geq t | x_i)$$

where y_i is classification decision for x_i, $t=1,\ldots,m$

- All axioms of probability are supposed to be satisfied, e.g.:

$\Pr(y_i \geq 1 | x_i) = 1$

$\Pr(y_i \leq t | x_i) = 1 - \Pr(y_i \geq t+1 | x_i)$

$\Pr(y_i \geq t | x_i) \leq \Pr(y_i \geq t' | x_i)$ for $t \geq t'$

- These probabilities are unknown, but can be estimated from data

Probabilistic model for DRSA

- For each class t=2,…,m, we have a binary problem of estimating the conditional probabilities $\Pr(y_i \geq t | x_i)$ and $\Pr(y_i < t | x_i)$
- It is solved by isotonic regression
  - let $y_{it}=1$ if $y_i \geq t$, otherwise $y_{it}=0$
  - let p_{it} be the estimate of probability $\Pr(y_i \geq t | x_i)$
- Choose estimates p^*_{it} which minimize the squared distance to class assignment y_{it}, subject to the monotonicity constraints:

$$\text{Min:} \quad \sum_{i=1}^{|U|} \left(y_{it} - p_{it} \right)^2$$

$$\text{s.t.} \quad x_i \succeq x_j \rightarrow p_{it} \geq p_{jt} \quad \text{for all } x_i, x_j \in U$$

Probabilistic model for DRSA

- Estimates obtained from isotonic regression satisfy all axioms of probability
- Although estimates of $\Pr(y_i \geq t | x_i)$ and $\Pr(y_i < t | x_i)$, respectively p^*_{it} and $1 - p^*_{it}$, are obtained in m separate problems ($t=2,\ldots,m$), they are consistent with respect to t:

  $$p^*_{it} \leq p^*_{it'} \qquad \text{for } t \geq t'$$

  (in analogy to $\Pr(y_i \geq t | x_i) \leq \Pr(y_i \geq t' | x_i)$ for $t \geq t'$)
- Solving isotonic regression requires $O(|U|^4)$ time, but a good heuristic needs only $O(|U|^2)$

Probabilistic model for DRSA

- Stochastic α-lower approximations for classes „at least t", „at most t-1":

$$\underline{P}\left(Cl_t^{\geq}\right) = \left\{x_i \in U\colon \Pr\left(y_i \geq t \mid x_i\right) \geq \alpha\right\}$$

$$\underline{P}\left(Cl_{t-1}^{\leq}\right) = \left\{x_i \in U\colon \Pr\left(y_i < t \mid x_i\right) \geq \alpha\right\}$$

- We replace the unknown probabilities $\Pr(y_i \geq t | x_i)$ and $\Pr(y_i < t | x_i)$ by their estimates p^*_{it} obtained from isotonic regression:

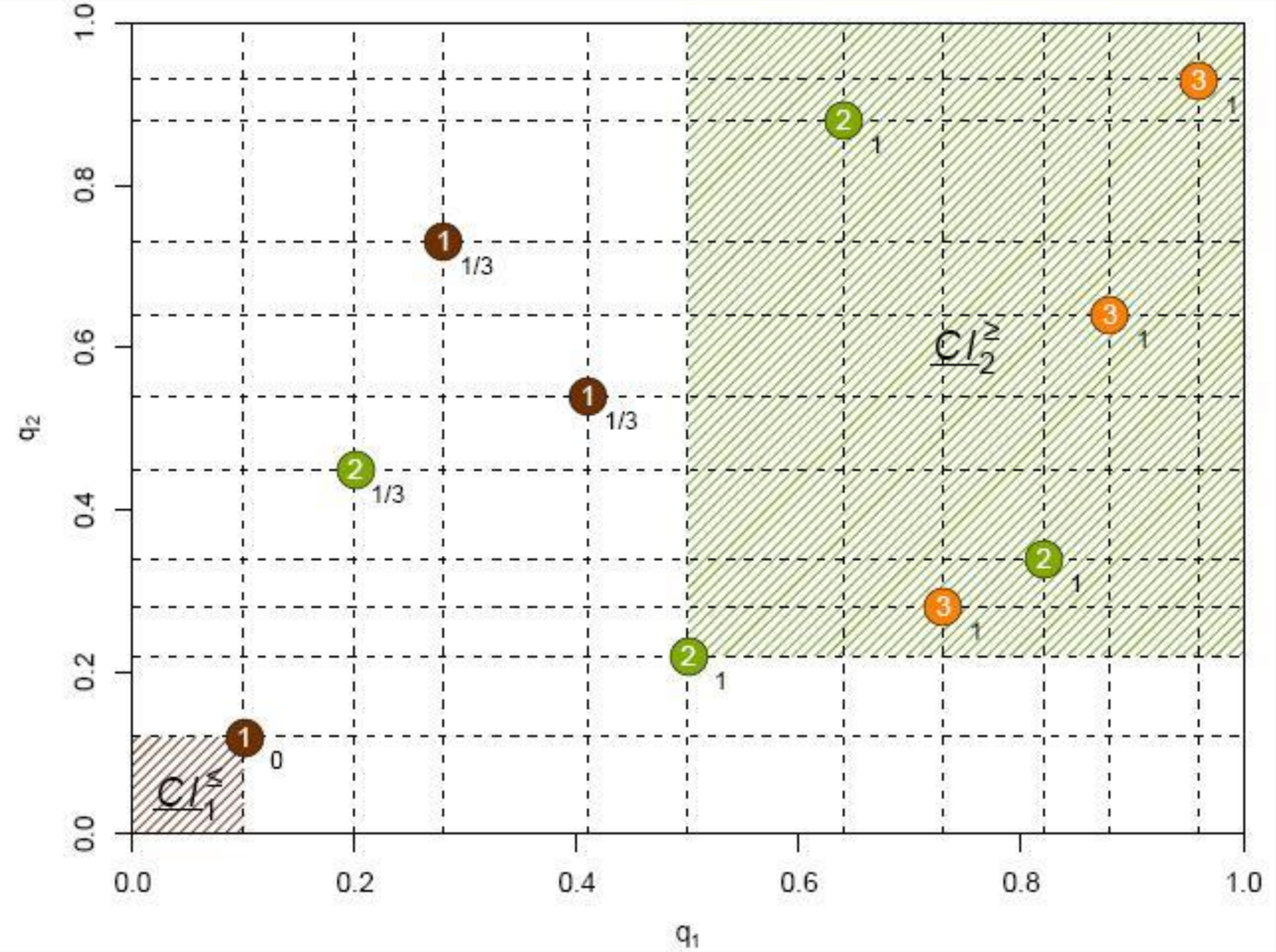
$$\underline{P}\left(Cl_t^{\geq}\right) = \left\{x_i \in U\colon p_{it}^* \geq \alpha\right\}$$

$$\underline{P}\left(Cl_{t-1}^{\leq}\right) = \left\{x_i \in U\colon p_{it}^* \geq 1 - \alpha\right\}$$

- Parameter $\alpha \in [0.5, 1]$ controls the allowed amount of inconsistency
- For $\alpha = 1$, stochastic lower approximations boil down to the classical lower upproximations

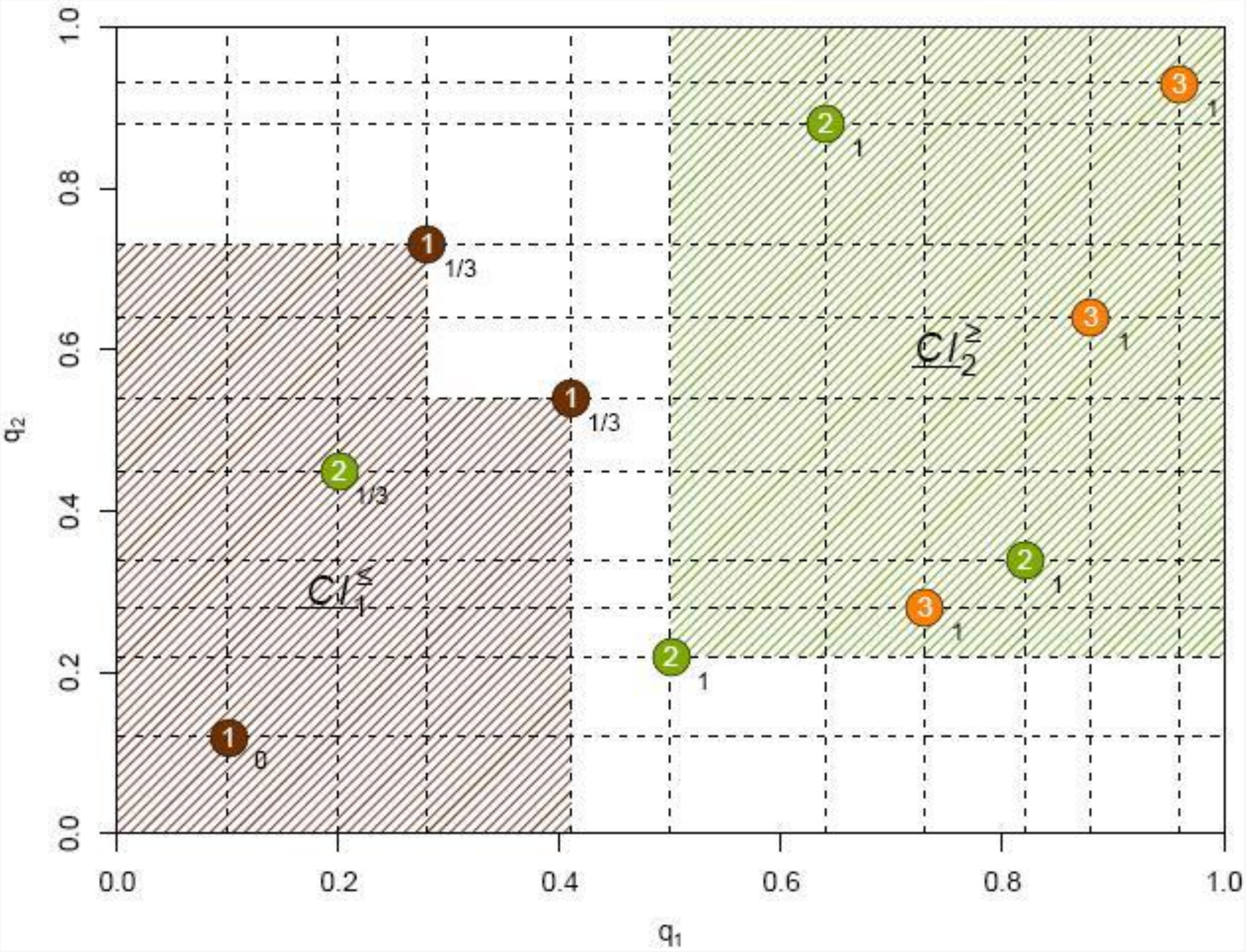
Probabilistic model for DRSA

$\alpha = 1$



Probabilistic model for DRSA

α=0.6



Probabilistic model for DRSA

- Do we really need to know the probability estimates to obtain stochastic lower approximations ?

$$\underline{P}\left(Cl_t^{\geq}\right) = \left\{x_i \in U\colon\ p_{it}^{*} \geq \alpha\right\}$$

$$\underline{P}\left(Cl_{t-1}^{\leq}\right) = \left\{x_i \in U\colon\ p_{it}^{*} \geq 1-\alpha\right\}$$

- In fact, we only need to know for which object x_i, $p^*_{it} \geq \alpha$ and for which x_i, $p^*_{it} \leq 1-\alpha$
- This can be found via linear programming (reassignment problem)

W.Kotłowski, K.Dembczyński, S.Greco, R.Słowiński: Stochastic dominance-based rough set model for ordinal classification. *Information Sciences*, 178 (2008) 4019-4037

Probabilistic model for DRSA

- Reassignment problem
  - let $y_{it}=1$ if $y_i \geq t$, otherwise $y_{it}=0$
  - let d_{it} be the decision variable (new class assignment)
- Reassign objects from „class" y_{it} to „class" d^*_{it}, such that new class assignments are consistent w.r.t. dominance principle:

$$\text{Min:} \quad \sum_{i=1}^{|U|} w_{y_{it}} \left| y_{it} - d_{it} \right|$$

$$\text{s.t.} \quad x_i \succeq x_j \to d_{it} \geq d_{jt} \quad \text{for all } x_i, x_j \in U$$

- Due to unimodularity of the constraint matrix, the optimal solution of this LP problem is always integer $d^*_{it} \in \{0, 1\}$
- For all objects consistent w.r.t. dominance principle, $d^*_{it}=y_{it}$

Probabilistic model for DRSA

- Reassignment problem

$$\text{Min:} \quad \sum_{i=1}^{|U|} w_{y_{it}} \left| y_{it} - d_{it} \right|$$

$$\text{s.t.} \quad x_i \succeq x_j \rightarrow d_{it} \geq d_{jt} \quad \text{for all} \quad x_i, x_j \in U$$

- If we set $w_0=\alpha$ and $w_1=1-\alpha$, then the optimal solution satisfies:

  $d^*_{it}=1 \;\Leftrightarrow\; p^*_{it} \geq \alpha$

- If we set $w_0=1-\alpha$ and $w_1=\alpha$, then the optimal solution satisfies:

  $d^*_{it}=0 \;\Leftrightarrow\; p^*_{it} \leq 1-\alpha$

- For each class $t=2,\ldots,m$, solving the reassignment problem twice, we obtain $\underline{P}\left(Cl_t^{\geq}\right)$, $\underline{P}\left(Cl_{t-1}^{\leq}\right)$ without knowing the probability estimates!

Results of computational experiments with Stoch-DRSA

- 8 data sets, for which it is known from a domain knowledge that monotonicity constraints are present

Data set	#attributes	#objects	#classes
ESL	4	488	8
SWD	10	1000	4
LEV	4	1000	5
Housing	8	506	4
Wisconsin	9	699	2
Ljubljana	8	286	2
Car	6	1728	4
CPU	6	209	4

- Stoch-DRSA compared with 3 standard „of-the-shelf" classifiers: C4.5 (decision trees), Naïve Bayes, Support Vector Machines (SVM)

- α set to 0.5 makes the class assignments univocal

Results of computational experiments with Stoch-DRSA

- Mean absolute error ± standard deviation from 10-fold cross-validation repeated 10 times to improve the replicability of the experiment

Dataset	Stoch-DRSA	J48	SVM	Naïve Bayes
ESL	**0.328**±0.023	0.369±0.022	0.355±0.023	**0.333**±0.024
SWD	**0.442**±0.018	**0.442**±0.016	**0.435**±0.016	0.457±0.016
LEV	**0.398**±0.017	0.415±0.018	0.444±0.016	0.441±0.017
Housing	**0.286**±0.02	0.332±0.023	0.314±0.025	0.506±0.033
CPU	**0.099**±0.02	**0.1**±0.019	0.371±0.03	0.18±0.033
Ljubljana	**0.241**±0.024	**0.259**±0.021	0.299±0.023	**0.252**±0.025
Wisconsin	**0.03**±0.007	0.046±0.009	**0.03**±0.007	0.037±0.007
Car	**0.045**±0.006	0.09±0.008	0.078±0.007	0.177±0.008

(results within one standard deviation from the best marked in **bold**)

- Stoch-DRSA which exploits solely the dominance relation outperforms standard classifiers in most of the cases!

Decision rule approach
to multiple criteria choice and ranking

Rough approximation of binary relations: DRSA for multiple criteria choice & ranking

- Preference information of the DM in form of pairwise comparisons of reference objects is put in a pairwise comparison table (PCT)

- Comparing objects $a,b \in A^R$ on

  - a <u>cardinal criterion</u>, one puts in PCT the value $\Delta_i(a,b) = g_i(a) - g_i(b)$

  - an <u>ordinal criterion</u>, one puts in PCT the ordered pair $(g_i(a),g_i(b))$

$B \subseteq A^R \times A^R$

Pairwise Comparison Table (PCT)

Pair of reference objects	Evaluations on criteria			Preference information
	g_1	...	g_n	
(a,b)	$\Delta_1(a,b)$	...	$(g_n(a),g_n(b))$	aSb
(b,a)	$\Delta_1(b,a)$	...	$(g_n(b),g_n(a))$	bS^ca
(b,c)	$\Delta_1(b,c)$	...	$(g_n(b),g_n(c))$	bSc
...	...	...	...	...
(d,e)	$\Delta_1(d,e)$	...	$(g_n(d),g_n(e))$	dS^ce

S – ***outranking***

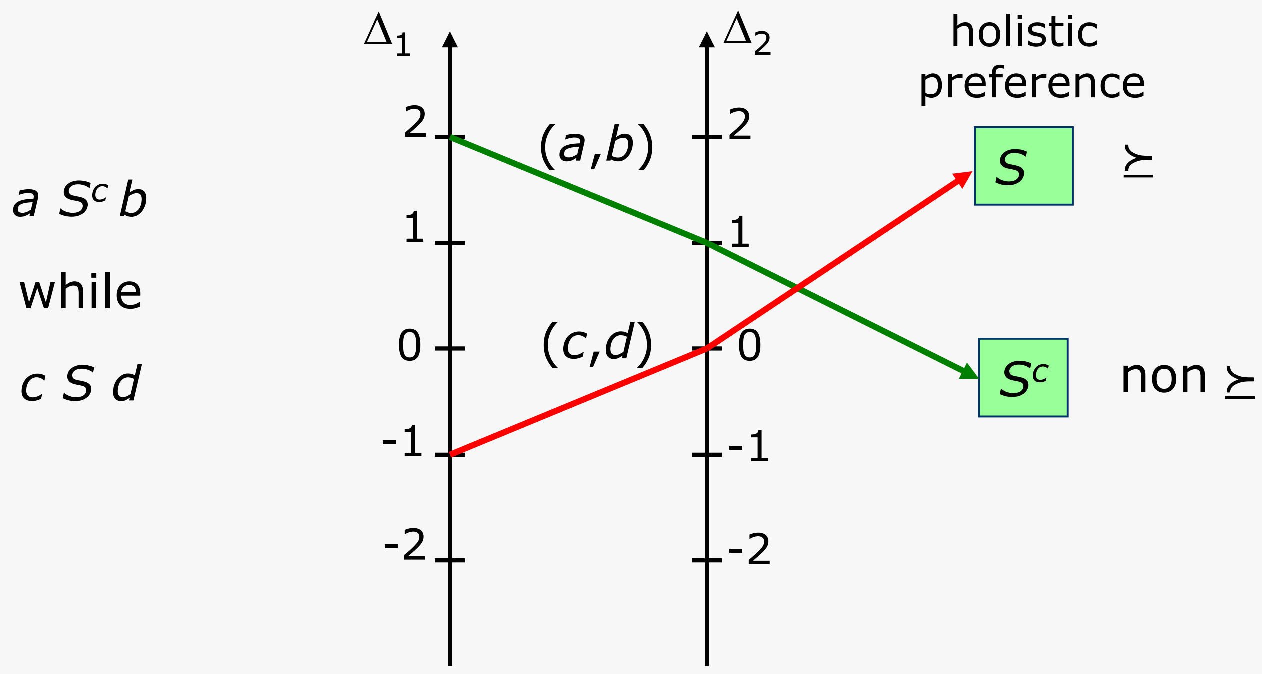
S^c – ***non-outranking***

$G = \{g_1,\dots,g_n\}$

g_1-cardinal; g_n-ordinal

DRSA for multiple criteria choice & ranking – inconsistency

- Problem $\rightarrow$ inconsistencies in the preference information, due to:
  - uncertainty of information – hesitation, unstable preferences,
  - incompleteness of the set of criteria,
  - granularity of information.
- Inconsistency w.r.t. dominance principle:



Syntax of decision rules

choice ranking **cardinal** criteria

$$\begin{cases} \textit{if } (x \succ_{q1}^{\geq h(q1)} y) \textit{ and } (x \succ_{q2}^{\geq h(q2)} y) \textit{ and} \ldots (x \succ_{qp}^{\geq h(qp)} y), \textit{ then } xSy \\ \textit{if } (x \succ_{q1}^{\leq h(q1)} y) \textit{ and } (x \succ_{q2}^{\leq h(q2)} y) \textit{ and} \ldots (x \succ_{qp}^{\leq h(qp)} y), \textit{ then } xS^c y \end{cases}$$

choice ranking **ordinal** criteria

$$\begin{cases} \textit{if } \boxed{x_{g1} \succeq_{g1} r_{q1} \;\&\; y_{g1} \preceq_{g1} r'_{q1}} \;\&\ldots x_{gp} \succeq_{gp} r_{gp} \;\&\; y_{gp} \preceq_{gp} r'_{gp}, \;\; \textit{then } xSy \\ \textit{if } \boxed{x_{g1} \preceq_{g1} r_{q1} \;\&\; y_{g1} \succeq_{g1} r'_{q1}} \;\&\ldots x_{gp} \preceq_{gp} r_{gp} \;\&\; y_{gp} \succeq_{gp} r'_{gp}, \;\; \textit{then } xS^c y \end{cases}$$

pair of objects x,y evaluated on criterion g_1

S.Greco, B.Matarazzo, R.Słowiński: Decision rule approach. Chapter 13 in: J.Figueira et al. (eds.), *Multiple Criteria Decision Analysis: State of the Art Surveys*, Springer, New York, 2005, pp. 507-562

DRSA for multiple criteria choice & ranking – dominance

- Marginal dominance relation D_2^i for pairs of objects $(a,b),(c,d)\in A\times A$:

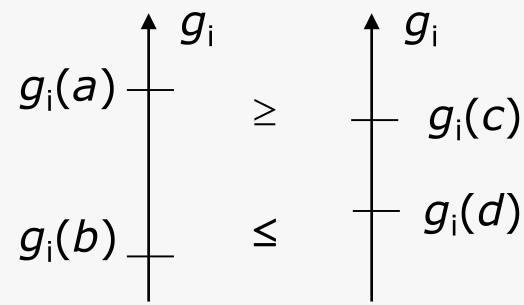
For cardinal criterion $g_i \in G$:

$(a,b)\, D_2^i\, (c,d)$ if

$\Delta_i(a,b) \geq \Delta_i(c,d)$

For ordinal criterion $g_i \in G$:

$(a,b)\, D_2^i\, (c,d)$ if



- Dominance relation D_2 for pairs of objects $(a,b),(c,d)\in A\times A$:

$$(a,b)D_2(c,d) \text{ if } (a,b)\, D_2^i\, (c,d) \text{ for all } g_i \in G, \text{ i.e.,}$$

if a is preferred to b at least as much as c is preferred to d for all $g_i \in G$

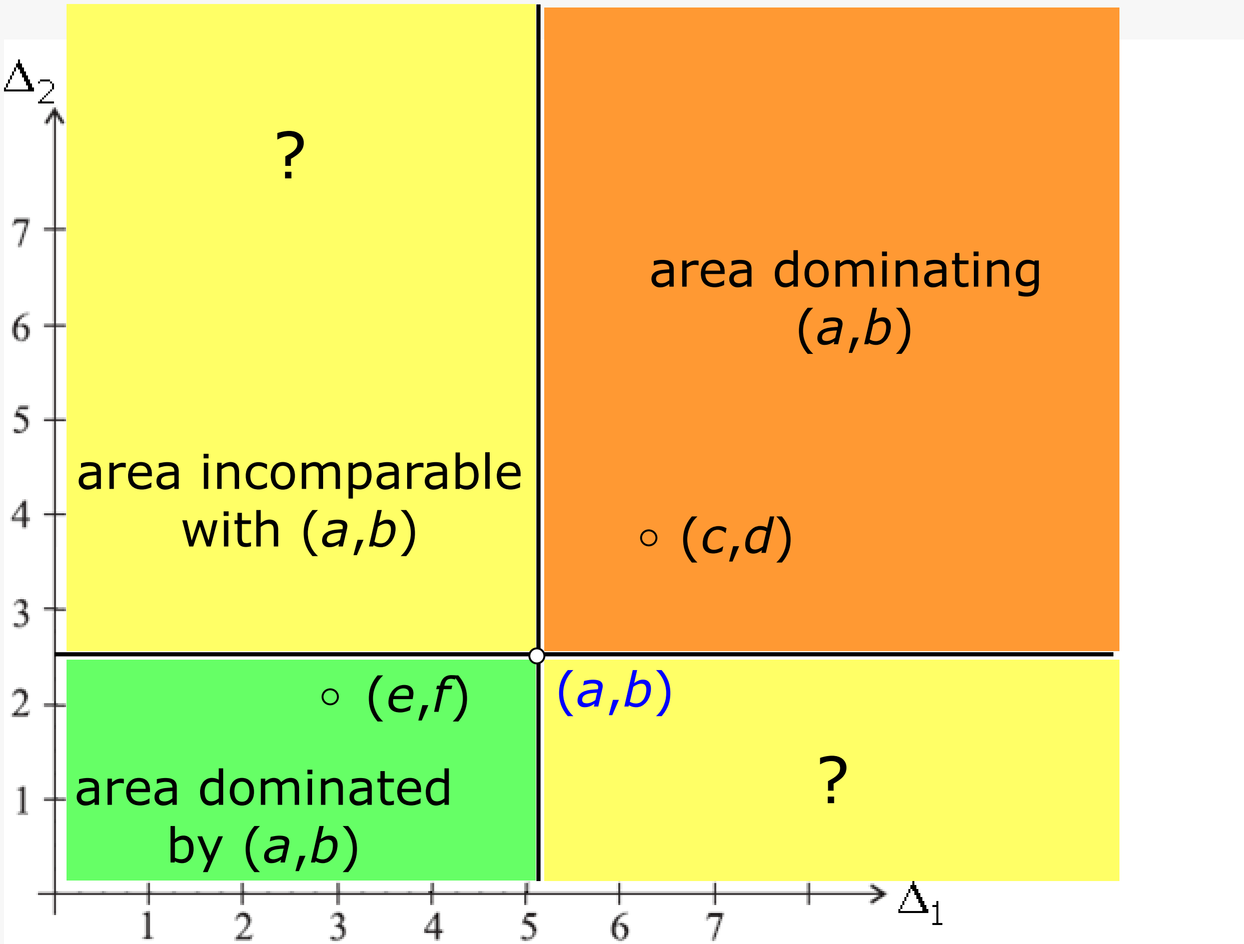
- D_2^i is reflexive, transitive, but not necessarily complete (partial preorder)

- $D_2 = \bigcap_{g_i \in G} D_2^i$ is a partial preorder on $A\times A$

DRSA – positive and negative dominance cones w.r.t. (*a*,*b*)

positive dominance cone: $D_2^+(a,b) = \{(c,d) \in A \times A : (c,d) D_2 (a,b)\}$

negative dominance cone: $D_2^-(a,b) = \{(e,f) \in A \times A : (a,b) D_2 (e,f)\}$



DRSA for multiple criteria choice & ranking – rough approximarions

- lower and upper approximations of outranking relation S:

$$\underline{S} = \left\{(a,b) \in B : D_2^+(a,b) \subseteq S\right\}$$
$$\overline{S} = \left\{(a,b) \in B : D_2^-(a,b) \cap S \neq \emptyset\right\} = \bigcup_{(a,b)\in S} D_2^+(a,b)$$

- lower and upper approximations of non-outranking relation S^c:

$$\underline{S^c} = \left\{(a,b) \in B : D_2^-(a,b) \subseteq S^c\right\}$$
$$\overline{S^c} = \left\{(a,b) \in B : D_2^+(a,b) \cap S^c \neq \emptyset\right\} = \bigcup_{(a,b)\in S^c} D_2^-(a,b)$$

- boundaries of S and S^c:

$$Bn(S) = \overline{S} - \underline{S}, \qquad Bn(S^c) = \overline{S^c} - \underline{S^c}$$
$$Bn(S) = Bn(S^c)$$

DRSA for multiple criteria choice & ranking – properties

- Basic properties:

$$\underline{S} \subseteq S \subseteq \overline{S}, \qquad \underline{S^c} \subseteq S^c \subseteq \overline{S^c}$$

$$\underline{S} = B - \overline{S^c}, \qquad \overline{S} = B - \underline{S^c}$$

$$\underline{S^c} = B - \overline{S}, \qquad \overline{S^c} = B - \underline{S}$$

- Quality of approximation of S and S^c:

$$\gamma = \frac{card\left(\underline{S} \cup \underline{S^c}\right)}{card(B)}$$

- (S,S^c)-reduct and (S,S^c)-core

DRSA for multiple criteria choice & ranking – VC-DRSA

- Variable Consistency DRSA (VC-DRSA) is relaxing the strict definitions of lower approximations of S and S^c as

$$\underline{S} = \{(a,b) \in S : \varepsilon_S(a,b) \le \theta_S\}, \qquad \overline{S} = B - \underline{S^c}$$

$$\underline{S^c} = \{(a,b) \in S^c : \varepsilon_{S^c}(a,b) \le \theta_{S^c}\}, \qquad \overline{S^c} = B - \underline{S}$$

where cost-type consistency measures $\varepsilon_S, \varepsilon_{S^c} : B \to [0,1]$ are defined as

$$\varepsilon_S(a,b) = \frac{card\left(D_2^+(a,b) \cap S^c\right)}{card\left(S^c\right)}$$

$$\varepsilon_{S^c}(a,b) = \frac{card\left(D_2^-(a,b) \cap S\right)}{card(S)}$$

and thresholds $\theta_S, \theta_{S^c} \in [0,1)$ (if $\theta_S = \theta_{S^c} = 0$, then VC-DRSA = DRSA)

Błaszczyński J., Greco S., Słowiński R., Szeląg M.: Monotonic Variable Consistency Rough Set Approaches, *International J. of Approximate Reasoning*, 50 (2009) 979-999

Induction of decision rules from rough approximations of S and S^c

- Decision rules
  - S-decision rules (induced from $\underline{S}$)

    if $(\Delta_{i1}(a,b) \geq \delta_{i1})$ *and* ... *and* $(\Delta_{ip}(a,b) \geq \delta_{ip})$ and

    $(g_{i(p+1)}(a) \geq r_{i(p+1)}$ *and* $g_{i(p+1)}(b) \leq s_{i(p+1)})$ *and* ... *and*

    $(g_{iz}(a) \geq r_{iz}$ *and* $g_{iz}(b) \leq s_{iz})$, *then* aSb

  - S^c-decision rules (induced from $\underline{S^c}$)

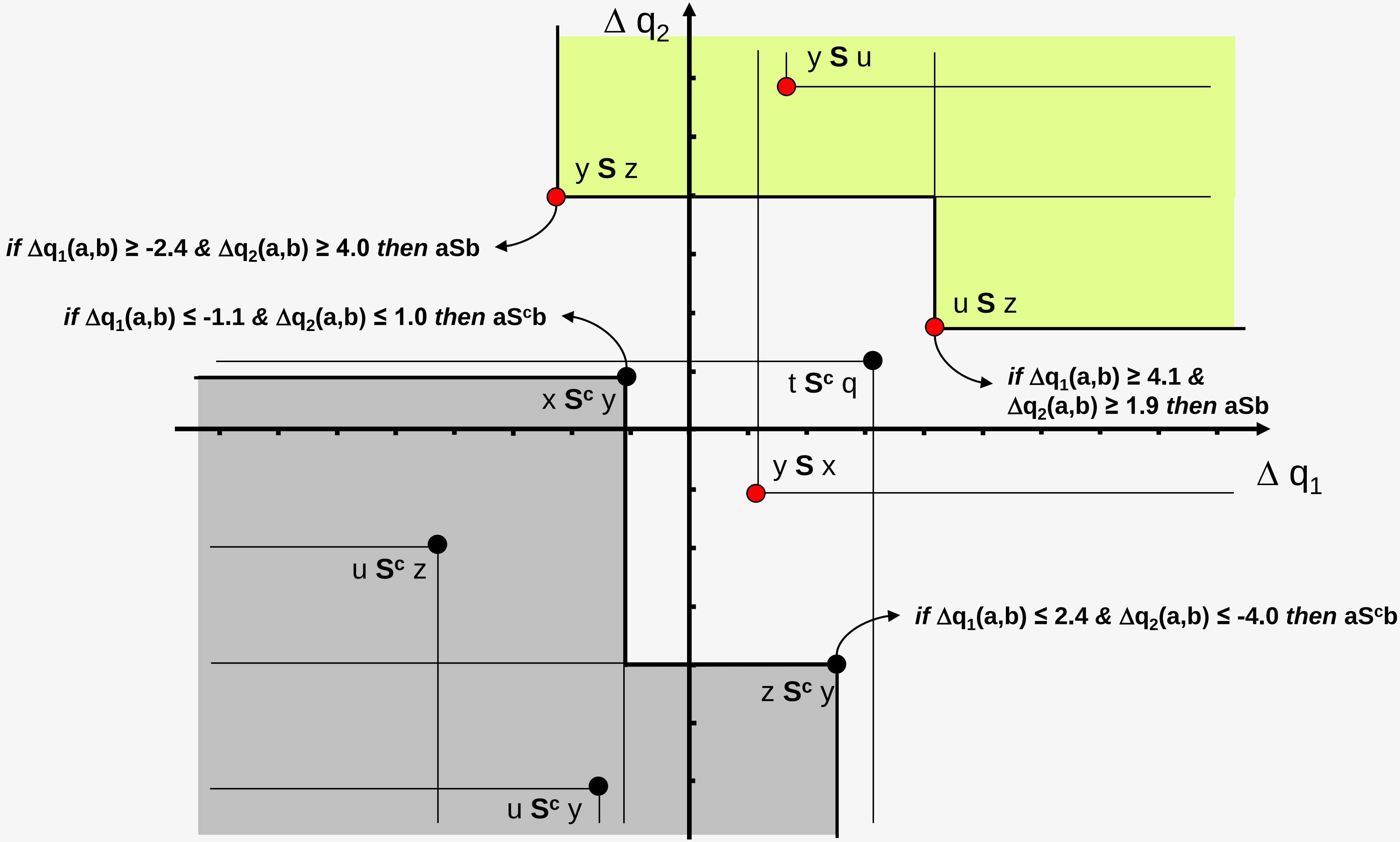
    if $(\Delta_{i1}(a,b) \leq \delta_{i1})$ *and* ... *and* $(\Delta_{ip}(a,b) \leq \delta_{ip})$ *and*

    $(g_{i(p+1)}(a) \leq r_{i(p+1)}$ *and* $g_{i(p+1)}(b) \geq s_{i(p+1)})$ *and* ... *and*

    $(g_{iz}(a) \leq r_{iz}$ *and* $g_{iz}(b) \geq s_{iz})$, *then* aS^cb

e.g., if car *a* has max speed at least 25 km/h greater than car b (cardinal criterion) and car *a* has comfort at least 3 while car *b* has comfort at most 2 (ordinal criterion), then car *a* outranks car *b* (aSb)

Induction of decision rules from rough approximations of *S* and *S*^{c}



Błaszczyński J., Słowiński R., Szeląg M.: Sequential Covering Rule Induction Algorithm for Variable Consistency Rough Set Approaches, *Information Sciences*, 181, 2011, 987-1002

DRSA for multiple criteria choice & ranking – decision rules

- Induction of rules using VC-DomLEM sequential covering algorithm, which generates a minimal set of decision rules
- Each generated rule is minimal and sufficiently consistent.
  Rule consistency is measured by cost-type rule consistency measure
  $\hat{\varepsilon}_T : R_T \rightarrow [0,1]$ defined as:

$$\hat{\varepsilon}_T(r_T) = \frac{card(\|r_T\| \cap \neg T)}{card(\neg T)}$$

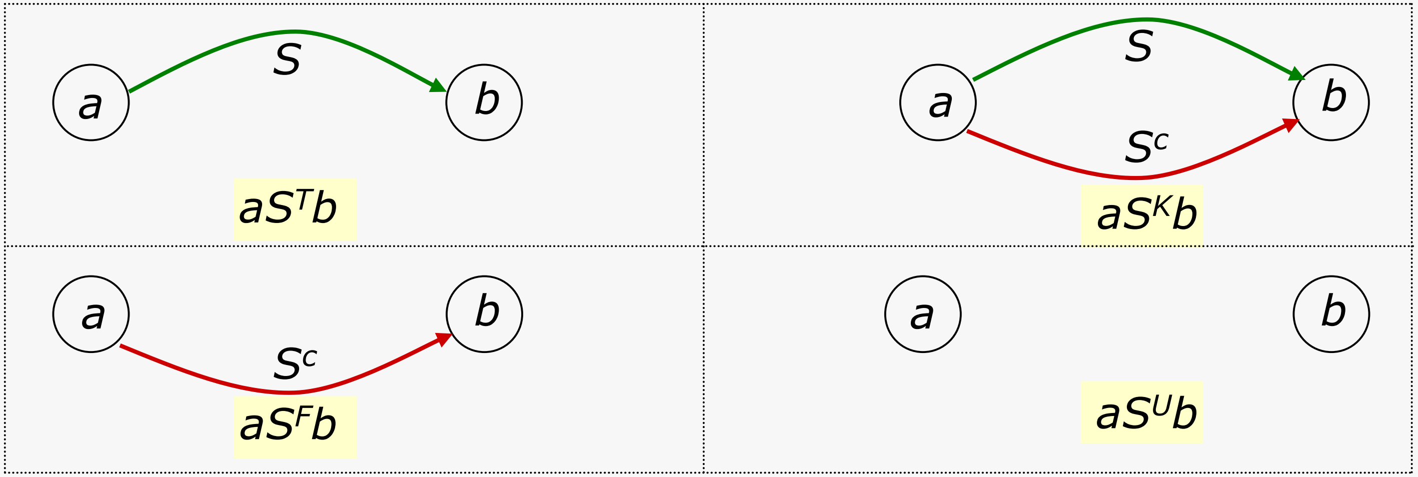
where: $T \in \{S, S^c\}$,
R_T = set of rules suggesting assignment to relation T,
$r_T \in R_T$,
$\|r_T\|$ = *set of pairs of objects covered by* r_T,
$\neg T = B - T$.

For each $r_T \in R_T$, we require that $\hat{\varepsilon}_T(r_T) \leq \theta_T$

Błaszczyński J., Słowiński R., Szeląg M.: Sequential Covering Rule Induction Algorithm for Variable Consistency Rough Set Approaches, *Information Sciences*, 181 (2011) 987-1002

Application of decision rules to multiple criteria choice & ranking

- Application of decision rules on the whole set A induces a specific preference structure on A (represented by directed multigraph)
- Any pair of objects $(a,b)\in A\times A$ can match the decision rules in one of four ways:
  - aSb and *not* aS^cb, that is *true* outranking (aS^Tb)
  - aS^cb and *not* aSb, that is *false* outranking (aS^Fb)
  - aSb and aS^cb, that is *contradictory* outranking (aS^Kb)
  - *not* aSb and *not* aS^cb, that is *unknown* outranking (aS^Ub)



The 4-valued outranking underlines the presence and the absence of positive and negative reasons of outranking.

Application of decision rules to multiple criteria choice & ranking

- The 4-valued outranking relation can be faithfully represented by three-valued fuzzy relation R_{3v}:

$$R_{3v}(a,b) = \begin{cases} 0 & \text{if } aS^F b \\ 1/2 & \text{if } aS^U b \text{ or } aS^K b \\ 1 & \text{if } aS^T b \end{cases}$$

Greco S., Matarazzo B., Słowiński R., Tsoukias A.: Exploitation of a rough approximation of the outranking relation in multicriteria choice and ranking. [In]: LNE&MS 465, Springer, Berlin, 1998, pp.45–60.

- or, more directly, as:

$$R_{3v}(a,b) = \frac{[aSb] + (1 - [aS^c b])}{2}$$

where [] denotes indicator function (0-1)

DRSA for multiple criteria choice & ranking – ranking methods

- In order to obtain final recommendation, relation R_{3v} is exploited using a ranking method. We consider the following ranking methods:
  - *Net Flow Rule* (*NFR*) – yields weak order using scoring function $SD : A \to \Re$ defined as: $SD(a) = \sum_{b \in A \setminus \{a\}} R_{3v}(a,b) - R_{3v}(b,a)$.
  - *Iterative Net Flow Rule* (*It.NFR*) – yields weak order by iterative application of scoring function *SD*.
  - *Min in Favor* (*MiF*) – yields weak order using scoring function *mF* defined as: $mF(a) = \min_{b \in A \setminus \{a\}} R_{3v}(a,b)$.
  - *Iterative Min in Favor* (*It.MiF*) – yields weak order by iterative application of scoring function *mF*.
  - *Leaving and Entering Flows* (*L/E*) – yields a partial preorder being the intersection of two weak orders obtained using scoring functions *SF* and –*SA*, defined as:

$$SF(a) = \sum_{b \in A \setminus \{a\}} R_{3v}(a,b) \qquad -SA(a) = -\sum_{b \in A \setminus \{a\}} R_{3v}(b,a)$$

Desirable properties of ranking methods

- We consider the following 10 desirable properties of ranking methods (in order of importance):
  - neutrality (property *N*),
  - monotonicity (property *M*),
  - covering compatibility (property *CC*),
  - discrimination (property *D*),
  - faithfulness (property *F*),
  - data-preservation (property *DP*),
  - independence of non-discriminating objects (property *INDO*),
  - independence of circuits (property *IC*),
  - ordinality (property *O*),
  - greatest faithfulness (property *GF*).

Szeląg M, Greco S, Słowiński R, Rule-based approach to multicriteria ranking. Chapter 6 in: M.Doumpos, E.Grigoroudis (eds.), *Multicriteria Decision Aid and Artificial Intelligence: Links, Theory and Applications*, Wiley-Blackwell, London, 2013, pp. 127-160.

Desirable properties of ranking methods

Property / RM	***NFR***	***It.NFR***	***MiF***	***It.MiF***	***L/E***
N	**T**	**T**	**T**	T	T
M	T	F	T	F	**T**
CC	**T**	**T**	T	T	T
D	T	T	F	T	T
F	**T**	**T**	**F**	**T**	T
DP	**T**	**T**	T	T	T
INDO	T	T	F	F	T
IC	**T**	F	**F**	F	F
O	**F**	F	**T**	**T**	F
GF	**F**	F	**T**	**T**	T

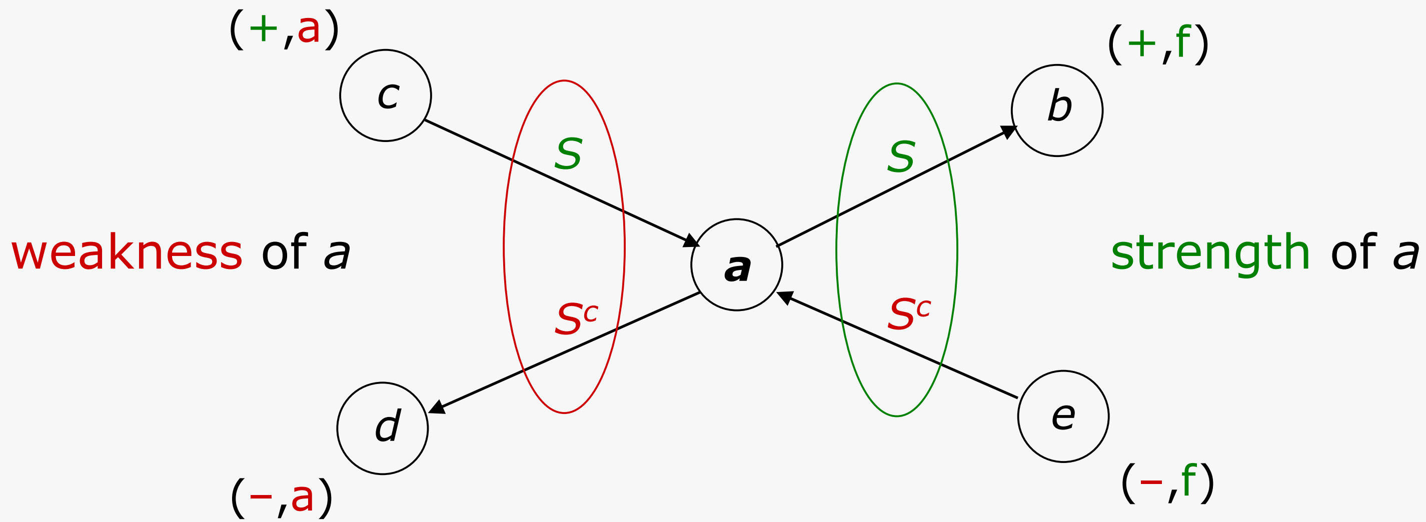
where: **T**/**F** – proof in the literature, T/F – proven by the authors

The best ranking method w.r.t. the considered properties is ***NFR***

DRSA for multiple criteria choice & ranking – *Net Flow Score*

aSb – positive (+) argument in favor of a but against b

aS^cb – negative (–) argument against a *but* in favor of b



$$NFS(a) = \text{strength}(a) - \text{weakness}(a)$$

$$NFS(a) = \sum_{b \in A \setminus \{a\}} [aSb] - [bSa] - [aS^cb] + [bS^ca]$$

where $a \in A$ and [] denotes indicator function

DRSA for multiple criteria choice & ranking – final recommendation

- Final recommendation:

  ranking: weak order over A determined by NFS

  best choice: object(s) $a^* \in A$ such that $NFS(a^*) = \max_{a \in A} \{NFS(a)\}$

Fortemps Ph, Greco S, Słowiński R, Multicriteria decision support using rules that represent rough-graded preference relations, *European J. Operational Research*, 188 (2008) 206-223.

DRSA for multiple criteria choice & ranking – illustrative example

- Mrs Brown is a scientist and wants to buy a notebook for personal use
- She would like to spend no more than 1700 EUR
- She is going to use it for: writing scientific papers, programming, performing computational experiments, and watching movies
- She considers 22 high-end notebooks (set A) that have Intel Core i7 processor with four cores, at least 4 MB of RAM (DDR3, 1333MHz), and monitor at least 15 inch with Full HD resolution (1920 x 1080 pixels)
- She evaluates the notebooks by three cardinal criteria:
  - price in EUR (g_1, to be minimized),
  - diagonal of a monitor in inches (g_2, to be maximized),
  - weight in kilograms (g_3, to be minimized).
- In the past, she tested 6 notebooks $n_1, n_4, n_{10}, n_{12}, n_{14}, n_{18}$ (reference objects), and she ranks them as follows: $n_4 \succ n_1 \succ n_{12} \succ n_{14} \succ n_{10} \succ n_{18}$

DRSA for multicriteria choice & ranking – illustrative example

Multicriteria evaluation of 6 reference notebooks (objects)

id	model	price g_1↓	diagonal g_2↑	weight g_3↓
n_1	Asus N75SF-V2G-TZ025V	865	17.3	3.4
…	…	…	…	…
n_4	DELL XPS L502X	1031	15.6	2.7
…	…	…	…	…
n_{10}	Samsung NP700G7A-S02PL	1656	17.3	3.81
…	…	…	…	…
n_{12}	Asus G53SX-IX059V	1372	15.6	3.92
…	…	…	…	…
n_{14}	Asus G73SW-91037V	1538	17.3	3.9
…	…	…	…	…
n_{18}	Lenovo ThinkPad T520	1467	15.6	2.5
…	…	…	…	…

DRSA for multicriteria choice & ranking – illustrative example

- The ranking of reference objects by the DM is a source of preference information: $n_4 \succ n_1 \succ n_{12} \succ n_{14} \succ n_{10} \succ n_{18}$
  - aSb whenever notebook a is ranked not lower than notebook b,
  - aS^cb whenever notebook a is ranked lower than notebook b.
- In this way we get $B=A^R \times A^R$
- Given the preference information, the following calculations are performed using jRank*

*http://www.cs.put.poznan.pl/mszelag/Software/jRank/jrank.pdf

DRSA for multicriteria choice & ranking – illustrative example

- The preference information in the form of pairwise comparisons of six reference objects yields a PCT composed of 36 pairs of objects

(*a*,*b*)	$\Delta_1\downarrow$	$\Delta_2\uparrow$	$\Delta_3\downarrow$	**relation**
(n_4,n_4)	0	0	0	S
(n_4,n_1)	166	-1.7	-0.7	S
…	…	…	…	…
(n_1,n_4)	-166	1.7	0.7	S^c
…	…	…	…	…
(n_{12},n_{14})	-166	-1.7	0.02	S
…	…	…	…	…
(n_{12},n_{18})	-95	0.0	1.42	S
…	…	…	…	…
(n_{18},n_{10})	-189	-1.7	-1.31	S^c
…	…	…	…	…

PCT contains in total 10 inconsistent pairs of objects

card(S)=21

card(S^c)=15

DRSA for multicriteria choice & ranking – illustrative example

- Inconsistent pairs of objects in the PCT

dominating pair $\in S^c$

dominated pair $\in S$

	(n_1,n_4)	(n_{14},n_{12})	(n_{18},n_{12})	(n_{18},n_{14})	(n_{18},n_{10})
(n_4,n_1)			*	*	*
(n_{12},n_{14})					*
(n_{12},n_{18})	*				
(n_{14},n_{18})	*				
(n_{10},n_{18})	*	*			

- We apply VC-DRSA, setting thresholds $\theta_S = \theta_{S^c} = 0.1$. In this way:
  - a pair of objects $(a,b)\in S$ is included in $\underline{S}$ if it is dominated by <u>at most 1</u> out of 15 pairs of objects belonging to S^c
  - a pair of objects $(a,b)\in S^c$ is included in $\underline{S^c}$ if it dominates <u>at most 2</u> out of 21 pairs of objects belonging to S

DRSA for multicriteria choice & ranking – illustrative example

- Decision rules induced by VCDomLEM from $\underline{S}$ and $\underline{S^c}$:

Decision rule r_T	***supp***	$\hat{\varepsilon}_T(r_T)$
if $\Delta_1(a,b) \le -284$, *then* aSb	9	0
if $\Delta_1(a,b) \le -166$ and $\Delta_3(a,b) \le 0.02$, *then* aSb	7	0.067
if $\Delta_1(a,b) \le -71$ and $\Delta_2(a,b) \ge 0$, *then* aSb	15	0.067
if $\Delta_1(a,b) \ge 95$, *then* aS^cb	12	0.095
if $\Delta_1(a,b) \ge -189$ and $\Delta_2(a,b) \le -1.7$, *then* aS^cb	4	0.095

where *supp* denotes the number of pairs of objects supporting rule r_T

- E.g., the 1st rule is read as: „*if* the difference of price for notebook *a* and notebook *b* is at most -284, then *a* is weakly preferred to *b*"
- The induced rules are relatively short and the number of rules is small w.r.t. the size of the PCT

Ranking of all 22 notebooks by the *Net Flow Score* procedure

Rank	notebook(s)	score
1	$\boldsymbol{n_1}$	39
2	n_2	38
3	n_6	37
4	n_3	30
5	$\boldsymbol{n_4}$, n_5	24
…	…	…
12	n_7, $\boldsymbol{n_{12}}$	-8
13	$\boldsymbol{n_{14}}$	-14
14	n_{20}	-16
15	n_{22}	-20
16	$\boldsymbol{n_{18}}$	-27
17	n_{17}, n_{19}	-32
18	n_9, $\boldsymbol{n_{10}}$	-35

Reference ranking

$\boldsymbol{n_4}$

$\boldsymbol{n_1}$

$\boldsymbol{n_{12}}$

$\boldsymbol{n_{14}}$

$\boldsymbol{n_{10}}$

$\boldsymbol{n_{18}}$

Kendal's $\tau = 0.733$

„Inverted" pairs:
(n_4, n_1), (n_{10}, n_{18})
were inconsistent
in the PCT

Indirect preference information – „Thierry’s choice”

(data from [Bouyssou et al. 2006])

- **Objects**: 14 cars; **Criteria**: Crit_1,…,Crit_5

Car	↓ Cost	↓ Accel	↓ Pick-up	↑ Brakes	↑ Road-h
1. Fiat Tipo	18 342	30.7	37.2	2.33	3.00
2. Alfa 33	15 335	30.2	41.6	2.00	2.50
3. Nissan Sunny	16 973	29.0	34.9	2.66	2.50
4. Mazda 323	15 460	30.4	35.8	1.66	1.50
5. Mitsubishi Colt	15 131	29.7	35.6	1.66	1.75
6. Toyota Corolla	13 841	30.8	36.5	1.33	2.00
7. Honda Civic	18 971	28.0	35.6	2.33	2.00
8. Opel Astra	18 319	28.9	35.5	1.66	2.00
9. Ford Escort	19 800	29.4	34.7	2.00	1.75
10. Renault 19	16 966	30.0	37.7	2.33	3.25
11. Peugeot 309 16V	17 537	28.3	34.8	2.33	2.75
12. Peugeot 309	15 980	29.6	35.3	2.33	2.75
13. Mitsubishi Galant	17 219	30.2	36.9	1.66	1.25
14. Renault 21	21 334	28.9	36.7	2.00	2.25

Indirect preference information – „Thierry's choice"

(data from [Bouyssou et al. 2006])

- 5 reference objects ranked by the DM: $11 \succ 3 \succ 13 \succ 9 \succ 14$

	obj1	obj2	diff_price	diff_accel	diff_pick_up	diff_brakes	diff_road_h	relation
1.	11	11	0	0	0	0	0	S
2.	11	3	564	-0,7	-0,1	-0,33	0,25	S
3.	11	13	318	-1,9	-2,1	0,67	1,5	S
4.	11	9	-2263	-1,1	0,1	0,33	1	S
5.	11	14	-3797	-0,6	-1,9	0,33	0,5	S
6.	3	11	-564	0,7	0,1	0,33	-0,25	Sc
7.	3	3	0	0	0	0	0	S
8.	3	13	-246	-1,2	-2	1	1,25	S
9.	3	9	-2827	-0,4	0,2	0,66	0,75	S
10.	3	14	-4361	0,1	-1,8	0,66	0,25	S
11.	13	11	-318	1,9	2,1	-0,67	-1,5	Sc
12.	13	3	246	1,2	2	-1	-1,25	Sc
13.	13	13	0	0	0	0	0	S
14.	13	9	-2581	0,8	2,2	-0,34	-0,5	S
15.	13	14	-4115	1,3	0,2	-0,34	-1	S
16.	9	11	2263	1,1	-0,1	-0,33	-1	Sc
17.	9	3	2827	0,4	-0,2	-0,66	-0,75	Sc
18.	9	13	2581	-0,8	-2,2	0,34	0,5	Sc
19.	9	9	0	0	0	0	0	S
20.	9	14	-1534	0,5	-2	0	-0,5	S
21.	14	11	3797	0,6	1,9	-0,33	-0,5	Sc
22.	14	3	4361	-0,1	1,8	-0,66	-0,25	Sc
23.	14	13	4115	-1,3	-0,2	0,34	1	Sc
24.	14	9	1534	-0,5	2	0	0,5	Sc
25.	14	14	0	0	0	0	0	S

Pairwise Comparison Table (PCT)

S = outranking

S^c = non-outranking

Illustrative example – Thierry's choice by DRSA (inductive learning of rules)

- All minimal rules (based on pairs of objects) induced from PCT

Number	Condition	Decision	Strength	Relative Strength
1.	(diff_price >= 1534)	relation at most Sc	7	70,00 %
2.	(diff_accel >= 1,9)	relation at most Sc	1	10,00 %
3.	(diff_brakes <= -0,66)	relation at most Sc	4	40,00 %
4.	(diff_road_h <= -1,25)	relation at most Sc	2	20,00 %
5.	(diff_price >= -564) & (diff_accel >= 0,7)	relation at most Sc	4	40,00 %
6.	(diff_price >= -564) & (diff_pick_up >= 0,1)	relation at most Sc	6	60,00 %
7.	(diff_price >= -564) & (diff_road_h <= -0,25)	relation at most Sc	7	70,00 %
8.	(diff_accel >= 1,2) & (diff_pick_up >= 2)	relation at most Sc	2	20,00 %
9.	(diff_price <= -1534)	relation at least S	7	46,67 %
16.	(diff_price <= 564) & (diff_pick_up <= -0,1)	relation at least S	6	40,00 %
17.	(diff_price <= 0) & (diff_road_h >= 0)	relation at least S	10	66,67 %
18.	(diff_price <= 564) & (diff_road_h >= 0,25)	relation at least S	7	46,67 %
19.	(diff_accel <= -1,2) & (diff_pick_up <= -2)	relation at least S	2	13,33 %

Minimal set of rules covering all actions:

{1, 7, 9, 17, 18}

Supporting Examples:

	obj1	obj2	diff_price	diff_accel	diff_pick_up	diff_brakes	diff_road_h	relation
16.	9	11	2263	1,1	-0,1	-0,33	-1	Sc
17.	9	3	2827	0,4	-0,2	-0,66	-0,75	Sc
18.	9	13	2581	-0,8	-2,2	0,34	0,5	Sc
21.	14	11	3797	0,6	1,9	-0,33	-0,5	Sc
22.	14	3	4361	-0,1	1,8	-0,66	-0,25	Sc
23.	14	13	4115	-1,3	-0,2	0,34	1	Sc
24.	14	9	1534	-0,5	2	0	0,5	Sc

Illustrative example – Thierry's choice by DRSA (inductive learning of rules)

- Ranking of all 14 objects by *Net Flow Score* exploitation procedure

rank	object	score
1	6	24
2	2	22
3	5, 12	
5	10	
6	4	
7	**11**	0
8	**3**	-2
9	1	-4
10	**13**	-10
11	8	-13
12	7	-17
13	**9**	-22
14	**14**	-26

Reference ranking:

$11 \succ 3 \succ 13 \succ 9 \succ 14$

Decision rule approach
to decision under risk & uncertainty

DRSA for decision under risk and uncertainty

- ***ST***$=\{st_1, st_2, st_3, \ldots\}$ – set of elementary states of the world
- Pr – a priori probability distribution over ***ST***
  e.g.: $pr_1=0.25$, $pr_2=0.30$, $pr_3=0.35$, ...
- ***A***$=\{A_1, A_2, A_3, A_4, A_5, A_6, \ldots\}$ – set of acts
- ***X***$=\{0, 10, 15, 20, 30, \ldots\}$ – set of possible outcomes (gains)
- ***Cl***$=\{Cl_1, Cl_2, Cl_3, \ldots\}$ – set of quality classes of the acts,
  e.g.: Cl_1=bad acts, Cl_2=medium acts, Cl_3=good acts
- $\rho(A_i,\pi)=x$ means that by act A_i one can gain *at least* x with at least probability $\pi=Pr(\boldsymbol{W})$, where $\boldsymbol{W}\subseteq \boldsymbol{ST}$ is an event
- There is a partial preorder on probabilities π of events
- Act A_i ***stochastically dominates*** A_j iff $\rho(A_i,\pi) \geq \rho(A_j,\pi)$ for each probability π

DRSA for decision under risk and uncertainty

- Preference information given by a Decision Maker:

  assignment of some acts to quality classes

- Example:

π/*Act*	A_1	A_2	A_3	A_4	A_5	A_6
.25	30	20	20	20	20	20
.35	10	20	20	20	20	20
.40	10	20	20	20	20	20
.60	10	20	15	15	20	20
.65	10	20	15	15	20	20
.75	10	20	0	15	10	20
1	10	0	0	0	10	10
Class	good	medium	medium	bad	medium	good

DRSA for decision under risk and uncertainty

- Decision rules induced from rough approximations of quality classes

$$\textit{if}\ \ \rho(A_i, 0.75)\geq 20\ \textit{and}\ \rho(A_i, 1)\geq 10,\ \ \textit{then}\ A_i \in Cl_3^{\geq} \qquad (A_6)$$

"if the probability of gaining <u>at least</u> 20 is ≥0.75, and the probability of gaining <u>at least</u> 10 is 1, then act A_i is at least good"

$$\textit{if}\ \ \rho'(A_i, 0.25)\leq 20\ \textit{and}\ \rho'(A_i, 0.75)\leq 15,\ \ \textit{then}\ A_i \in Cl_2^{\leq} \qquad (A_3, A_4, A_5)$$

"if the probability of gaining <u>at most</u> 20 is ≥0.25, and the probability of gaining <u>at most</u> 15 is ≥0.75, then act A_i is at most medium"

- Generalization:
  DRSA for decision under risk with outcomes distributed over time
  (decision under uncertainty and time preference)

Greco S., Matarazzo B., Slowinski R., Dominance-based rough set approach to decision under uncertainty and time preference. *Annals of Operations Research*, 176 (2010) 41-75

DRSA-PCT to decision under risk and uncertainty

- Decision rules induced from rough approximations of binary preference relations on pairs of acts A_i, A_j :

„if the probability of gaining <u>at least 20\$ more</u> is ≥0.75,
and the probability of gaining <u>at least 10\$ more</u> is 1,
then act A_i is better than act A_j"

„if the probability of gaining <u>at least 10\$ less</u> is ≥0.5,
and the probability of gaining <u>at least 5\$ less</u> is ≥0.8,
then act A_i is worse than act A_j"

What other monotonic relationships can be handled by DRSA?

- DRSA – dominance relation:

  „The more, the better”

- DRSA for decision under uncertainty – stochastic dominance relation:

  „The more and the more probable, the better”

- DRSA for time preferences – time dominance relation:

  „The more and the earlier, the better”

- DRSA for time preference & uncertainty – time stochastic dominance:

  „The more, the more probable and the earlier, the better”

- DRSA can be applied to a large collection of operational research problems, such as portfolio selection, scheduling under uncertainty, inventory management, interactive (robust) multiobjective optimization, …

Decision rule approach
to interactive multiobjective optimization

DRSA to Interactive Multiobjective Optimization

$$\begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{bmatrix} \to \text{Min}$$

subject to the constraints :

$$g_1(\boldsymbol{x}) \{\leq, =, \geq\} b_1$$

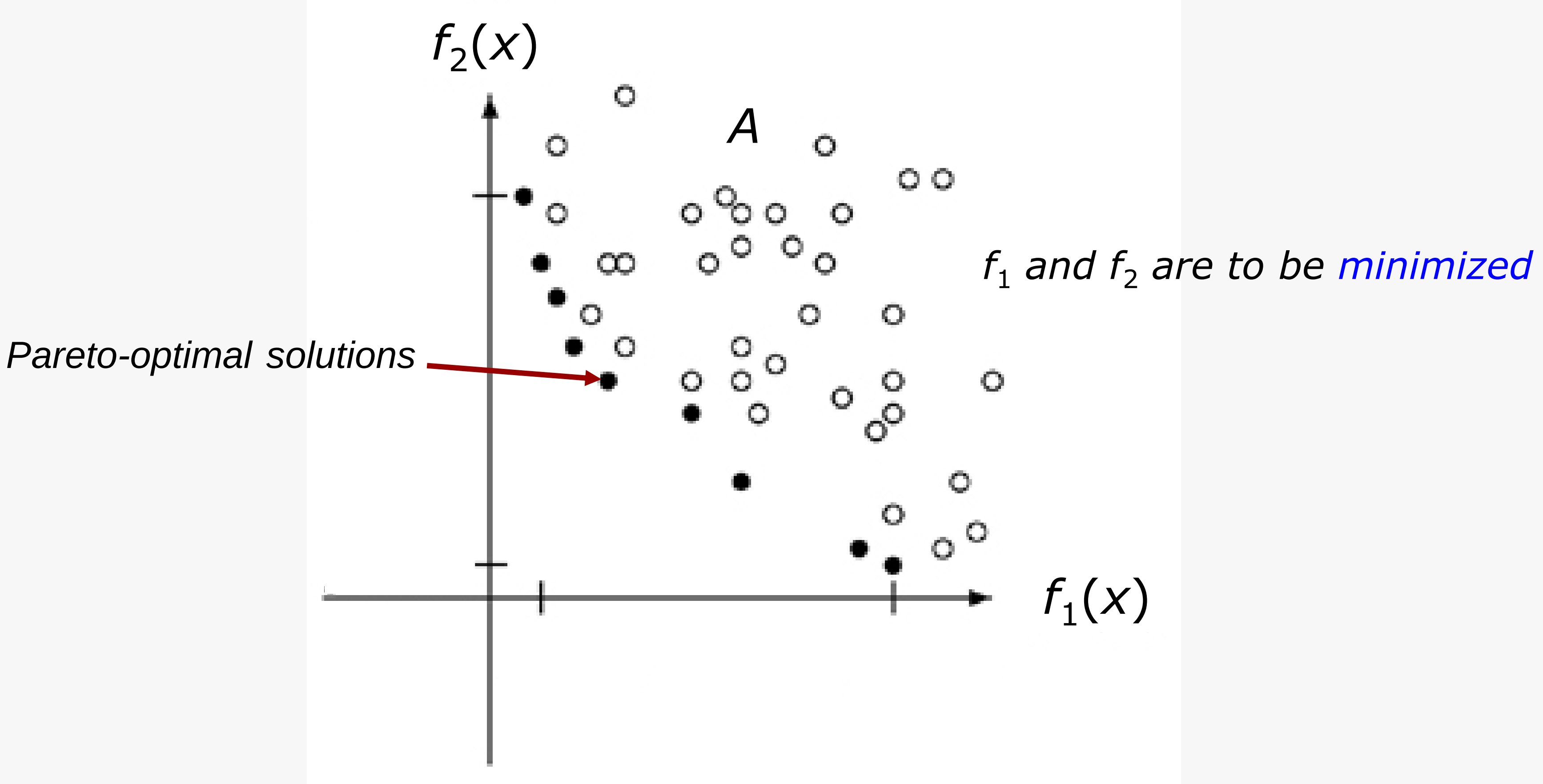
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$g_m(\boldsymbol{x}) \{\leq, =, \geq\} b_m$$

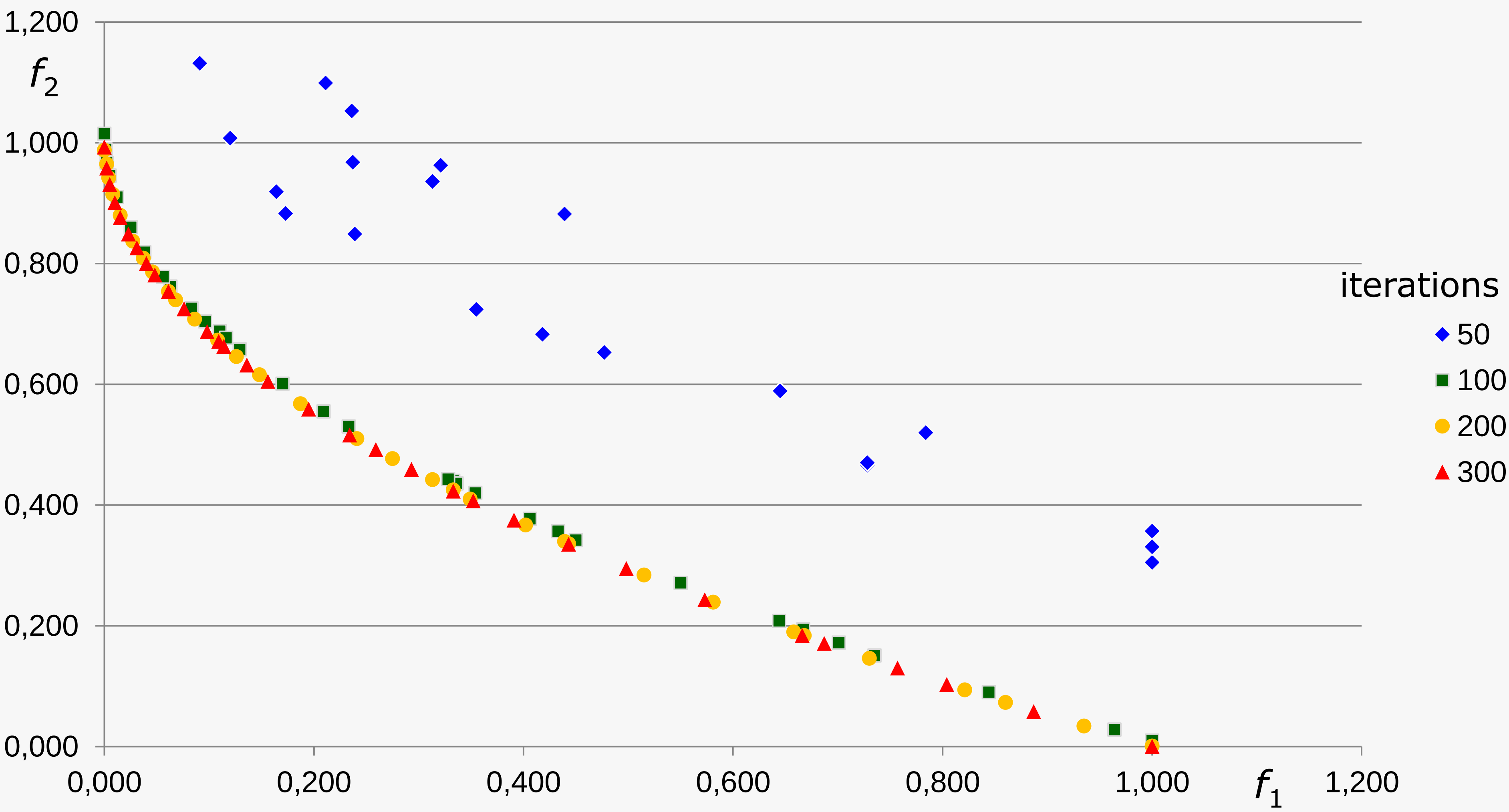
where $x=[x_1,\ldots,x_k]$ is a vector of decision variables
$f_j(x)$, $j=1,\ldots,n$, are real-valued objective functions
$g_i(x)$, $i=1,\ldots,m$, are real-valued functions of the constraints
b_i, $i=1,\ldots,m$, are constant RHS of the constraints

Multiobjective Optimization – dominance relation

- Solution $a \in A$ is Pareto-optimal (non-dominated) if and only if there is no other solution $b \in A$ such that $f_i(b) \succeq f_i(a)$, $i \in \{1,\dots,n\}$, and on at least one objective $j \in \{1,\dots,n\}$, $f_j(b) \succ f_j(a)$



Evolutionary Multiobjective Optimization (EMO)



Dominance-based association rules describing the Pareto optimal set

- Relationships between attainable values of different objective functions (criteria) in the set of Pareto-optimal solutions

- Formal syntax (in case of maximization of objectives):

  If $f_{i1}(x) \geq r_{i1}$ *and* $f_{i2}(x) \geq r_{i2}$ *and* … $f_{ip}(x) \geq r_{ip}$,

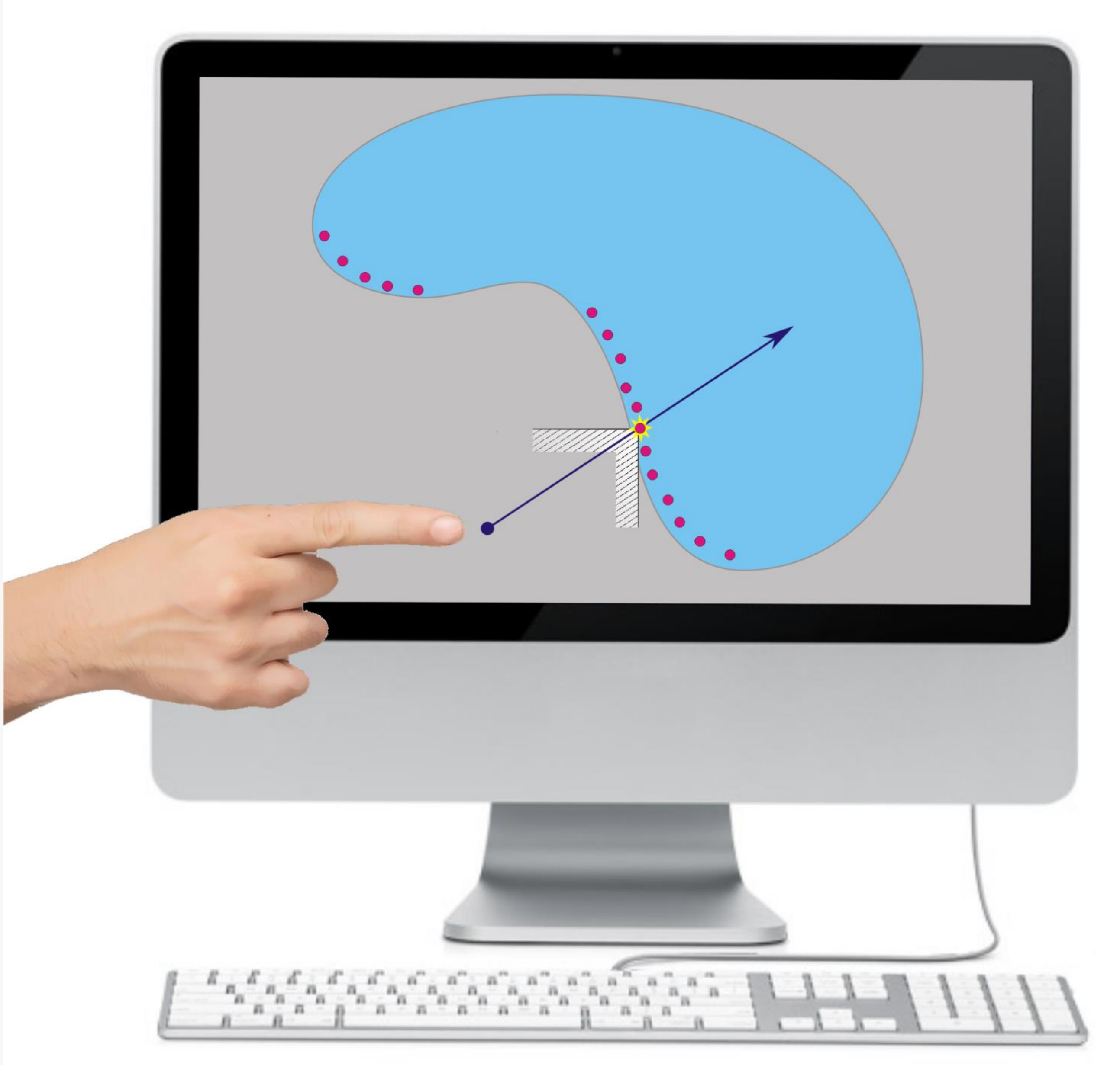
  then $f_{ip+1}(x) \leq r_{ip+1}$ *and* $f_{ip+2}(x) \leq r_{ip+2}$ *and* … $f_{iq}(x) \leq r_{iq}$

- Example from product-mix problem:
  - „*if* profit ≥ 148 & time_machine ≤ 150,

    then *amount_product_*$x_B \leq 2$"

Greco, S., Matarazzo, B., Słowiński, R.: Dominance-Based Rough Set Approach to Interactive Multiobjective Optimization, Chapter 5 in J.Branke, K.Deb, K.Miettinen, R.Słowiński (eds.), *Multiobjective Optimization: Interactive and Evolutionary Approaches*. Springer, State-of-the-Art Surveys, LNCS 5252, Berlin, 2008, pp.121-156

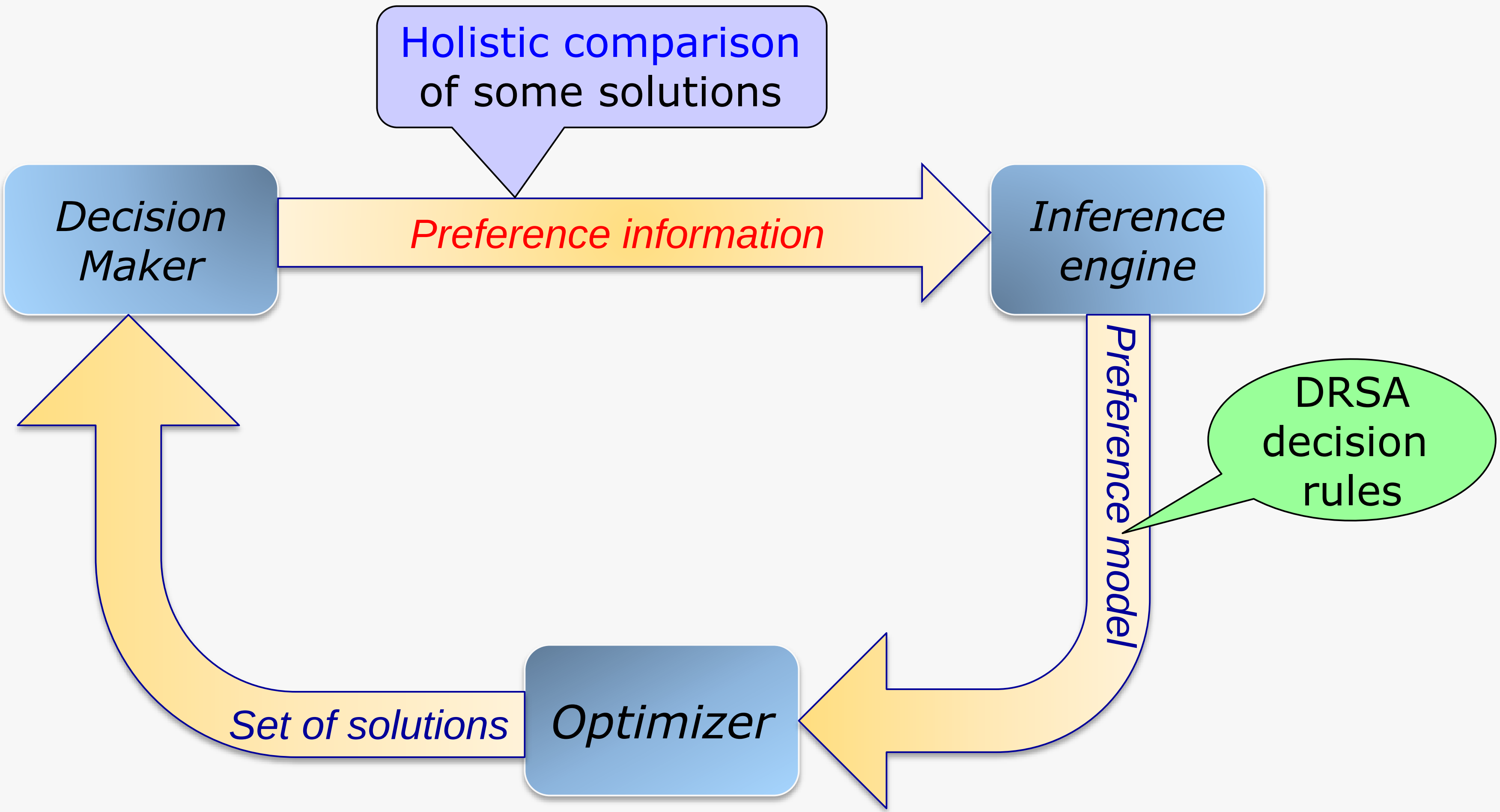
Multiobjective Optimization – interactive procedures



What preference information and preference model should be used ?

- The traditional interactive methods appear to be too demanding of the cognitive effort of their users
- We advocate for „easy” preference information = natural and partial
- The most natural is a **holistic comparison** of some solutions
- The preference model should be intelligible and comprehensible
- We advocate for **decision rules**

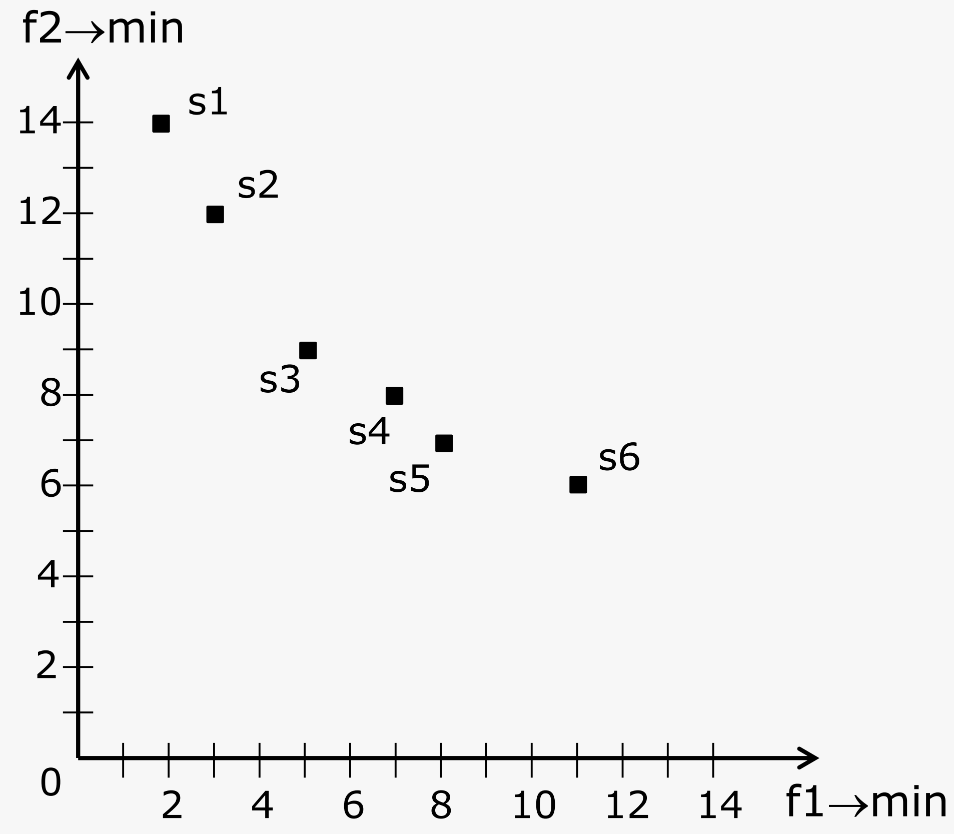
Interactive cycle with elicitation of preferences



Example

Sample of 6 non-dominated solutions submitted to evaluation of the DM

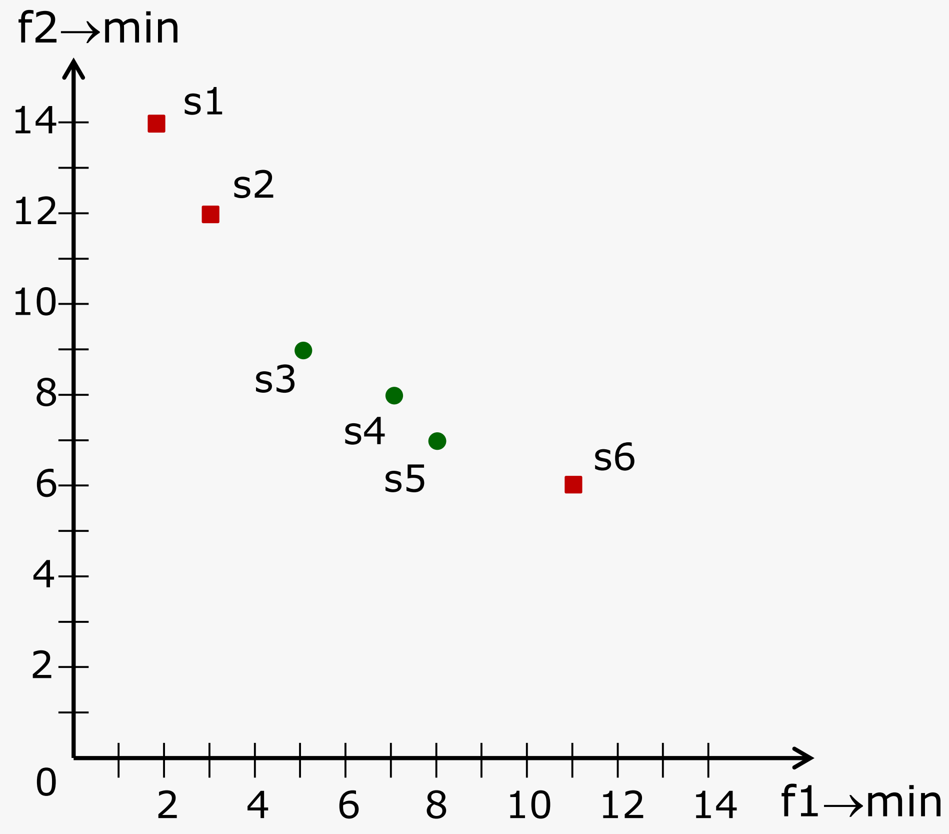
solution	f1	f2	DM
s1	2	14	
s2	3	12	
s3	5	9	
s4	7	8	
s5	8	7	
s6	11	6	



Example

Sample of 6 non-dominated solutions submitted to evaluation of the DM

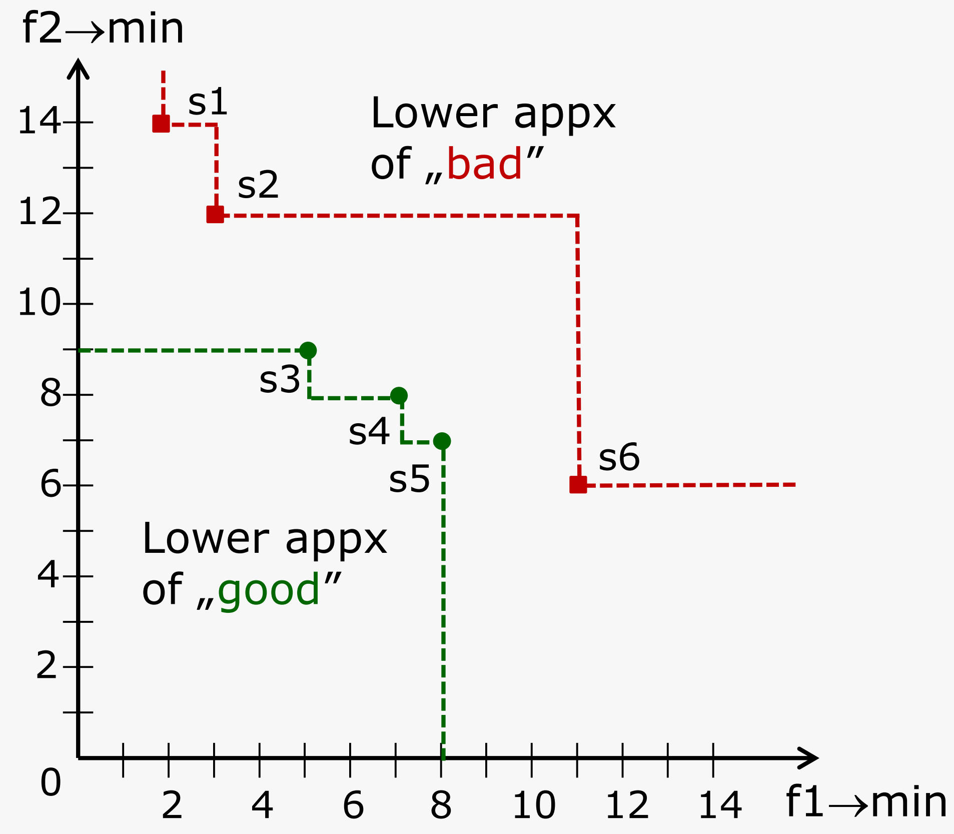
solution	f1	f2	DM
s1	2	14	bad
s2	3	12	bad
s3	5	9	good
s4	7	8	good
s5	8	7	good
s6	11	6	bad



Example

Sample of 6 non-dominated solutions submitted to evaluation of the DM

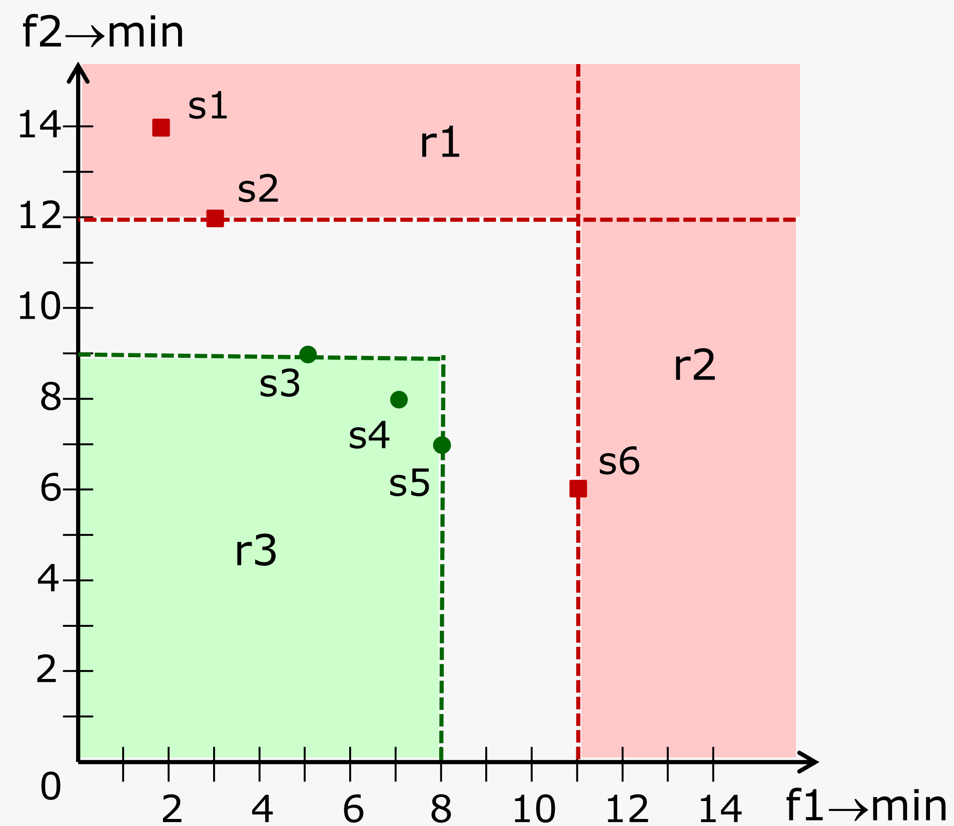
solution	f1	f2	DM
s1	2	14	bad
s2	3	12	bad
s3	5	9	good
s4	7	8	good
s5	8	7	good
s6	11	6	bad



Example

Sample of 6 non-dominated solutions submitted to evaluation of the DM

solution	f1	f2	DM
s1	2	14	bad
s2	3	12	bad
s3	5	9	good
s4	7	8	good
s5	8	7	good
s6	11	6	bad



r1: if f2(s)≥12, then s is bad supported by {s1,s2}

r2: if f1(s)≥11, then s is bad supported by {s6}

r3: if f1(s)≤8 & f2(s)≤9, then s is good supported by {s3,s4,s5}

DRSA to Interactive Multiobjective Optimization – DRSA-IMO

1) Present to DM a representative set of efficient (Pareto-optimal) solutions
2) Present association rules showing relationships between the attainable values of the objective functions and relationships between decision variables and objective functions in the Pareto-optimal set
3) If DM finds a satisfactory solution, then stop, otherwise go to step 4)
4) DM selects efficient solutions judged as (relatively) **good** and **bad**
5) DRSA „*if...,then...*" decision rules are induced from info got in step 4)
6) The most interesting decision rules are presented to DM
7) The DM selects one or more decision rules being the most adequate to his/her preferences
8) Constraints relative to these decision rules are included in the set of constraints
9) Go back to step 1)

Example of Production Mix Problem: data

- Three products: A, B, C
- Produced quantity (decision variables): x_A , x_B , x_C
- Price: $p_A=20$, $p_B=30$, $p_C=25$
- Time machine 1: $t^1_A=5$, $t^1_B=8$, $t^1_C=10$
- Time machine 2: $t^2_A=8$, $t^2_B=6$, $t^2_C=2$
- Raw material 1: $r^1_A=1$, $r^1_B=2$, $r^1_C=0.75$; unit cost: 6
- Raw material 2: $r^2_A=0.5$, $r^2_B=1$, $r^2_C=0.5$; unit cost: 8
- Market limit: $x_A^* =10$, $x_B^* =20$, $x_C^* =10$

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Example of Production Mix Problem: mathematical formulation

- Max $\rightarrow$ Profit
- Min $\rightarrow$ Total time (machine 1 + machine 2)
- Max $\rightarrow$ Produced quantity of A
- Max $\rightarrow$ Produced quantity of B
- Max $\rightarrow$ Produced quantity of C
- Max $\rightarrow$ Sales

Example of Production Mix Problem: objectives and constraints

- Max $\rightarrow$ $20x_A + 30x_B + 25x_C - (1x_A + 2x_B + 0.75x_C)6 +$
  $-(0.5x_A + x_B + 0.5x_C)8$ [Profit]
- Min $\rightarrow$ $5x_A + 8x_B + 10x_C + 8x_A + 6x_B + 2x_C$
  [Total time machine 1 + machine 2]
- Max $\rightarrow$ x_A [Produced quantity of A]
- Max $\rightarrow$ x_B [Produced quantity of B]
- Max $\rightarrow$ x_C [Produced quantity of C]
- Max $\rightarrow$ $20x_A + 30x_B + 25x_C$ [Sales]
- $x_A \leq 10$, $x_B \leq 20$, $x_C \leq 10$ [Market Limits]
- $x_A \geq 0$, $x_B \geq 0$, $x_C \geq 0$ [Non-negativity]

Set of representative efficient solutions of a production mix problem

Efficient solutions	Profit	Total time	Prod. x_A	Prod. x_B	Prod. x_C	Sales
S1	165	120	0	0	10	250
S2	172.692	130	0.769	0	10	265.385
S3	180.385	140	1.538	0	10	280.769
S4	141.125	140	3	3	4.917	272.916
S5	148.375	150	5	2	4.75	278.75
S6	139.125	150	5	3	3.583	279.583
S7	188.077	150	2.308	0	10	296.154
S8	159	150	6	0	6	270
S9	140.5	150	6	2	3.667	271.667
S10	209.25	200	6	2	7.833	375.833
S11	189.375	200	5	5	5.417	385.417
S12	127.375	130	3	3	4.083	252.083
S13	113.625	120	3	3	3.25	231.25

Sorting of representative efficient solutions

Efficient solutions	Profit	Total time	Prod. x_A	Prod. x_B	Prod. x_C	Sales	Class
S1	165	120	0	0	10	250	*
S2	172.692	130	0.769	0	10	265.385	*
S3	180.385	140	1.538	0	10	280.769	**Good**
S4	141.125	140	3	3	4.917	272.916	**Good**
S5	148.375	150	5	2	4.75	278.75	**Good**
S6	139.125	150	5	3	3.583	279.583	*
S7	188.077	150	2.308	0	10	296.154	*
S8	159	150	6	0	6	270	*
S9	140.5	150	6	2	3.667	271.667	**Good**
S10	209.25	200	6	2	7.833	375.833	*
S11	189.375	200	5	5	5.417	385.417	*
S12	127.375	130	3	3	4.083	252.083	*
S13	113.625	120	3	3	3.25	231.25	*

The most interesting DRSA decision rules

- If profit$\geq$140.5 and time$\leq$150 and $x_B \geq 2$,
  then product mix is good (s4,s5,s9)
- If time$\leq$140 and $x_A \geq 1.538$ and $x_C \geq 10$,
  then product mix is good (s3)
- If time$\leq$150 and $x_B \geq 2$ and $x_C \geq 4.75$,
  then product mix is good (s4,s5)
- If time$\leq$140 and sales$\geq$272.917,
  then product mix is good (s3,s4)
- If time$\leq$150 and $x_B \geq 2$ and $x_C \geq 3.667$ and sales$\geq$271.667,
  then product mix is good (s4,s5,s9)

Selected decision rules

- DM selected **two** rules as the most adequate to preferences:

r1: If profit $\geq$ 140.5 and time $\leq$ 150 and $x_B \geq 2$,
then product mix is good (s4,s5,s9)

r2: If time $\leq$ 140 and sales $\geq$ 272.917,
then product mix is good (s3,s4)

Added constraints for decision rule r1

- First selected decision rule r1:

  If profit $\geq$ 140.5 and time $\leq$ 150 and $x_B \geq 2$,
  then product mix is good (s4,s5,s9)

- Added constraints to the production mix problem:
- $20x_A + 30x_B + 25x_C - (1x_A + 2x_B + 0.75x_C)6 +$
  $- (0.5x_A + x_B + 0.5\,x_C)8 + (1-\delta_1)M \geq 140.5$ [profit $\geq$ 140.5]
- $5x_A + 8x_B + 10x_C + 8x_A + 6x_B + 2x_C - (1-\delta_1)M \leq 150$
  [time $\leq$ 150]
- $x_B + (1-\delta_1)M \geq 2$ [produced quantity of B $\geq$ 2]

where $\delta_1 \in \{0,1\}$, M big number (10^6); if δ_1=1, then r1 is satisfied;
if δ_1=0, then r1 is **not** satisfied because each above constraint is satisfied whatever value of decision variables

Added constraints for decision rule r2

- Second selected decision rule r2:

  If time $\leq$ 140 and sales $\geq$ 272.917,

  then product mix is good (s3,s4)

- Added constraints to the production mix problem:
- $5x_A + 8x_B + 10x_C + 8x_A + 6x_B + 2x_C - (1-\delta_2)M \leq 140.5$

  [time $\leq$ 140]
- $20x_A + 30x_B + 25x_C + (1-\delta_2)M \geq 272.917$ [sales $\geq$ 272.9167]

where $\delta_2 \in \{0,1\}$, M big number (10^6); if $\delta_2=1$, then r2 is satisfied

Condition for filtering good solutions

- A solution is good if it satisfies at least one of decision rules r1 & r2:

$$\boxed{\delta_1 + \delta_2 \geq 1}$$

Set of representative efficient solutions (second iteration)

Efficient solution	Profit	Total time	Prod. x_A	Prod. x_B	Prod. x_C	Sales	δ_1	δ_2
S1’	210.7143	150	0	2.142857	10	314.2857	1	0
S2’	140.5	150	0	9.469565	1.452174	320.3913	1	0
S3’	140.5	103.5676	0	2	6.297297	217.4324	1	0
S4’	140.5	150	5.097923	2	4.643917	278.0564	1	0
S5’	120	140	0	10	0	300	0	1
S6’	191.875	134.8959	1.145835	0	10	272.9167	0	1
S7’	150	109.7297	0	2	6.810811	230.2703	1	0
S8’	150	127.9459	2	2	6.162162	254.0541	1	0
S9’	150	135	2	3.134783	5.426087	269.6957	1	0
S10’	160.875	135	2	2	6,75	268.75	1	0
S11’	192.7143	135	1	0.142857	10	274.2857	0	1
S12’	184.5	135	1	1	9	275	0	1

Association rules describing relationships between objectives

- If profit≥140.5, then $x_B \leq 9.4696$ and sales≤320.3913

  (s1′,s2′,s3′,s4′,s6′,s7′,s8′,s9′,s10′,s11′,s12′)

- If time≤140, then $x_B \leq 10$ and sales≤300

  (s3′, s5′,s6′,s7′,s8′,s9′,s10′,s11′,s12′)

- If time≤135, then sales≤275

  (s3′, s6′,s7′,s8′,s9′,s10′,s11′,s12′)

- If $x_C \leq 1.4522$, then $x_B \leq 9.4696$ and sales≤320.391

  (s1′,s2′,s3′,s4′,s6′,s7′,s8′,s9′,s10′,s11′,s12′)

- If time≤135, then profit≤192.7143

  (s3′, s6′,s7′,s8′,s9′,s10′,s11′,s12′)

- If profit≥150, then $x_A \leq 2$ and $x_C \leq 3.1348$

  (s1′,s2′,s3′,s4′,s6′,s7′,s8′,s9′,s10′,s11′,s12′)

Association rules describing relationships between objectives

- If time$\leq$135, then $x_A \leq 2$ and $x_B \leq 3.1348$

  (s3′,s6′, s7′,s8′,s9′,s10′,s11′,s12′)

- If profit$\geq$150, then time$\geq$109.7297

  (s1′, s6′,s7′,s8′,s9′,s10′,s11′,s12′)

- If $x_C \geq 5.4261$, then $x_A \leq 2$ and $x_B \leq 3.1348$

  (s1′,s3′,s6′, s7′,s8′,s9′,s10′,s11′,s12′)

- If sales$\geq$272.9167, then time$\geq$134.8959

  (s1′,s2′, s4′,s5′,s6′,s11′,s12′)

- If profit$\geq$150 and sales$\geq$254.0541, then time$\geq$127.9459 and $x_A \leq 2$

  (s1′, s6′,s8′,s9′,s10′,s11′,s12′)

- If sales$\geq$230.2703, then time$\geq$109.7297

  (s1′,s2′,s4′,s5′,s6′,s7′,s8′, s9′,s10′,s11′,s12′)

Association rules describing relationships between objectives

- If sales≥230.2703, then time≥109.7297

  (s1′,s2′,s4′,s5′,s6′,s7′,s8′,s9′,s10′,s11′,s12′)

- If sales≥254.0541, then time≥127.9459

  (s1′,s2′,s4′,s5′,s6′,s8′,s9′,s10′,s11′,s12′)

Association rules describing relationships between decision variables

- Describe the Pareto-optimal set in terms of decision variables:
- If $x_A \geq 2$, then $x_B \leq 3.1348$

  (s4',s8',s9',s10')
- If $x_C \geq 5.4261$, then $x_A \leq 3$ and $x_B \leq 4$

  (s1',s3',s6',s7',s8',s9',s10',s11',s12')

Association rules describing relationships between decision variables and objectives

- If $x_C \leq 6.16$, then $x_B \geq 2$ and sales ≥ 254.05

  (s2′,s4′,s5′,s8′,s9′)

- If $x_C \leq 6.81$, then $x_B \geq 2$

  (s2′,s3′,s4′,s5′,s7′,s8′,s9′,s10′)

- If $x_A \geq 2$, then $x_B \geq 2$ and sales ≥ 254.05

  (s4′,s8′,s9′,s10′)

- If $x_C \geq 6.81$, then profit ≥ 150 and sales ≥ 230.27

  (s1′,s6′,s7′,s11′,s12′)

- If $x_B \geq 2$ and $x_C \geq 6.75$, then time ≤ 135

  (s6′,s7′,s10′,s11′,s12′)

- If $x_B \geq 3.15$, then sales ≥ 269.7

  (s2′,s5′,s9′)

Association rules describing relationships between decision variables and objectives

- If $x_C \geq 10$, then profit ≥ 191.875 and sales ≥ 279.92
  (s1′,s6′,s11′)
- If $x_A \geq 1$ and $x_C \geq 9$, then profit ≥ 184.5 and time ≤ 135
  (s6′,s11′,s12′)
- If $x_A \geq 2$ and $x_C \geq 5.43$, then profit ≥ 150 and time ≤ 135
  (s8′,s9′,s10′)
- If $x_A \leq 1$ and $x_C \geq 9$, then profit ≥ 184.5
  (s1′,s11′,s12′)

Sorting of representative efficient solutions (second iteration)

Efficient solution	Profit	Total time	Prod. x_A	Prod. x_B	Prod. x_C	Sales	Class
S1′	210.7143	150	0	2.142857	10	314.2857	*
S2′	140.5	150	0	9.469565	1.452174	320.3913	**Bad**
S3′	140.5	103.5676	0	2	6.297297	217.4324	**Bad**
S4′	140.5	150	5.097923	2	4.643917	278.0564	*
S5′	120	140	0	10	0	300	**Bad**
S6′	191.875	134.8959	1.145835	0	10	272.9167	*
S7′	150	109.7297	0	2	6.810811	230.2703	*
S8′	150	127.9459	2	2	6.162162	254.0541	*
S9′	150	135	2	3.134783	5.426087	269.6957	*
S10′	160.875	135	2	2	6.75	268.75	**Good**
S11′	192,7143	135	1	0.142857	10	274.2857	*
S12′	184.5	135	1	1	9	275	**Good**

DRSA decision rules describing «good» solutions

- If profit≥160.875 and $x_A \geq 2$,
  then product mix is good (s10')

- If profit≥160.875 and $x_B \geq 2$,
  then product mix is good (s10')

- If profit≥184.5 and time≤135 and $x_B \geq 1$,
  then product mix is good (s12')

- If profit≥184.5 and $x_A \geq 1$ and $x_B \geq 1$,
  then product mix is good (s12')

- If $x_A \geq 2$ and $x_C \geq 5.75$,
  then product mix is good (s10')

- If time≤135 and $x_B \geq 1$ and $x_C \geq 9$,
  then product mix is good (s12')

DRSA decision rules describing «good» solutions

- If $x_A \geq 1$ and $x_B \geq 1$ and $x_C \geq 9$,
  then product mix is good (s12′)
- If time≤135 and sales≥275,
  then product mix is good (s12′)
- If profit≥184.5 and $x_A \geq 1$ and sales≥275,
  then product mix is good (s12′)
- If $x_A \geq 1$ and $x_C \geq 9$ and sales≥275,
  then product mix is good (s12′)
- If time≤135 and $x_B \geq 2$ and $x_C \geq 6.75$ and sales≥275,
  then product mix is good (s10′)

DRSA decision rules describing «bad» solutions

- If profit≤120,
  then product mix is bad (s5′)
- If profit≤140.5 and $x_A \leq 0$,
  then product mix is bad (s2′,s3′,s5′)
- If $x_C \leq 1.452174$,
  then product mix is bad (s2′,s5′)
- If $x_A \leq 0$ and $x_C \leq 6.297$,
  then product mix is bad (s2′,s3′,s5′)
- If sales≤217.43245,
  then product mix is bad (s3′)
- If time≥140 and $x_A \leq 0$ and sales≤300,
  then product mix is bad (s5′)

Selected decision rules

- DM selected **four** rules as the most adequate to preferences:

r3: If profit $\geq$ 184.5 and time $\leq$ 135 and $x_B \geq 1$,
then product mix is good (s12′)

r4: If time $\leq$ 135 and $x_B \geq 2$ and $x_C \geq 6.75$ and sales $\geq$ 275,
then product mix is good (s10′)

r5: If profit $\leq$ 140.5 and $x_A \leq 0$,
then product mix is bad (s2′,s3′,s5′)

r6: If time $\geq$ 140 and $x_A \leq 0$ and sales $\leq$ 300,
then product mix is bad (s5′)

Added constraints for decision rule r3

- Decision rule r3:

  If profit $\geq$ 184.5 and time $\leq$ 135 and $x_B \geq 1$,
  then product mix is good (s12')

- Added constraints to the production mix problem:
- $20x_A + 30x_B + 25x_C - (1x_A + 2x_B + 0.75x_C)6 +$
  $- (0.5x_A + x_B + 0.5\,x_C)8 + (1-\delta_3)M \geq 184.5$ [profit $\geq$ 184.5]
- $5x_A + 8x_B + 10x_C + 8x_A + 6x_B + 2x_C - (1-\delta_3)M \leq 135$
  [time $\leq$ 135]
- $x_B + (1-\delta_3)M \geq 1$ [produced quantity of B $\geq$ 1]

where $\delta_3 \in \{0,1\}$, M big number (10^6); if $\delta_3=1$, then r3 is satisfied

Added constraints for decision rule r4

- Decision rule r4:

  If time $\leq$ 135 and $x_B \geq 2$ and $x_C \geq 6.75$ and sales $\geq$ 275,
  then product mix is good (s10′)

- Added constraints to the production mix problem:
- $5x_A + 8x_B + 10x_C + 8x_A + 6x_B + 2x_C - (1-\delta_4)M \leq 135$ [time $\leq$ 135]
- $x_B + (1-\delta_4)M \geq 2$ [produced quantity of B $\geq$ 2]
- $x_C + (1-\delta_4)M \geq 6.75$ [produced quantity of C $\geq$ 6.75]
- $20x_A + 30x_B + 25x_C + (1-\delta_4)M \geq 275$ [sales $\geq$ 275]

where $\delta_4 \in \{0,1\}$, M big number (10^6); if $\delta_4=1$, then r4 is satisfied

Condition for filtering good solutions

- A solution is good if it satisfies at least one of decision rules r3 & r4:

$$\boxed{\delta_3 + \delta_4 \geq 1}$$

Added constraints for decision rule r5

- Decision rule r5:

  If profit $\leq$ 140.5 and $x_A \leq 0$,
  then product mix is bad (s2′,s3′,s5′)

- Added constraints to the production mix problem:
- $20x_A + 30x_B + 25x_C - (1x_A + 2x_B + 0.75x_C)6 +$
  $-(0.5x_A + x_B + 0.5\,x_C)8 + (1-\delta_{51})M \geq 140.5+\boldsymbol{\varepsilon}$ [profit > 140.5]
- $x_A + (1-\delta_{51})M \geq \boldsymbol{\varepsilon}$ [produced quantity of A > 0]
- $\delta_{51}+\delta_{52} \geq 1$

where $\delta_{51},\delta_{52}\in\{0,1\}$, M big number (10^6), $\boldsymbol{\varepsilon}$ small positive (10^{-3})

- A solution is <u>not bad</u> if at least one condition of r5 does not hold

Added constraints for decision rule r6

- Decision rule r6:

  If time $\geq$ 140 and $x_A \leq 0$ and sales $\leq$ 300,
  then product mix is bad (s5')

- Added constraints to the production mix problem:
- $5x_A + 8x_B + 10x_C + 8x_A + 6x_B + 2x_C - (1-\delta_{61})M \leq 140 - \varepsilon$ [time < 140]
- $x_A + (1-\delta_{62})M \geq \varepsilon$ [produced quantity of A > 0]
- $20x_A + 30x_B + 25x_C + (1-\delta_{63})M \geq 300 + \varepsilon$ [sales > 300]
- $\delta_{61}+\delta_{62}+\delta_{63} \geq 1$

where $\delta_{61},\delta_{62},\delta_{63} \in \{0,1\}$, M big number (10^6), ε small positive (10^{-3})

- A solution is not bad if at least one condition of r6 does not hold

Set of representative efficient solutions (third iteration)

Efficient solution	Profit	Total time	Prod. x_A	Prod. x_B	Prod. x_C	Sales	δ_1	δ_2	δ_3	δ_4
s1''	197.86	135.00	0.00	1.07	10.00	282.14	0	1	1	0
s2''	167.38	130.58	0.00	3.54	6.75	275.00	1	0	0	1
s3''	171.16	135.00	0.00	3.86	6.75	284.46	0	1	0	1
s4''	164.97	135.00	1.20	2.74	6.75	275.00	1	0	0	1
s5''	171.16	135.00	0.00	3.86	6.75	284.46	0	1	0	1
s6''	197.46	135.00	0.08	1.00	10.00	281.54	0	1	1	0
s7''	184.50	135.00	1.00	1.00	9.00	275.00	0	1	1	0
s8''	174.92	135.00	1.00	2.00	7.83	275.83	1	0	0	1
s9''	170.00	135.00	1.00	2.51	7.23	276.26	1	0	0	1
s10''	170.00	134.43	1.00	2.42	7.29	275.00	1	0	0	1

Selected solution (third iteration)

Efficient solution	Profit	Total time	Prod. x_A	Prod. x_B	Prod. x_C	Sales	Class
s1''	197.86	135.00	0.00	1.07	10.00	282.14	*
s2''	167.38	130.58	0.00	3.54	6.75	275.00	*
s3''	171.16	135.00	0.00	3.86	6.75	284.46	*
s4''	164.97	135.00	1.20	2.74	6.75	275.00	*
s5''	171.16	135.00	0.00	3.86	6.75	284.46	*
s6''	197.46	135.00	0.08	1.00	10.00	281.54	*
s7''	184.50	135.00	1.00	1.00	9.00	275.00	**selected**
s8''	174.92	135.00	1.00	2.00	7.83	275.83	*
s9''	170.00	135.00	1.00	2.51	7.23	276.26	*
s10''	170.00	134.43	1.00	2.42	7.29	275.00	*

Decision rules explaining the choice

- r7: If profit $\geq$ 184.5 and $x_A \geq 1$,
  then product mix is <u>selected</u> (s7'')

- r8: If $x_A \geq 1$ and $x_C \geq 9$,
  then product mix is <u>selected</u> (s7'')

Summing up …

Efficient solution	Profit	Total time	Prod. x_A	Prod. x_B	Prod. x_C	Sales	δ_1 Rule r1	δ_2 Rule r2	δ_3 Rule r3	δ_4 Rule r4
s1''	197.86	135.00	0.00	1.07	10.00	282.14	0	1	1	0
s2''	167.38	130.58	0.00	3.54	6.75	275.00	1	0	0	1
s3''	171.16	135.00	0.00	3.86	6.75	284.46	0	1	0	1
s4''	164.97	135.00	1.20	2.74	6.75	275.00	1	0	0	1
s5''	171.16	135.00	0.00	3.86	6.75	284.46	0	1	0	1
s6''	197.46	135.00	0.08	1.00	10.00	281.54	0	1	1	0
s7''	184.50	135.00	1.00	1.00	9.00	275.00	0	1	1	0
s8''	174.92	135.00	1.00	2.00	7.83	275.83	1	0	0	1
s9''	170.00	135.00	1.00	2.51	7.23	276.26	1	0	0	1
s10''	170.00	134.43	1.00	2.42	7.29	275.00	1	0	0	1

Reasons for choosing solution s7”

Efficient solution	Profit	Total time	Prod. x_A	Prod. x_B	Prod. x_C	Sales
s7''	184.50	135.00	1.00	1.00	9.00	275.00

- S7” is good because profit$\geq$140.5 & time$\leq$150 & $x_B \geq 2$ (decision rule r2)
- S7” is good because profit$\geq$184.5 & time$\leq$135 & $x_B \geq 1$ (decision rule r3)
- S7” is not bad because profit>140.5 (decision rule r5)
- S7” is not bad because $x_A > 0$ (decision rules r5 & r6)
- S7” is not bad because time<140 (decision rule r6)
- S7” is good because profit$\geq$184.5 & $x_A \geq 1$ (decision rule r7)
- S7” is good because $x_A \geq 1$ & $x_C \geq 9$ (decision rule r8)

Main features of the DRSA-IMO interactive method

- The method is based on ordinal properties of values of objective functions (the weakest possible)
- At each step, the method does not aggregate the objective functions into a single value (no scalarization is involved)
- DM learns from association rules about the shape of Pareto-optimal set
- DM gives preference information by answering easy questions in terms of sorting into good and bad, without using any technical parameters, such as weights, tradeoffs, thresholds,...
- Both association and decision rules are easily understandable and intelligible for DM („glass box”) – DM can identify solutions supporting each rule & see relationships between decision variables & objectives
- They enable argumentation, explanation and justification of the final decision as a conclusion of a decision process

Decision rule approach to interactive multiobjective optimization under risk and uncertainty

DRSA to IMO under uncertainty – portfolio selection

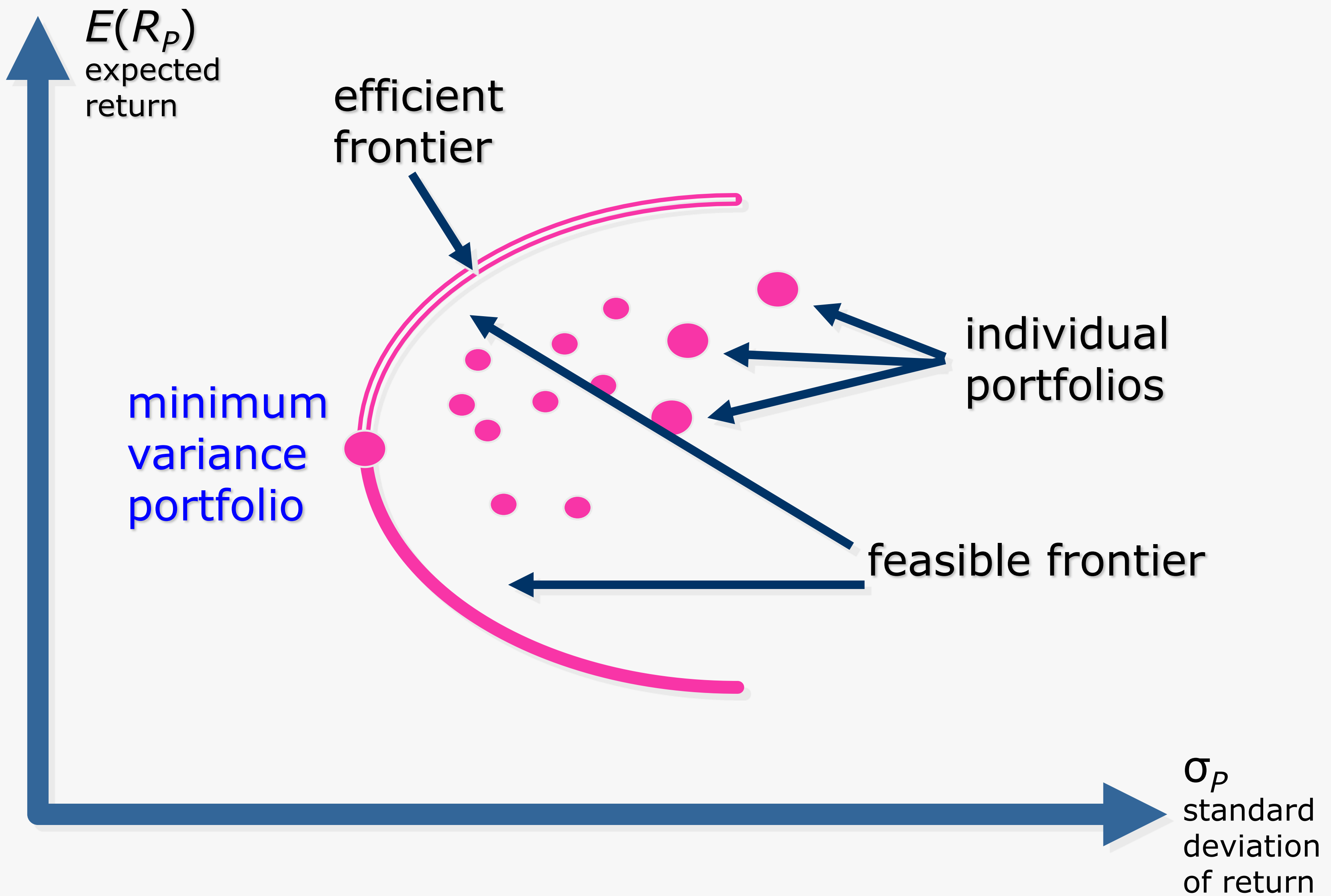
- Three securities: S_1, S_2, S_3 , with probability distributions on returns
- Expected returns of the securities:

$$R_1=12\%,\ \ R_2=14\%,\ \ R_3=16\%$$

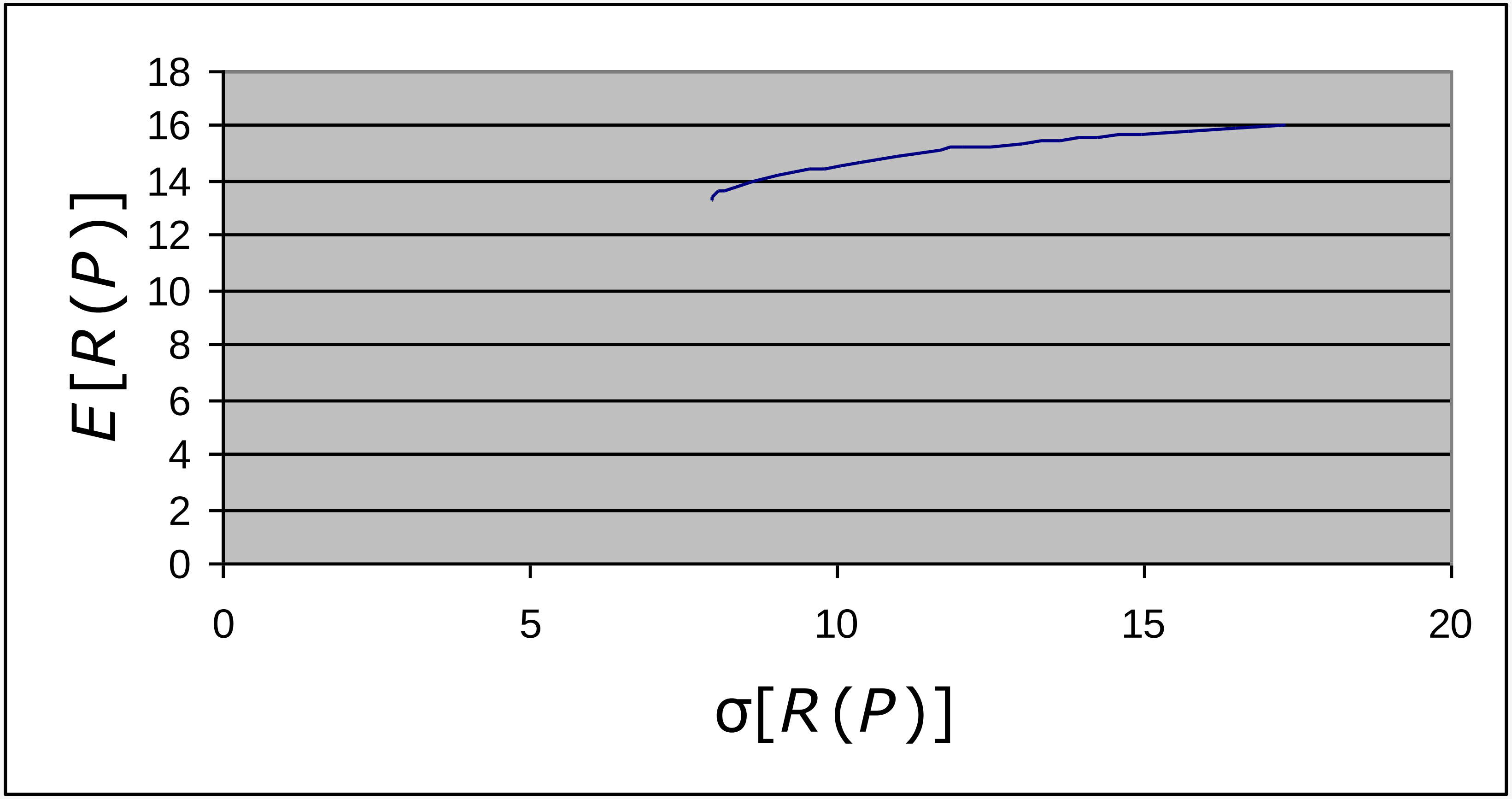
- Matrix of Variance-Covariance of return
  (yellow=variance; blue=covariance)

securities	S_1	S_2	S_3
S_1	100	50	-20
S_2	50	200	10
S_3	-20	10	300

The efficient frontier of risky portfolios



Efficient frontier of risky portfolios



The trap of standard deviation as a risk measure

- Consider the coin tossing (heads or tails) game

	heads	tails	mean	std dev
lottery 1	100 €	100 €	100 €	0
lottery 2	200 €	100 €	150 €	50

- If the utility function used for evaluating the lotteries:

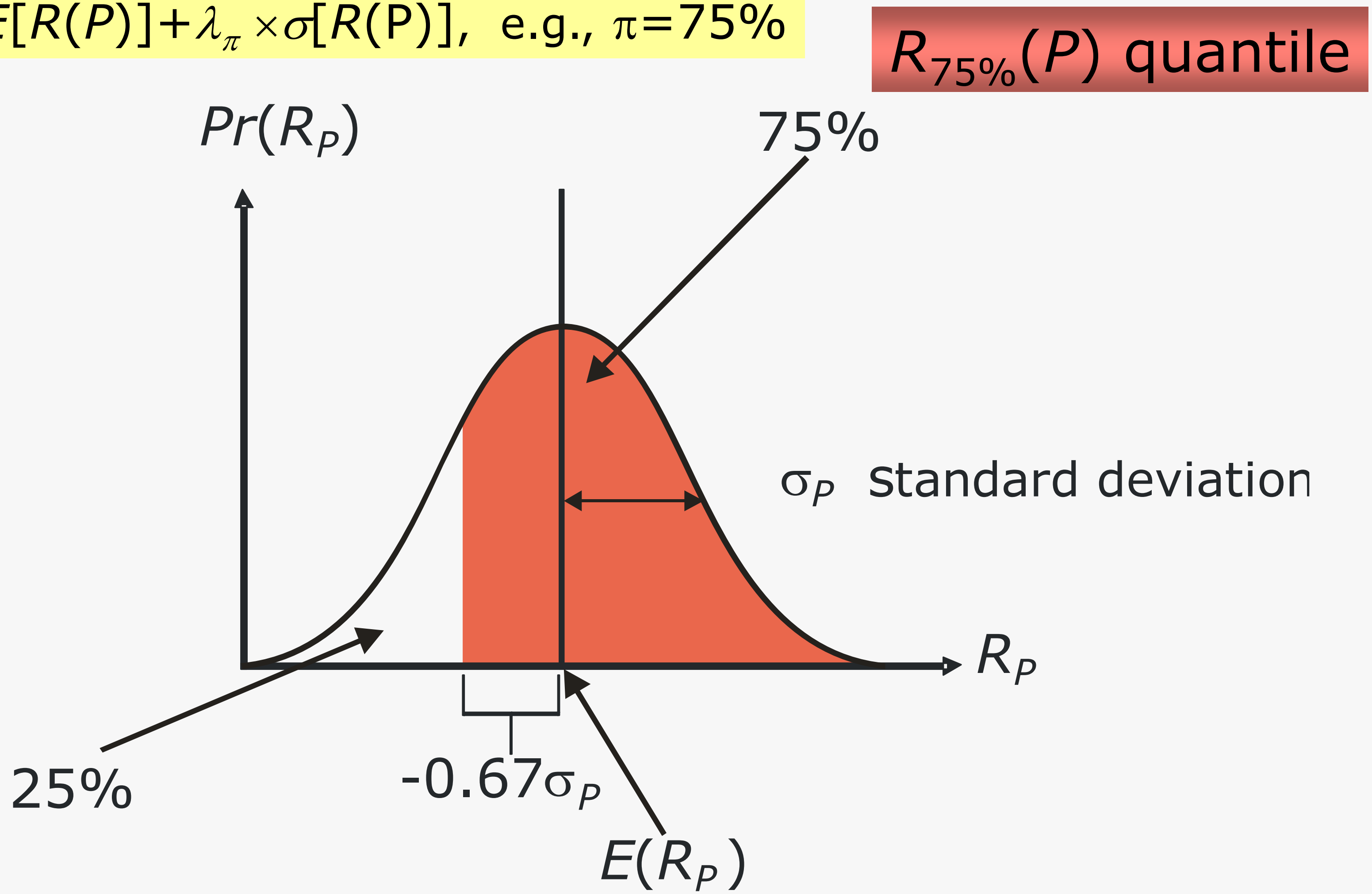
  U(lottery) = mean − λ × std dev, and, e.g., λ=2, then

  U(lottery 1) = 100−0=100, U(lottery 2) = 150−100=50, thus

  paradoxically, lottery 1 $\succ$ lottery 2

Distribution of return for portfolio P

$R_{\pi}(P)=E[R(P)]+\lambda_{\pi}\times\sigma[R(\mathrm{P})]$, e.g., $\pi=75\%$



In 75% of best cases, portfolio P will give return at least $E(Rp)-0.67\sigma_p$
or in 25% of worst cases, portfolio P will give return at most $E(Rp)-0.67\sigma_p$

Set of representative efficient solutions (first iteration)

Portfolio	w_1	w_2	w_3	$E[R(P)]$	$\sigma[R(P)]$	$R_{1\%}(P)$	$R_{25\%}(P)$	$R_{50\%}(P)$	$R_{75\%}(P)$	$R_{99\%}(P)$	Class
P1	0.39	0.29	0.32	13.86	8.43	33.50	19.51	13.86	8.21	-5.78	*
P2	0.21	0.22	0.57	14.71	10.64	39.49	21.84	14.71	7.58	-10.07	*
P3	0.01	0.48	0.51	15.01	11.39	41.55	22.64	15.01	7.37	-11.54	*
P4	0.61	0.04	0.35	13.50	8.30	32.82	19.05	13.50	7.94	-5.83	*
P5	0.43	0.39	0.18	13.52	8.58	33.51	19.27	13.52	7.77	-6.48	**Good**
P6	0.51	0.46	0.03	13.04	9.58	35.37	19.46	13.04	6.62	-9.29	*
P7	0.52	0.20	0.29	13.54	8.03	32.24	18.92	13.54	8.16	-5.16	**Good**
P8	0.54	0.04	0.42	13.75	8.70	34.03	19.58	13.75	7.92	-6.53	**Good**
P9	0.34	0.21	0.45	14.22	9.16	35.57	20.36	14.22	8.08	-7.13	*
P10	0.54	0.22	0.23	13.38	7.99	32.01	18.74	13.38	8.03	-5.24	**Good**
P11	0.60	0.15	0.25	13.28	7.94	31.78	18.60	13.28	7.97	-5.21	*
P12	0.53	0.19	0.28	13.5	8.00	32.14	18.86	13.5	8.14	-5.14	**Good**
P13	0.37	0.26	0.37	14	8.62	34.09	19.78	14	8.224	-6.09	**Good**
P14	0.21	0.34	0.46	14.5	9.79	37.30	21.06	14.5	7.94	-8.30	**Good**
P15	0.04	0.41	0.54	15	11.33	41.39	22.59	15	7.41	-11.39	*
P16	0	0.25	0.75	15.5	13.60	47.19	24.61	15.5	6.39	-16.19	*
P17	0	0	1	16	17.32	56.36	27.60	16	4.40	-24.36	**Good**

Induction of DRSA decision rules wrt stochastic dominance

- 19 rules were induced with the following frequency of the presence of objectives in the premise:
- $R_{1\%}(P)$: 6/19
- $R_{25\%}(P)$: 5/19
- $R_{50\%}(P)$: 5/19
- $R_{75\%}(P)$: 5/19
- $R_{99\%}(P)$: 12/19

The most interesting DRSA decision rules

- If $R_{1\%}(P) \geq 32.01\%$ and $R_{99\%}(P) \geq -5.24\%$, then portfolio is good (P7, P10, P12)
- If $R_{25\%}(P) \geq 18.74\%$ and $R_{99\%}(P) \geq -5.24\%$, then portfolio is good (P7, P10, P12)
- If $R_{50\%}(P) \geq 13.38\%$ and $R_{99\%}(P) \geq -5.24\%$, then portfolio is good (P7, P10, P12)
- If $R_{75\%}(P) \geq 8.03\%$ and $R_{99\%}(P) \geq -5.24\%$, then portfolio is good (P7, P10, P12)
- If $R_{1\%}(P) \geq 33.51\%$ and $R_{99\%}(P) \geq -6.48\%$, then portfolio is good (P5, P13)
- If $R_{1\%}(P) \geq 34.03\%$ and $R_{99\%}(P) \geq -6.53\%$, then portfolio is good (P8, P13)
- If $R_{50\%}(P) \geq 16\%$, then portfolio is good (P17)
- If $R_{50\%}(P) \geq 14.5\%$ and $R_{99\%}(P) \geq -8.3\%$, then portfolio is good (P14)

Selected decision rule and corresponding added constraints

- The DM selected the following rule as the most adequate to his preferences:

  If $R_{75\%}(P) \geq 8.03\%$ and $R_{99\%}(P) \geq -5.24\%$,
  then portfolio is good (P7, P10, P12)

- Added constraints to the portfolio selection problem:
  - $R_{75\%}(P) = E[R(P)] - 0.67 \times \sigma\,[R(P)] \geq 8.03\%$
  - $R_{99\%}(P) = E[R(P)] - 2.33 \times \sigma\,[R(P)] \geq -5.24\%$

Set of representative efficient solutions (second iteration)

Portfolio	w_1	w_2	w_3	$E[R(P)]$	$\sigma[R(P)]$	$R_{1\%}(P)$	$R_{25\%}(P)$	$R_{50\%}(P)$	$R_{75\%}(P)$	$R_{99\%}(P)$	Class
P1′	0.52	0.20	0.29	13.86	8.03	32.24	18.92	13.54	8.16	-5.16	*
P2′	0.54	0.19	0.27	14.71	7.98	32.04	18.80	13.45	8.11	-5.13	**Good**
P3′	0.54	0.20	0.26	15.01	7.98	32.05	18.80	13.45	8.10	-5.15	*
P4′	0.50	0.23	0.27	13.50	8.05	32.29	18.93	13.53	8.14	-5.22	**Good**
P5′	0.53	0.18	0.29	13.52	8.02	32.20	18.89	13.52	8.15	-5.16	**Good**
P6′	0.57	0.16	0.27	13.04	7.96	31.93	18.72	13.39	8.06	-5.14	**Good**
P7′	0.54	0.16	0.30	13.54	8.02	32.20	18.89	13.51	8.14	-5.18	*
P8′	0.52	0.21	0.27	13.75	8.01	32.14	18.85	13.49	8.12	-5.17	*
P9′	0.59	0.12	0.29	14.22	7.99	32.00	18.74	13.39	8.04	-5.22	*
P10′	0.59	0.12	0.30	13.38	8.00	32.06	18.78	13.42	8.05	-5.23	*
P11′	0.58	0.16	0.26	13.35	7.94	31.86	18.67	13.35	8.03	-5.16	*
P12′	0.49	0.20	0.30	13.62	8.10	32.49	19.05	13.62	8.20	-5.24	**Good**
P13′	0.57	0.17	0.27	13.40	7.96	31.94	18.73	13.4	8.07	-5.14	*
P14′	0.55	0.18	0.27	13.45	7.97	32.03	18.79	13.45	8.11	-5.13	**Good**
P15′	0.53	0.18	0.28	13.50	8.00	32.14	18.86	13.5	8.14	-5.14	*
P16′	0.50	0.20	0.30	13.60	8.07	32.41	19.01	13.6	8.19	-5.21	**Good**

Induction of DRSA decision rules wrt stochastic dominance

- 5 rules were induced with the following frequency of the presence of objectives in the premise:
- $R_{1\%}(P)$: 1/5
- $R_{25\%}(P)$: 1/5
- $R_{50\%}(P)$: 1/5
- $R_{75\%}(P)$: 1/5
- $R_{99\%}(P)$: 1/5

The most interesting DRSA decision rules

- If $R_{1\%}(P) \geq 32.29\%$,
  then portfolio is good (P4', P12', P16')
- If $R_{25\%}(P) \geq 18.93\%$,
  then portfolio is good (P4', P12', P16')
- If $R_{50\%}(P) \geq 13.6\%$,
  then portfolio is good (P12', P16')
- If $R_{75\%}(P) \geq 8.19\%$,
  then portfolio is good (P12', P16')
- If $R_{99\%}(P) \geq -5.13\%$,
  then portfolio is good (P2', P14')

Selected decision rule and corresponding added constraints

- The DM selected the following rule as the most adequate to his preferences:

  If $R_{25\%}(P) \geq 18.93\%$,
  then portfolio is good (P4', P12', P16')

- Added constraint to the portfolio selection problem:
  - $R_{25\%}(P) = E[R(P)] + 0.67 \times \sigma\,[R(P)] \geq 18.93\%$

Set of representative efficient solutions (third iteration)

Portfolio	w_1	w_2	w_3	$E[R(P)]$	$\sigma[R(P)]$	$R_{1\%}(P)$	$R_{25\%}(P)$	$R_{50\%}(P)$	$R_{75\%}(P)$	$R_{99\%}(P)$	Class
P1''	0.50	0.20	0.30	13.59	8.07	32.38	18.99	13.59	8.18	-5.20	*
P2''	0.49	0.20	0.30	13.62	8.09	32.48	19.04	13.62	8.20	-5.24	*
P3''	0.50	0.19	0.31	13.62	8.09	32.47	19.04	13.62	8.20	-5.23	*
P4''	0.51	0.20	0.29	13.55	8.03	32.27	18.93	13.55	8.17	-5.17	*
P5''	0.50	0.22	0.28	13.55	8.05	32.31	18.95	13.55	8.16	-5.20	*
P6''	0.50	0.21	0.28	13.55	8.04	32.29	18.94	13.55	8.16	-5.19	*
P7''	0.52	0.17	0.30	13.56	8.04	32.30	18.95	13.56	8.17	-5.19	*
P8''	0.50	0.21	0.29	13.59	8.07	32.38	18.99	13.59	8.18	-5.21	*
P9''	0.49	0.23	0.28	13.58	8.07	32.39	18.99	13.58	8.17	-5.23	*
P10'	0.50	0.20	0.30	13.56	8.05	32.33	18.96	13.56	8.16	-5.21	*
P11''	0.52	0.19	0.29	13.55	8.03	32.26	18.93	13.55	8.17	-5.17	*
P12''	0.49	0.20	0.30	13.62	8.10	32.49	19.05	13.62	8.20	-5.24	**Best**
P13''	0.51	0.20	0.29	13.57	8.05	32.33	18.96	13.57	8.18	-5.19	*
P14''	0.5	0.2	0.3	13.60	8.07	32.41	19.01	13.60	8.19	-5.21	*

Application of monotonic rules to non-ordinal classification

DRSA for classification with unknown monotonicity constraints

- Attributes with unknown monotonic relationship w.r.t. decision
  1. Ordinal (number-coded) attributes
     - qualitative (small (1), medium (2), ..., large (k): e.g., size)
     - quantitative (numerical: e.g., temperature)

     Each ordinal attribute a_i is replaced by 2 criteria:

     gain-type criterion q'_i and cost-type criterion q''_i

  2. Nominal (not ordered) attributes (blue, red, ..., white: e.g., color)

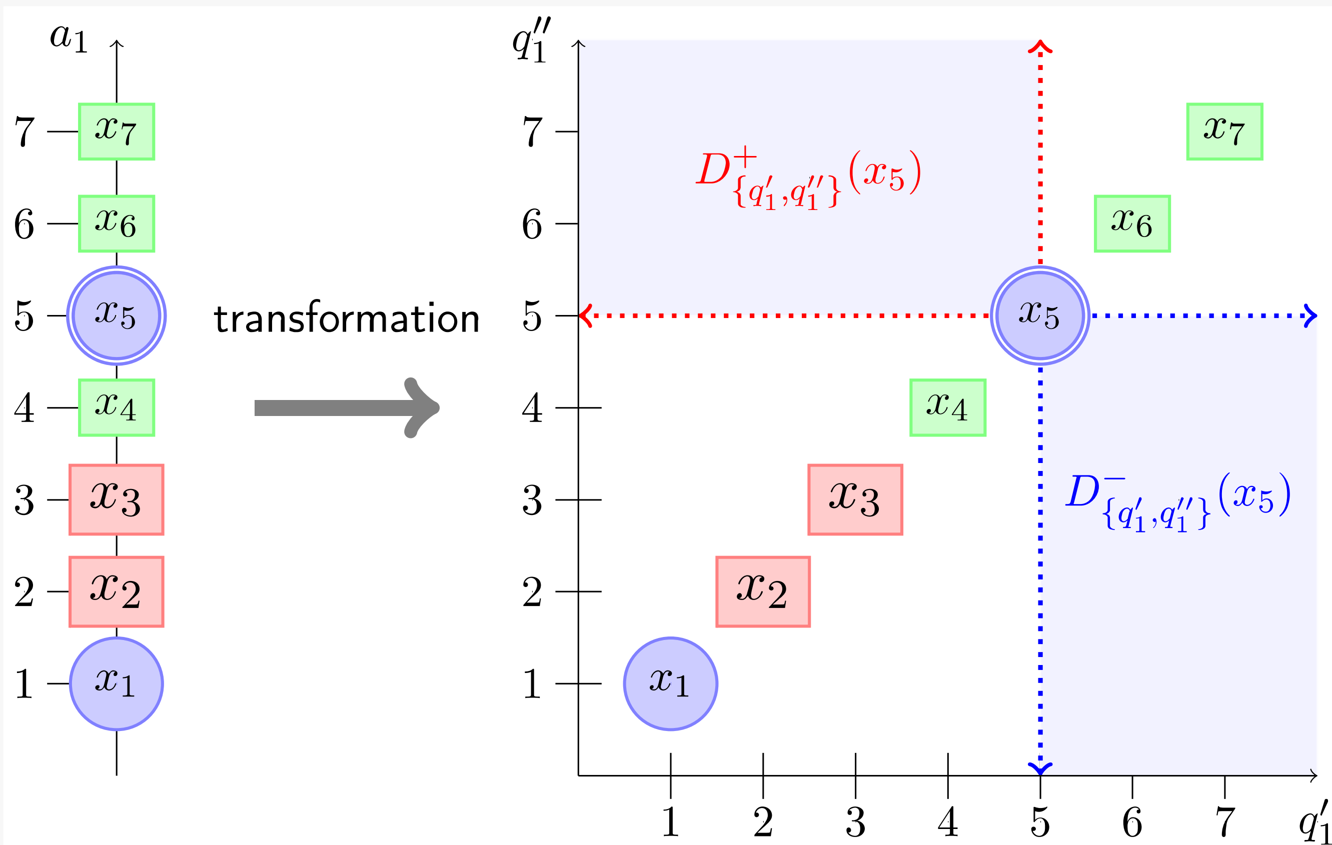
     Each nominal attribute a_i (taking 1 of k values, k>2)

     is replaced by $2\times k$ binary criteria: for each $h\in\{1,\dots,k\}$,

     gain-type 0-1 criterion $q'_i(h)$ and cost-type 0-1 criterion $q''_i(h)$

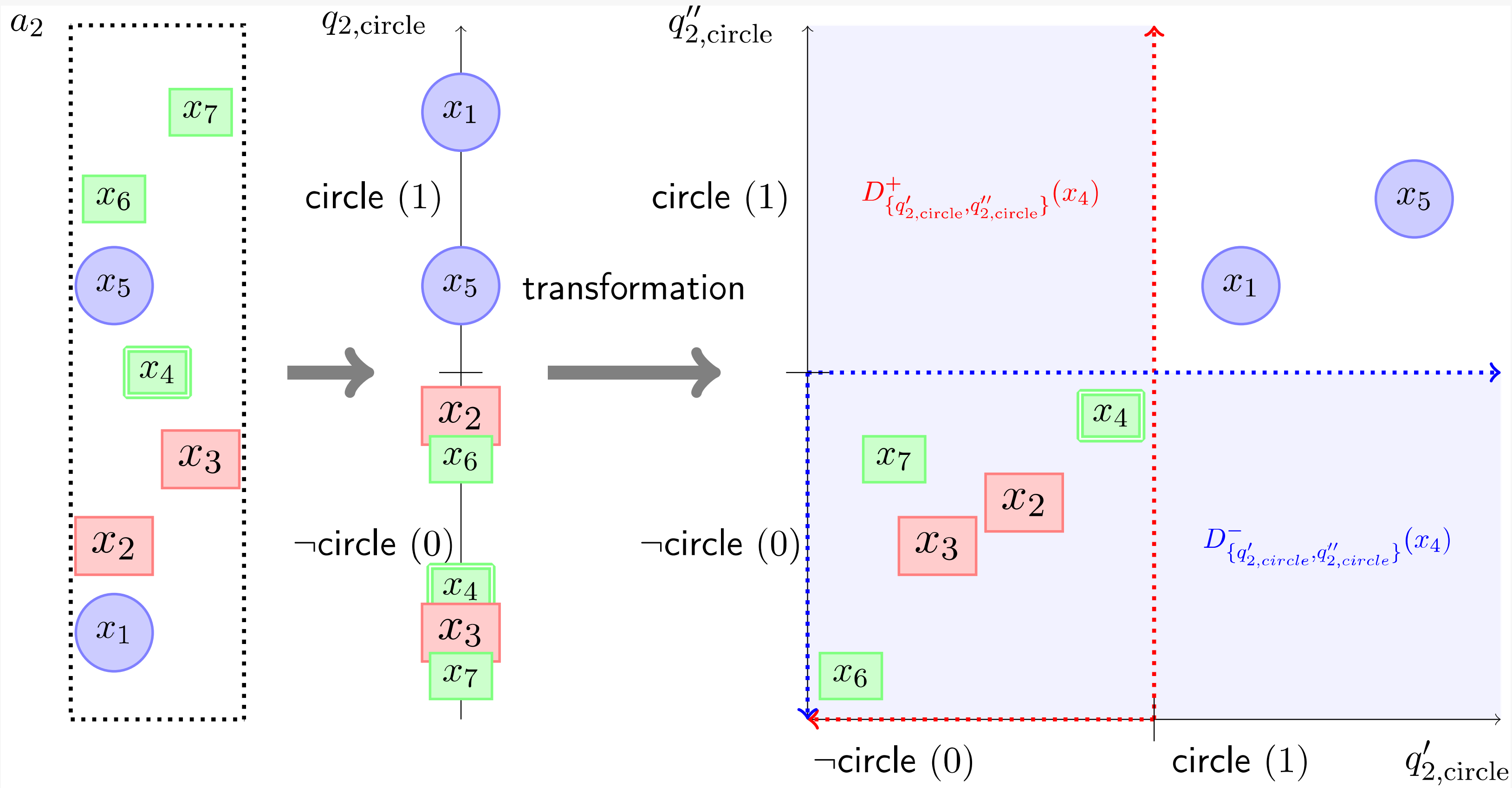
DRSA for classification with unknown monotonicity constraints

- Each ordinal (number-coded) attribute a_i is replaced by gain-type criterion q'_i and cost-type criterion q''_i
- Indiscernibility granules wrt a_1 $\Rightarrow$ dominance cones wrt $\{q'_1, q''_1\}$:



DRSA for classification with unknown monotonicity constraints

- Each one of k>2 values of a nominal attribute a_i is replaced by 0-1 gain-type criterion q'_i and 0-1 cost-type criterion q''_i
- Indiscernibility granules wrt a_1 ⇒ dominance cones wrt $\{q'_1, q''_1\}$:



DRSA for classification with unknown monotonicity constraints

- Decision attribute d makes partition of U into a finite number of non-ordered decision classes $\boldsymbol{Cl}=\{Cl_t,\ t=1,\dots,m\}$
- Using DRSA, one approximates:
  - in case of $m=2$ (binary classification): Cl_1 and $\neg Cl_1=Cl_2$
  - in case of $m>2$: Cl_t and $\neg Cl_t$, for each $t\in\{1,\dots,k\}$, i.e.

$$Cl_1,\ Cl_2,\ Cl_3,\dots,\ Cl_{t-1},\ Cl_t,\ Cl_{t+1},\dots,\ Cl_{m-1},\ Cl_m$$

$$\neg Cl_t$$

$$\boxed{\neg Cl_t \prec Cl_t}$$

Induction of monotonic decision rules for non-ordinal classification

- Induction of monotonic decision rules from rough approximations:
  - *positive decision rules*, supported by objects $\in \underline{P}^{\varepsilon_{Cl_t}}(Cl_t)$

    if $x_{q'1} \succeq_{q'1} r_{q'1}$ *and* $x_{q''1} \preceq_{q''1} r_{q''1}$ *and* $x_{q'2} \succeq_{q'2} r_{q'2}$ *and* $x_{q''2} \preceq_{q''2} r_{q''2}$ *and* … $x_{q'p} \succeq_{q'p} r_{q'p}$ *and* $x_{q''p} \preceq_{q''p} r_{q''p}$, *then* $x \in Cl_t$

  - *negative decision rules*, supported by objects $\in \underline{P}^{\varepsilon_{\neg Cl_t}}(\neg Cl_t)$

    if $x_{q'1} \preceq_{q'1} r_{q'1}$ *and* $x_{q''1} \succeq_{q''1} r_{q''1}$ *and* $x_{q'2} \preceq_{q'2} r_{q'2}$ *and* $x_{q''2} \succeq_{q''2} r_{q''2}$ *and* … $x_{q'p} \preceq_{q'p} r_{q'p}$ *and* $x_{q''p} \succeq_{q''p} r_{q''p}$, *then* $x \in \neg Cl_t$

- Consistency of induced monotonic decision rules is controlled by consistency measure ε

Example of application of DRSA to non-ordinal data

Set of patients after radical prostatectomy.

Id	Age	Gleason	PSA	Volume	Recurrence
1	60	10	2.0	large	other
2	20	7	1.2	large	local
3	40	4	0.1	medium	local
4	45	2	0.8	medium	no
5	50	3	0.3	small	local
6	50	3	0.3	small	no
7	40	7	0.5	small	no
8	25	5	0.4	small	no
9	25	2	0.5	small	no
10	40	4	0.5	small	no

Example of application of DRSA to non-ordinal data

Transformed set of patients after radical prostatectomy—binary classification into “no” and “¬ no”.

Id	Age′ ↑	Age″ ↓	Gleason′ ↑	Gleason″ ↓	PSA′ ↑	PSA″ ↓	V-s′ ↑	V-s″ ↓	V-m′ ↑	V-m″ ↓	V-l′ ↑	V-l″ ↓	R-no ↑
1	60	60	10	10	2.0	2.0	0	0	0	0	1	1	0
2	20	20	7	7	1.2	1.2	0	0	0	0	1	1	0
3	40	40	4	4	0.2	0.2	0	0	1	1	0	0	0
4	45	45	2	2	0.8	0.8	0	0	1	1	0	0	1
5	50	50	3	3	0.3	0.3	1	1	0	0	0	0	0
6	50	50	3	3	0.3	0.3	1	1	0	0	0	0	1
7	40	40	7	7	0.6	0.6	1	1	0	0	0	0	1
8	25	25	5	5	0.4	0.4	1	1	0	0	0	0	1
9	25	25	2	2	0.5	0.5	1	1	0	0	0	0	1
10	40	40	4	4	0.5	0.5	1	1	0	0	0	0	1

Example of application of DRSA to non-ordinal data

- Two decision rules are sufficient to cover all consistent objects from the table with binary classification „no” and „¬no” for recurrence

$$1: \ \textit{if} \ \text{Gleason}'' \geq 4 \ \textit{and} \ \text{V-s}' \leq 0, \ \textit{then} \ \text{R-no} \leq 0,$$

$$2: \ \textit{if} \ \text{PSA}' \geq 0.4 \ \textit{and} \ \text{PSA}'' \leq 0.8, \ \textit{then} \ \text{R-no} \geq 1.$$

- Elementary condition V-s’ ≤ 0 from the rule 1) is be read as: „Volume is not small”. After returning to original scales:

$$\begin{aligned} 1: \ & \textit{if} \ \text{Gleason} \geq 4 \ \textit{and} \ \text{Volume} \in \{\text{medium, large}\}, \\ & \textit{then} \ \text{Recurrence is} \ \neg \textit{no}, \end{aligned}$$

$$2: \ \textit{if} \ \text{PSA} \in [0.4, 0.8], \ \textit{then} \ \text{Recurrence is no}.$$

Example of application of DRSA to non-ordinal data

- Two decision rules are sufficient to cover all consistent objects from the table with binary classification „local” and „¬local” for recurrence

$$3:\ \textit{if}\ \text{Age}\ \geq 25\ \textit{and}\ \text{PSA}\ \geq 0.4,\ \textit{then}\ \text{Recurrence is}\ \neg\text{local},$$

$$4:\ \textit{if}\ \text{Age}\ \leq 40\ \textit{and}\ \text{Volume}\ \in \{\text{medium,large}\},\ \textit{then}\ \text{Recurrence is local}.$$

- Other two rules are sufficient to cover all consistent objects from the table with binary classification „other” and „¬other” for recurrence

$$5:\ \textit{if}\ \text{PSA}\ \leq 1.2,\ \textit{then}\ \text{Recurrence is}\neg\text{other},$$

$$6:\ \textit{if}\ \text{PSA}\ \geq 2,\ \textit{then}\ \text{Recurrence is other}.$$

Application of monotonic decision rules to non-ordinal classification

- Recommendation is based on a score coefficient that involves confidence and coverage of rules matching object x
- Let $\varphi_1 \rightarrow Cl_t$,..., $\varphi_k \rightarrow Cl_t$, be the rules matching x,

  $||\varphi_j||$ is a set of objects with property φ_j , $j=1,\ldots,k$
- For classified object x, the score is calculated for each Cl_t, $t=1,\ldots,m$

$$score(Cl_t, x) = \boxed{Pr(\varphi_1 \vee \ldots \vee \varphi_k \mid Cl_t)} \times \boxed{Pr(Cl_t \mid \varphi_1 \vee \ldots \vee \varphi_k)}$$

$$\boxed{Pr(\varphi_1 \vee \ldots \vee \varphi_k \mid Cl_t) = \frac{\left|(\|\varphi_1\| \cap Cl_t) \cup \ldots \cup (\|\varphi_k\| \cap Cl_t)\right|}{|Cl_t|} = conf(\varphi_1 \vee \ldots \vee \varphi_k \rightarrow Cl_t)}$$

$$\boxed{Pr(Cl_t \mid \varphi_1 \vee \ldots \vee \varphi_k) = \frac{\left|(\|\varphi_1\| \cap Cl_t) \cup \ldots \cup (\|\varphi_k\| \cap Cl_t)\right|}{\left|\|\varphi_1\| \cup \ldots \cup \|\varphi_k\|\right|} = cov(\varphi_1 \vee \ldots \vee \varphi_k \rightarrow Cl_t)}$$

Application of monotonic decision rules to non-ordinal classification

- $score(Cl_t) = -score(\neg Cl_t)$

Recommendation: $x \rightarrow Cl_t$

where $Cl_t = \underset{t \in \{1,\ldots,m\}}{\arg\max}\left(score\left(Cl_t, x\right)\right)$

J.Błaszczyński, S.Greco, R.Słowiński: Multi-criteria classification – a new scheme for application of dominance-based decision rules. *European J. Operational Research*, 181 (2007) 1030-1044

Example of application of DRSA to non-ordinal data

- Classification of patient (x_{11}) using the six rules

Id	Age′ ↑	Age″ ↓	Gleason′ ↑	Gleason″ ↓	PSA′ ↑	PSA″ ↓	V-s′ ↑	V-s″ ↓	V-m′ ↑	V-m″ ↓	V-l′ ↑	V-l″ ↓
11	30	30	2	2	0.6	0.6	1	1	0	0	0	0

- The patient is covered by the following rules:
  - rule 2, suggesting assignment to class "no",
  - rule 3, dissuading assignment to class "local" (i.e. suggesting assignment to "¬ local"),
  - rule 5, dissuading assignment to class "other" (i.e. suggesting assignment to "¬ other").

- The result of classification is as follows:

$$score_{r_{no}}(no, x_{11}) = \frac{5^2}{5 \times 5} = 1,$$

$$score_{r_{\neg local}}(\neg local, x_{11}) = \frac{6^2}{6 \times 6} = 1,$$

$$score_{r_{\neg other}}(\neg other, x_{11}) = \frac{9^2}{9 \times 9} = 1,$$

$$\boxed{\begin{aligned} &score(no, x_{11}) = 1, \\ &score(local, x_{11}) = -1, \\ &score(other, x_{11}) = -1. \end{aligned}} \Rightarrow \boxed{\text{No recurrence for } x_{11}}$$

Examples of Applications of DRSA

Intervention based on „at least” and „at most” rules

- The „at least” rules

  if $x_{q1} \succeq_{q1} r_{q1}$ *and* $x_{q2} \succeq_{q2} r_{q2}$ *and …* $x_{qp} \succeq_{qp} r_{qp}$, *then* $x \in Class_t^{\geq}$

  indicate **opportunities for improving** the assignment of object x to $Class_t$ or better, if it was not assigned as high, and its score on $q_1,…,q_p$ would grow to $r_{q1},…,r_{qp}$

- The „at most” rules

  if $x_{q1} \preceq_{q1} r_{q1}$ *and* $x_{q2} \preceq_{q2} r_{q2}$ *and …* $x_{qp} \preceq_{qp} r_{qp}$, *then* $x \in Class_t^{\leq}$

  indicate **threats for deteriorating** the assignment of object x to $Class_t$ or worse, if it was not assigned as low, and its score on $q_1,…,q_p$ would drop to $r_{q1},…,r_{qp}$

$$incr_{SS'}(\Psi) = \sum_{\emptyset \subset P \subseteq N} \left[cer_S(\Phi, \Psi) \times cer_{S'}\left(\neg\Psi, \bigwedge_{i \in P} \neg\Phi_i \wedge \bigwedge_{j \notin P} \Phi_j \right) \right] \times \frac{\left| \left\| \neg\Psi \right\|_{S'} \right|}{|U'|}$$

S.Greco, B.Matarazzo, N.Pappalardo, R.Słowiński: Measuring expected effects of interventions based on decision rules. *J. of Experimental & Applied Artificial Intelligence*, 17 (2005) 103-118

Intervention based on „at least” and „at most” rules - example

- Example: customer satisfaction analysis by a *Company*
- 44 questions and 3 classes of overall satisfaction: *High*, *Medium*, *Low*

Threats of deterioration of satisfaction	**Opportunities for improvement of satisfaction**
Deterioration from ***High*** or ***Medium*** to ***Low*** satisfaction	Improvement from ***Low*** to ***Medium*** or ***High*** satisfaction
Deterioration from ***High*** to ***Medium*** or ***Low*** satisfaction	Improvement from ***Low*** or ***Medium*** to ***High*** satisfaction

Factors for Consumer Channel

A: Product and product quality

- A1: Width of product range
- A2: Fulfilling Account’s shopper’s/the end consumer’s needs
- A3: Innovative product introducing
- A4: Quality of supplier's products from the consumer perspective
- A5: Price/quality ratio of the light bulbs from the consumer perspective
- A6: Packaging

B: Logistics

- B1: Quality of changes implementing process
- B2: Masterdata
- B3: Availability of ordered products
- B4: Accuracy of the delivery
- B5: The punctuality of the date of the delivery
- B6: Communication on delivery date changes
- B7: Condition of delivered products
- B8: Delivery documents

C: Customer service

- C1: Accessibility of KAM
- C2: KAM's knowledge about products
- C3: Professionalism of KAM
- C4: Speed of response to problems
- C5: Ease of order placing
- C6: Speed of order processing
- C7: Satisfaction with settlement of complaints
- C8: Timely credit/debit payments
- C9: Clarity of invoices
- C10: Accuracy of invoices
- C11: Logic of shelf display
- C12: Stock replenishment adjusted to the speed of product rotation)
- C13: Order/neatness of the shelf
- C14: Service merchandising costs

D: Marketing support

- D1: Sufficient number of promotions offered by the supplier
- D2: Contribution to Account's sales increase
- D3: Contribution to Account's strategy for the category
- D4: Availability of POS materials
- D5: Quality of POS materials
- D6: MarCom support during product introductions
- D7: Shoppers shelf perception
- D8: Optimizing the profit of the shelf
- D9: Satisfying Account's strategy
- D10: Possibility and effectiveness of training
- D11: Effectiveness of product information

F: Business relationship wih the supplier

- F1: Profit Margins
- F2: Product price / performance ratio
- F3: Focus on customer service
- F4: Focus on end consumers
- F5: Supplier's market leadership

44 factors organized into 5 groups (A-D, F)

Decision rules induced from customer data structured by DRSA

Certain **at least** rules

- If (F1 $\succeq$ 5) $\Rightarrow$ (SATISFACTION $\succeq$ HIGH)
- If (A1 $\succeq$ 4) & (E2 $\succeq$ 5) $\Rightarrow$ (SATISFACTION $\succeq$ HIGH)
- If (A3 $\succeq$ 5) & (C3 $\succeq$ 5) $\Rightarrow$ (SATISFACTION $\succeq$ HIGH)
- If (A1 $\succeq$ 5) & (C4 $\succeq$ 5) $\Rightarrow$ (SATISFACTION $\succeq$ HIGH)
- If (F1 $\succeq$ 4) $\Rightarrow$ (SATISFACTION $\succeq$ MEDIUM)
- If (A1 $\succeq$ 4) & (C3 $\succeq$ 3) $\Rightarrow$ (SATISFACTION $\succeq$ MEDIUM)

Certain **at most** rules

- If (C4 $\preceq$ 2) $\Rightarrow$ (SATISFACTION $\preceq$ LOW)
- If (F1 $\preceq$ 2) $\Rightarrow$ (SATISFACTION $\preceq$ LOW)
- If (A1 $\preceq$ 2) $\Rightarrow$ (SATISFACTION $\preceq$ MEDIUM)
- If (C1 $\preceq$ 2) $\Rightarrow$ (SATISFACTION $\preceq$ MEDIUM)
- If (B2 $\preceq$ 2) $\Rightarrow$ (SATISFACTION $\preceq$ MEDIUM)
- If (E3 $\preceq$ 3) $\Rightarrow$ (SATISFACTION $\preceq$ MEDIUM)
- If (A3 $\preceq$ 4) & (A4 $\preceq$ 4) $\Rightarrow$ (SATISFACTION $\preceq$ MEDIUM)
- If (A3 $\preceq$ 4) & (C3 $\preceq$ 4) $\Rightarrow$ (SATISFACTION $\preceq$ MEDIUM)

Intervention based on monotonic rules - example

At least rule:

If (A3 ≥ 4) & (C3 ≥ 3), then Satisfaction $\succeq$ *Medium*

$$incr_{SS'}(Medium) = 77\%$$

Opportunity: if

- **A3 ≥ 4**, and
- **C3 ≥ 3**, then

satisfaction of 77% of customers with Satisfaction = *Low* will **improve** to ***Medium*** or ***High***

Intervention based on monotonic rules - example

At most rule:

If (A2 ≤ 3) & (E4 ≤ 4) , then Satisfaction $\preceq$ *Low*

$$incr_{SS'}(Low) = 89\%$$

Threat: if

- **A2 ≤ 3**, and
- **E4 ≤ 4**, then

satisfaction of 89% of customers with Satisfaction = *High* or *Medium* will **deteriorate** to ***Low***

Intervention based on monotonic rules

- In practice, the choice of rules used for intervention can be supported by additional measures, like:
  - length of the rule – the shorter the better,
  - cost of intervention on attributes present in the rule,
  - priority of intervention on some types of attributes, like: short-term before long-term actions

Mobile ***E***mergency ***T***riage System - ***MET*** System

- MET – Mobile Emergency Triage
  - Facilitates triage disposition for presentations of acute pain (abdominal and scrotal pain, hip pain)
  - Supports triage decision with or without complete clinical information
  - Provides mobile support through handheld devices
  - http://www.mobiledss.uottawa.ca

W. Michalowski, University of Ottawa

K. Farion, Children's Hospital of Eastern Ontario

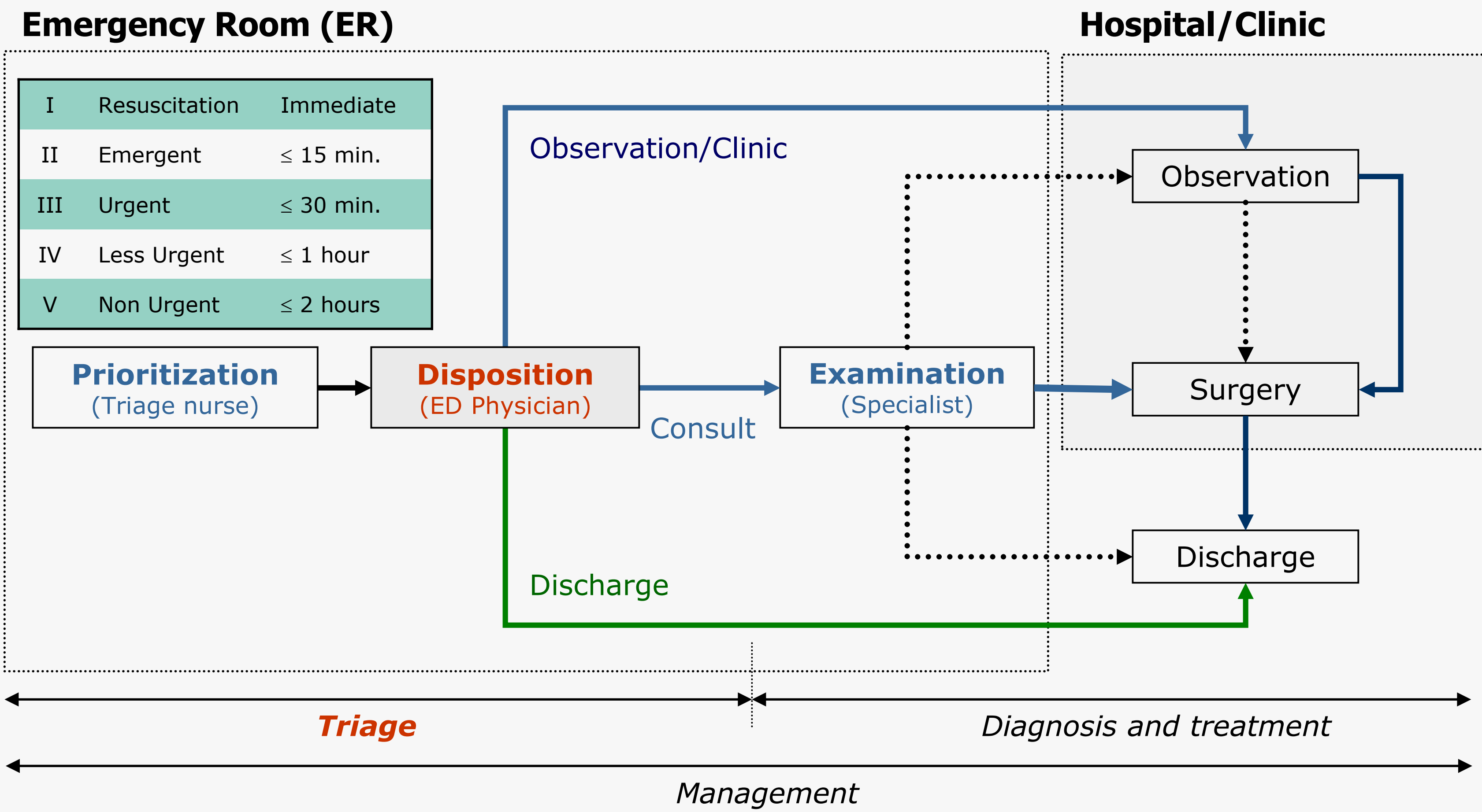
Sz. Wilk, R. Słowiński, Poznań University of Technology

Trial Location

Children's Hospital of Eastern Ontario
Centre hospitalier pour enfants de l'est de l'Ontario

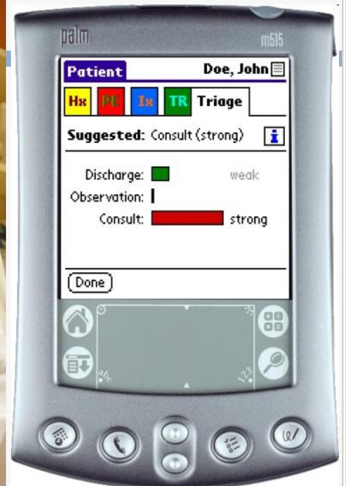
- Total pediatric population >400,000
- 55,000 patient visits in the ER per year
- 3 pediatric general surgeons (supported by emergency physicians and residents)

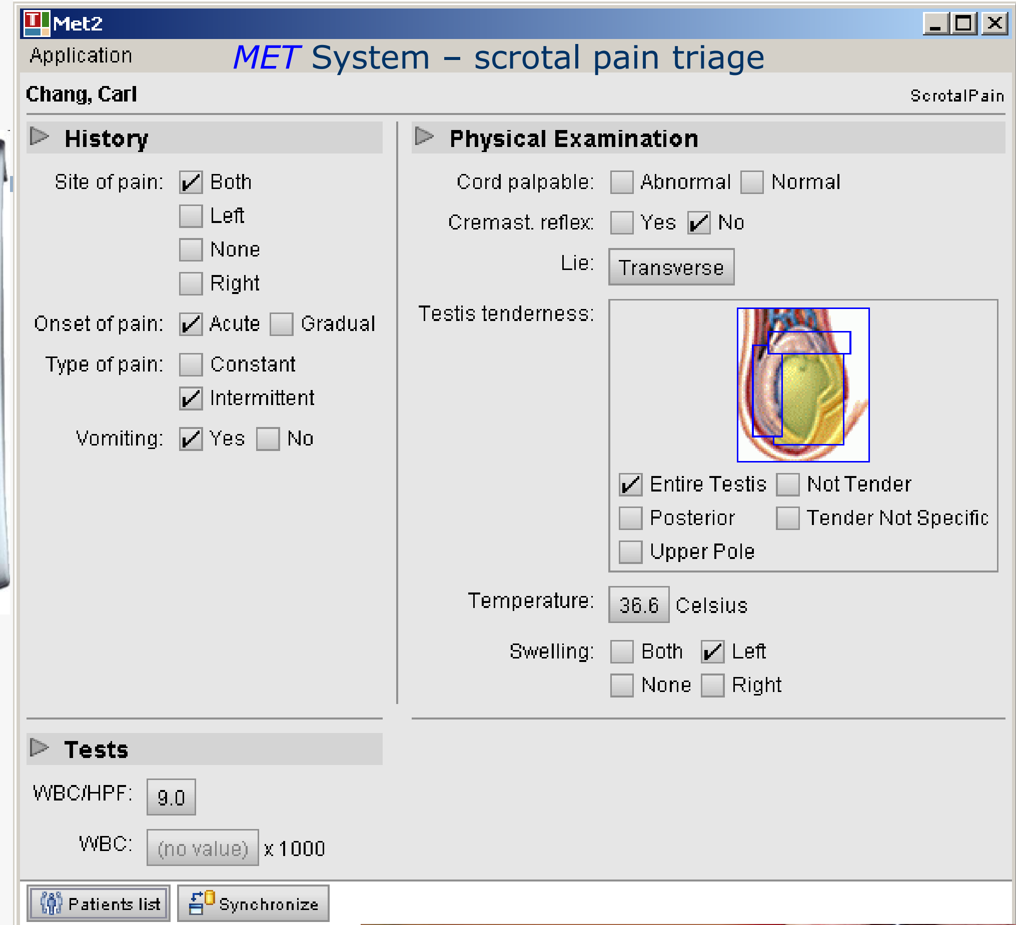
Triage Process

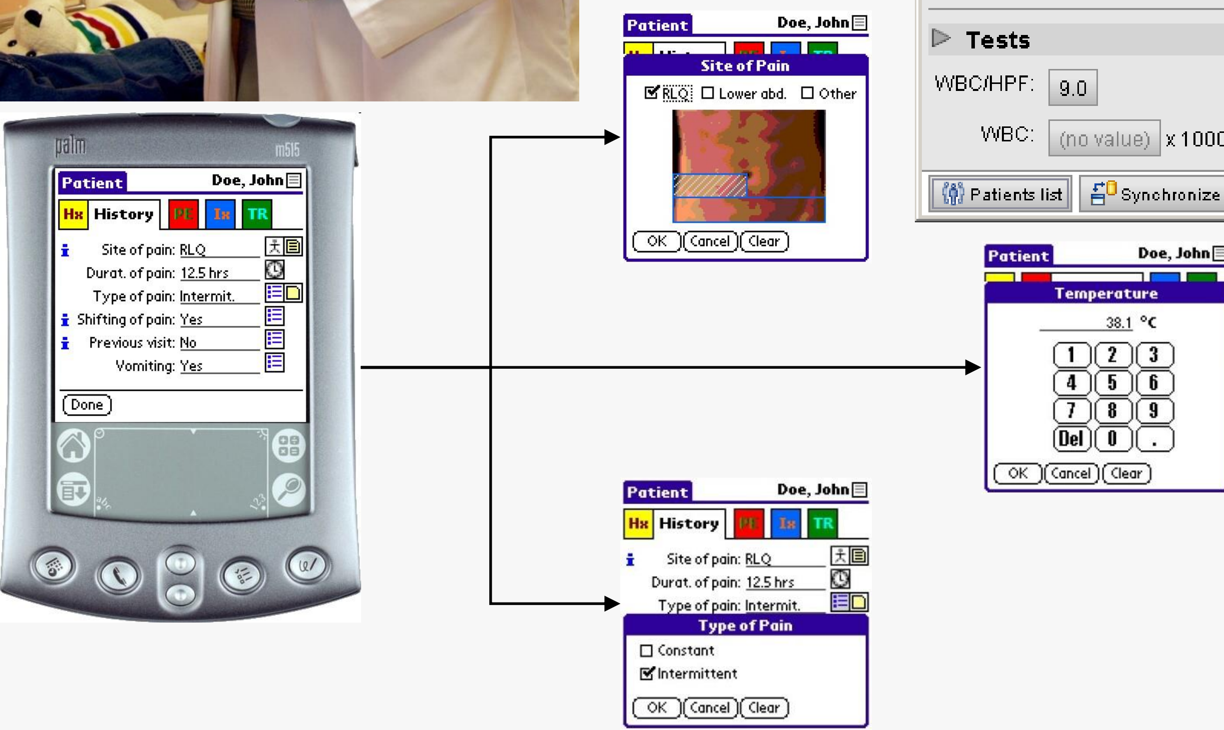


I	Resuscitation	Immediate
II	Emergent	≤ 15 min.
III	Urgent	≤ 30 min.
IV	Less Urgent	≤ 1 hour
V	Non Urgent	≤ 2 hours


CHEO
Dr. K. Farion


palm
m515
Patient
Doe, John
Hx
PE
Ix
TR
Triage
Suggested: Consult (strong)
Discharge:
weak
Observation:
Consult:
strong
Done


Met2
Application
MET System – scrotal pain triage
Chang, Carl
ScrotalPain
History
Site of pain: Both Left None Right
Onset of pain: Acute Gradual
Type of pain: Constant Intermittent
Vomiting: Yes No
Physical Examination
Cord palpable: Abnormal Normal
Cremast. reflex: Yes No
Lie: Transverse
Testis tenderness:
Entire Testis Not Tender
Posterior Tender Not Specific
Upper Pole
Temperature: 36.6 Celsius
Swelling: Both Left None Right
Tests
WBC/HPF: 9.0
WBC: (no value) x 1000
Patients list
Synchronize


palm
m515
Patient
Doe, John
Hx
History
PE
Ix
TR
Site of pain: RLQ
Durat. of pain: 12.5 hrs
Type of pain: Intermit.
Shifting of pain: Yes
Previous visit: No
Vomiting: Yes
Done
Site of Pain
RLQ
Lower abd.
Other
OK
Cancel
Clear
Temperature
38.1 °C
1 2 3 4 5 6 7 8 9 Del 0 .
Type of Pain
Constant
Intermittent

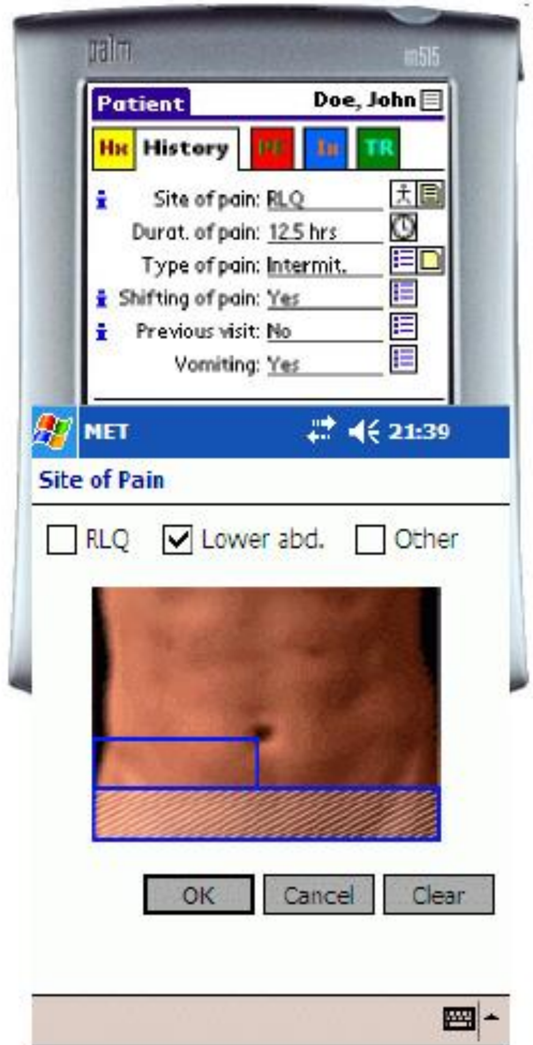
Decision Rules (examples)

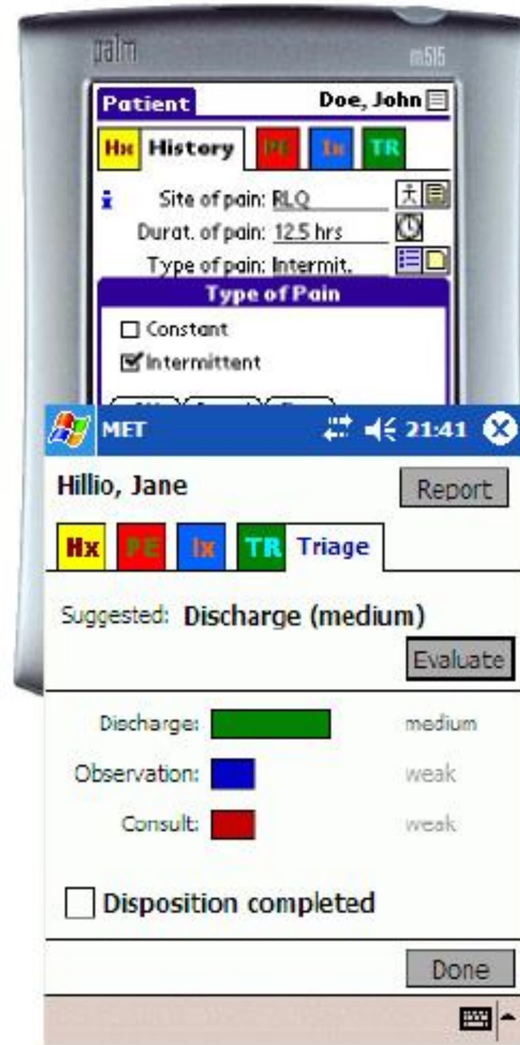
- **if** (Age < 5 years) **and** (PainSite = lower_abdomen)
  and (RebTend = yes) **and** (4 < WBC < 12)
  then (Triage = discharge)

- **if** (PainDur > 7 days) **and** (PainSite = lower_abdomen)
  and (37 ≤ Tempr ≤ 39) **and** (TendSite = lower_abdomen)
  then (Triage = observation)

- **if** (Sex = male) **and** (PainSite = lower_abdomen)
  and (PainType = constant) **and** (RebTend = yes)
  and (WBCC ≥ 12) **then** (Triage = consult)

System MET-AP





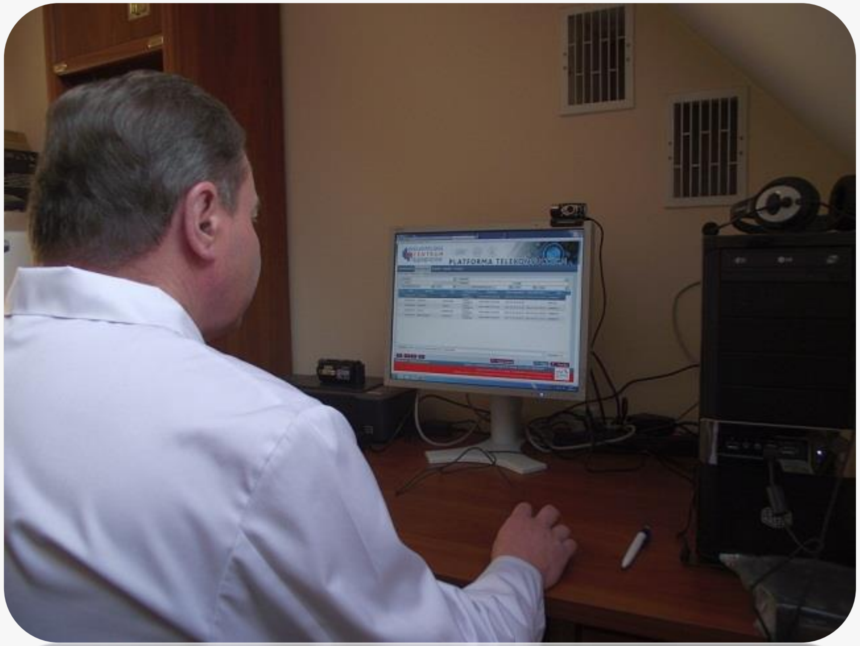
Wielkopolska Center of Telemedicine - WCT

- Platforma (organizacyjna i techniczna) oferująca systemy i usługi z zakresu telekonsultacji, teleedukacji i wspomagania decyzji
- Obecnie ograniczona do chirurgii urazowej
- Przeznaczona do obsługi przypadków stabilnych
- Obejmuje 5 szpitali uniwersyteckich z klinikami oraz 21 szpitali partnerskich
- Współpracujące ośrodki
  - Klinika Chirurgii Urazowej, Leczenia Oparzeń i Chirurgii Plastycznej
  - Poznańskie Centrum Superkomputerowo-Sieciowe
  - Politechnika Poznańska

https://www.telemedycyna.wlkp.pl

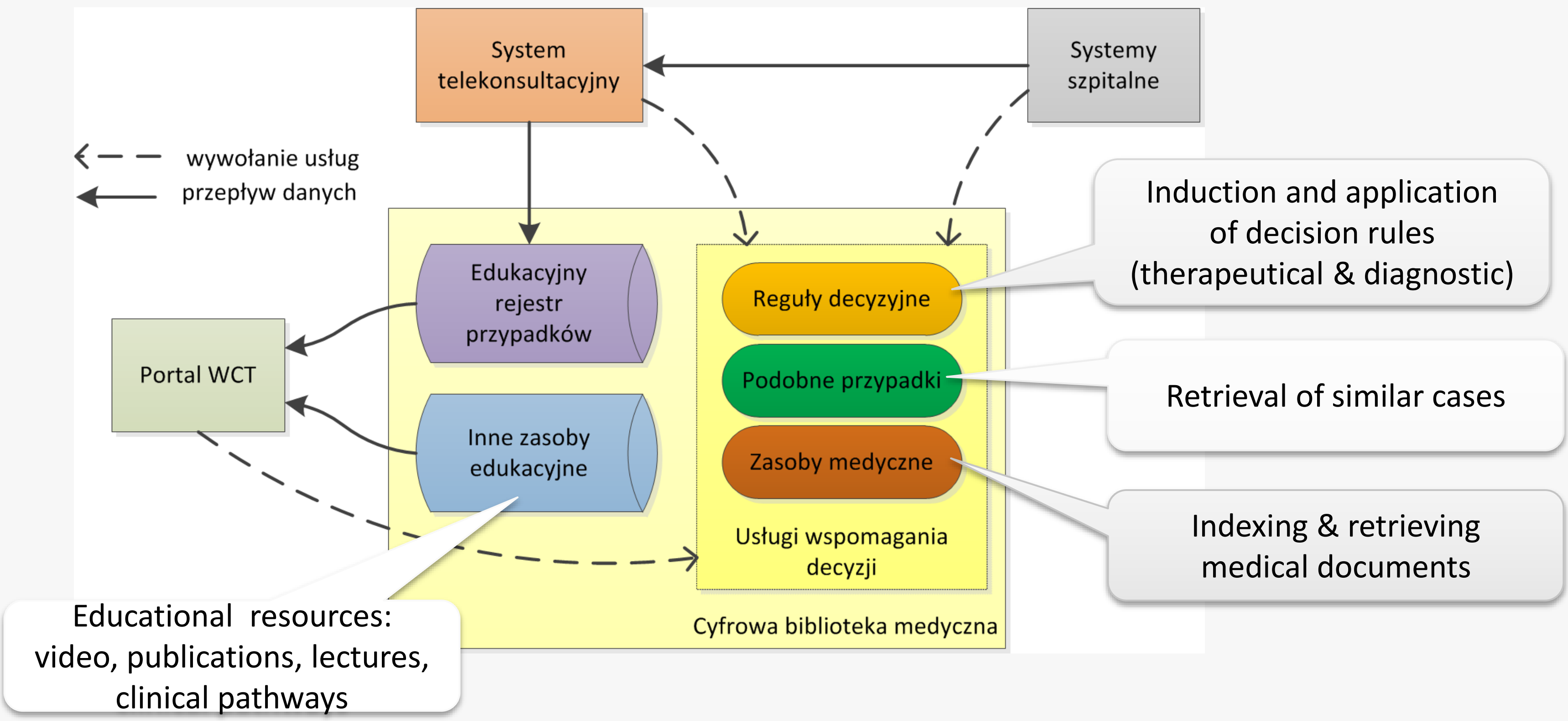
WCT – goals of the project

- Standardization and increase of efficiency of communication between regional hospitals and reference clinics
- Increase of security of trauma patients with multiple injuries
- Efficient use of scarce human resources (specialists-consultants)
- Increase of competence in regional hospitals of Wielkopolska
- Contribution to education of medical students

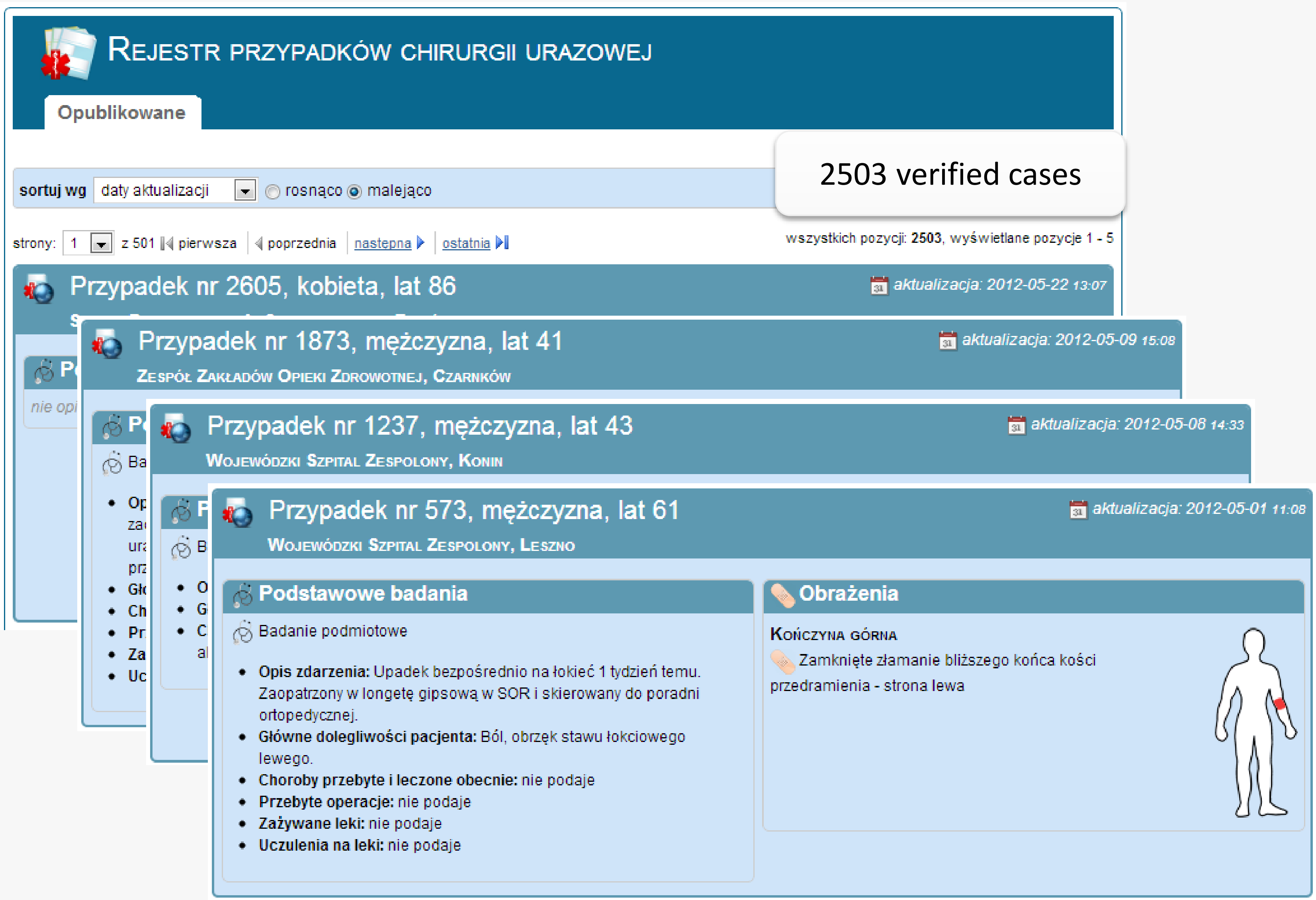


WCT – architecture

Structure of the system and available services of clinical decision support



Database of trauma patients



2503 verified cases

Badania przedmiotowe

- **Głowa i szyja:** Głowa kształtna, symetryczna, przy palpacji i opukiwaniu niebolesna. Gałki oczne osadzone prawidłowo. Źrenice równe okrągłe, prawidłowo reagujące na światło.
  Szyja niebolesna o ruchomości prawidłowej.
- **Klatka piersiowa:** Klatka piersiowa symetryczna, wysklepiona prawidłowo.
- **Brzuch:** Brzuch wysklepiony na poziomie klatki piersiowej, miękki, niebolesny.
- **Miednica:** Bez odchyleń od stanu prawidłowego.
- **Kończyny górne:** Bez odchyleń od stanu prawidłowego.
- **Kończyny dolne:** Lewy staw skokowy-rozległe rany pooperacyjne nad oboma kostkami,niezagojone,brzegi ran martwiczo zmienione.Zmiany martwicze skóry w okolicy kostki bocznej,duży krwiak w okolicy kostki bocznej.
  Prawa kończyna dolna prawidłowa.

Badania obrazowe

- **zdjęcie rentgenowskie:**

1144.123.1186.1157.11.159.2011

1144.123.1186.1157.11.159.2011

Badania laboratoryjne

Grupa krwi

- **Grupa krwi:** A
- **Rh:** +

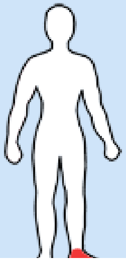
Elektrolity

	Wyniki	
Sód (Na):	136	mmol/l
Potas (K):	4,6	mmol/l

WIĘCEJ »

Obrażenia

Kończyna dolna



Zamknięte złamanie kostek goleni - strona lewa

- **Złamania na wysokości więzozrostu:** z uszkodzeniem strony przyśrodkowej (kostka lub więzadło)

Rozpoznanie i leczenie

Rozpoznanie

ICD-10	Opis
S82.3	Stan po operacyjnym leczeniu złamania podudzia lewego z przemieszczeniem.

Zastosowane leczenie operacyjne

ICD-9	Opis
78.62	Usunięcie 2 drutów Kirschnera.

Przeżywalność

- **Klasyfikacja przeżywalności:** pacjent przeżył

Wypis

- **Pacjent:** wypisany do domu
- **Liczba dni hospitalizacji:** 7
- **Dalsza kontrola w poradni:** chirurgii urazowej
- **Liczba dni do najbliższej kontroli:** 2

WIĘCEJ »

Induction and application of *decision rules*

- Decision rules („*if…, then…*”) describe strong relationships and patterns discovered in verified database of trauma patients
- The rules are concise representation of knowledge discovered from data (important for education)
- Types of rules:
  - Diagnostic – expected severity of injury on ISS scale (*Injury Severity Score*)
  - Therapeutic – suggested treatment of bone fractures (conservative treatment or surgery)

Induction and application of *decision rules*

Rule matching to patient #63

Zalecane postępowanie: **operacyjne (7)** zachowawcze (0)

Opis reguły

Jeżeli	Krwinki czerwone	=	>=3,<=5 mln/mm^3
oraz	Liczba obrażeń w lokalizacji: 6.1.1	>=	1.0
oraz	Ciśnienie skurczowe	=	>=100,<=190 mmHg
to	**Leczenie**	=	**operacyjne**

Hip fracture

Miary oceny reguły

Antywsparcie	0.85%
Wsparcie	13.00%
Wiarygodność	93.84%

Przypadki poprawnie dopasowane (274)

Przypadki błędnie dopasowane (18)

Matching rules

Patients ma
surgery

Jeżeli	Krwinki czerwone
oraz	Liczba obrażeń w lokal
oraz	Ciśnienie skurczowe
to	**Leczenie**

Miary oceny reguły

Przypadki poprawnie dopasowar

7, 15, 19, 55, 58, 63, 66, 80, 81,
204, 218, 269, 324, 344, 353, 37
518, 524, 528, 531, 543, 546, 54
651, 686, 696, 699, 712, 748, 76
854, 855, 857, 858, 894, 900, 91
1040, 1044, 1046, 1047, 1080, 1
1196, 1202, 1203, 1216, 1251, 1
1397, 1401, 1415, 1420, 1423, 1
1529, 1530, 1540, 1542, 1547, 1
1604, 1606, 1636, 1683, 1684, 1
1750, 1752, 1754, 1765, 1778, 1
1896, 1899, 1902, 1912, 1924, 1
2039, 2042, 2054, 2061, 2070, 2
2186, 2202, 2213, 2225, 2239, 2
2317, 2320, 2322, 2330, 2342, 2
2412, 2430, 2444, 2445, 2454, 2
2543, 2564, 2582, 2586, 2592, 2
2658, 2703, 2705, 2739, 2741, 2

Przypadki błędnie dopasowane (

PRZYPADEK NR 196
KLINIKA CHIRURGII URAZOWEJ, LECZENIA OPARZEŃ I CHIRURGII PLASTYCZNEJ
(UNIWERSYTET MEDYCZNY IM. KAROLA MARCINKOWSKIEGO W POZNANIU)

EKSPORTUJ ▾ UMIEŚĆ ▾

Podsumowanie | Podstawowe badania | Obrażenia | Rozpoznanie i leczenie

Dane podstawowe

WIEK 68 **PŁEĆ** kobieta

Podstawowe badania

BADANIE PODMIOTOWE

- **Opis zdarzenia:** Upadek na ulicy. Przywieziona przez ambulans.
- **Główne dolegliwości pacjenta:** Ból biodra lewego.
- **Choroby przebyte i leczone obecnie:** Nie podaje.
- **Przebyte operacje:** Strumektomia. Cholecystektomia.
- **Zażywane leki:** Apirin Protect.
- **Uczulenia na leki:** Nie podaje.

OCENA CIĘŻKOŚCI OBRAŻEŃ CIAŁA

- **Kończyny:** amputacja poniżej łokcia; zwichnięcie barku; złamanie kości ramiennej bez przemieszczenia; złamanie obu kości przedramienia; amputacja stopy; zwichnięcie kolana; złamanie k. udowej bez przemieszczenia; złamanie kości podudzia

OCENA STANU PRZYTOMNOŚCI

- **Oczy:** samoistne otwieranie oczu
- **Najlepsza reakcja ruchowa:** spełnianie poleceń
- **Najlepsza odpowiedź słowna:** świadoma rozmowa

PARAMETRY ŻYCIOWE

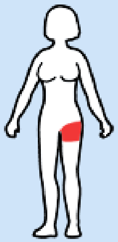
	Wyniki	
Ciśnienie skurczowe:	180	mmHg
Ciśnienie rozkurczowe:	98	mmHg
Tętno:	115	/min

Obrażenia

KOŃCZYNA DOLNA

ZAMKNIĘTE ZŁAMANIE SZYJKI KOŚCI UDOWEJ - STRONA LEWA

- **Klasyfikacja złamań szyjki kości udowej wg Gardena:** III (złamanie z przemieszczeniem, ale odłamy kostne stykają się)



WIĘCEJ »

Rozpoznanie i leczenie

ROZPOZNANIE

ICD-10	Opis
S72.0	Złamanie szyjki kości udowej

ZASTOSOWANE LECZENIE OPERACYJNE

ICD-9	Opis
00.781	Operacje stawu biodrowego - oba elementy mocowane bezcementowo
81.521	Częściowa pierwotna wymiana stawu biodrowego - endoproteza bipolarna

ZASTOSOWANE LECZENIE NIEOPERACYJNE

ICD-9 Opis

Matching

Patients m… conservati…

Informacja! Poniższa lista zawiera…

Zalecane postępowanie: oper…

Jeżeli	Krwinki czerwone
oraz	Liczba obrażeń w lok…
oraz	Ciśnienie skurczowe
to	**Leczenie**

Miary oceny reguły

Przypadki poprawnie dopasow…

Przypadki błędnie dopasowane

44, 435, 443, 743, 758, 781, 7…
2364, 2686

Przypadek nr 1132
Wojewódzki Szpital Zespolony, Konin

Eksportuj ▾ Umieść ▾

Podsumowanie | Podstawowe badania | Obrażenia | Rozpoznanie i leczenie

Dane podstawowe

Wiek 93 **Płeć** kobieta

Podstawowe badania

Badanie podmiotowe

- **Opis zdarzenia:** 3-4 dni wczesniej uraz biodra, upadek na pozimie 0. wywiad niemożliwy do przeprowadzenia
- **Główne dolegliwości pacjenta:** ból biodra prawego
- **Choroby przebyte i leczone obecnie:** Niewydolność serca NYCHA III, Stan po zawale m sercowego, Nadciśnienie tetnicze, zylaki odbytu, Uchyłkowatość jelit
- **Zażywane leki:** tritace vivacor

Ocena stanu przytomności

- **Oczy:** samoistne otwieranie oczu
- **Najlepsza reakcja ruchowa:** zginanie-wycofywanie kończyny na bodziec bólowy
- **Najlepsza odpowiedź słowna:** niezrozumiałe dźwięki

Parametry życiowe

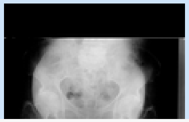
	Wyniki	
Ciśnienie skurczowe:	120	mmHg
Ciśnienie rozkurczowe:	70	mmHg
Tętno:	82	/min

Badania przedmiotowe

- **Głowa i szyja:** Kontakt niemożliwy. Stan ogólny zły
- **Kończyny dolne:** ból biodra ze skróceniem i rotacją zew,

Badania obrazowe

- **zdjecie rentgenowskie:**

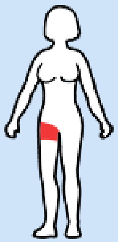


Obrażenia

Kończyna dolna

Zamknięte krętarzowe złamanie kości udowej - strona prawa

- **Klasyfikacja złamań krętarzowych kości udowej wg Boyda i Griffina:** II (Złamanie wieloodłamowe. Główna linia złamania przebiega wzdłuż linii międzykrętarzowej, ale ze znacznym uszkodzeniem bliższego odłamu. Krętarze większy i mniejszy mogą być złamane.)



Więcej »

Rozpoznanie i leczenie

Rozpoznanie

ICD-10	Opis
I11.0	Choroba nadciśnieniowa z zajęciem serca - Choroba nadciśnieniowa z zajęciem serca, z (zastoinową) niewydolnością serca
I25.9	Przewlekła choroba niedokrwienna serca - Przewlekła choroba niedokrwienna serca, nie określona
I50.1	Niewydolność serca - Niewydolność serca lewokomorowa
K57.3	Choroba uchyłkowa jelita grubego bez przedziurawienia lub ropnia
S72.1	Złamanie przezkrętarzowe

Zastosowane leczenie nieoperacyjne

ICD-9	Opis
93.44	Wyciąg układu kostnego

Violinmakers competition

Jury's assessment

Sound recording

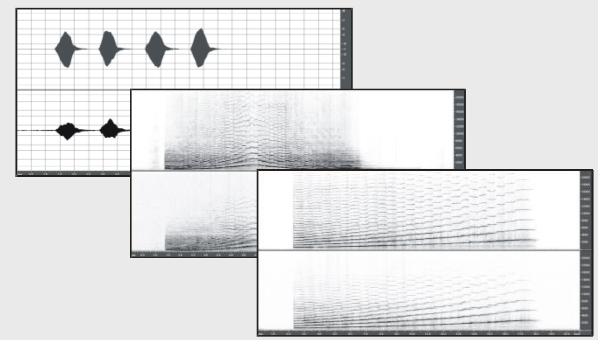
Criteria:
- **volume of sound (X)**,
- **timbre of sound (Y)**,
- ease of sound emission,
- **equal sound volume of strings (Z)**,
- accuracy of assembly,
- individual qualities

Ranking of violins based on the criterion **X**

Ranking of violins based on the criterion **Y**

Ranking of violins based on the criterion **Z**

The violin's acoustic data:
- individual sounds played on open strings, G,D,A,E,
- successive sounds of chromatic scale,



Acoustic features:
- power spectrum of chromatic scale sounds,
- wavelets,
- harmonic based spectral parameters (tristimuli, brightness, odd/even harmonics content...),
- psychoacoustic features
- cepstral coefficients.

Dominance-based Rough Set Approach

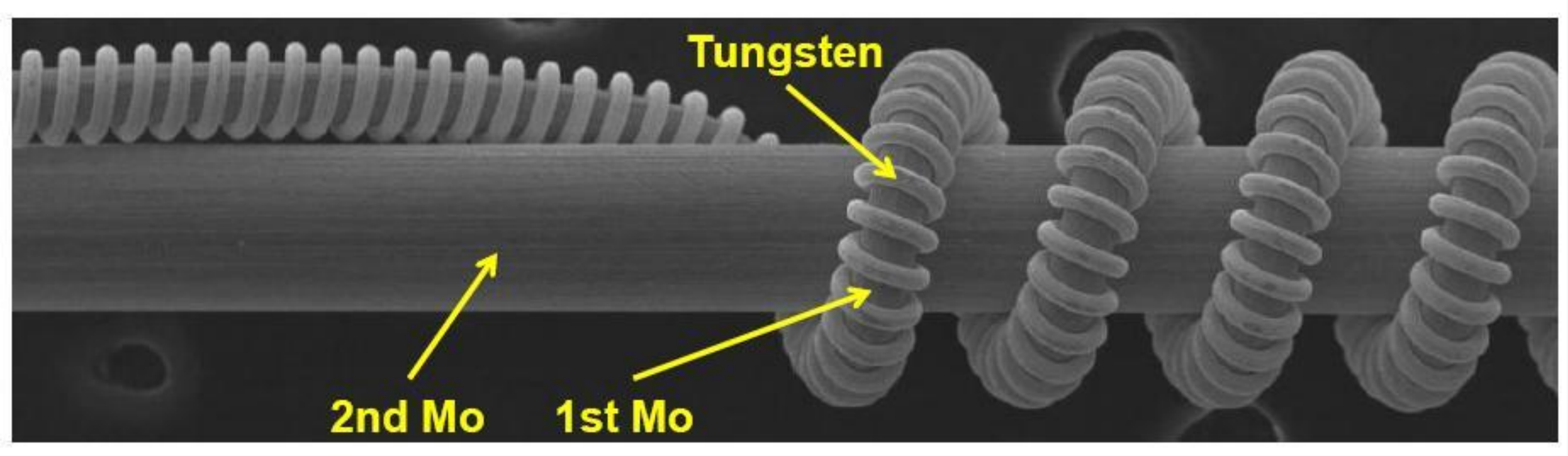
Violinmakers competition – DRSA results

- Reconstructing the expert's rankings of a set of 23 violins
- Three rankings: volume, timbre and inter-string equality
- Feature space - cepstral coefficients

Ranking according to	Best subset of acoustic features	Number of rules	Ranking fit
volume	A14, E13, D12, G16	62	87%
timbre	E13, D15, G4, G17, D5	99	92%
inter-string equality	D20, D15, A24, D10	64	79%

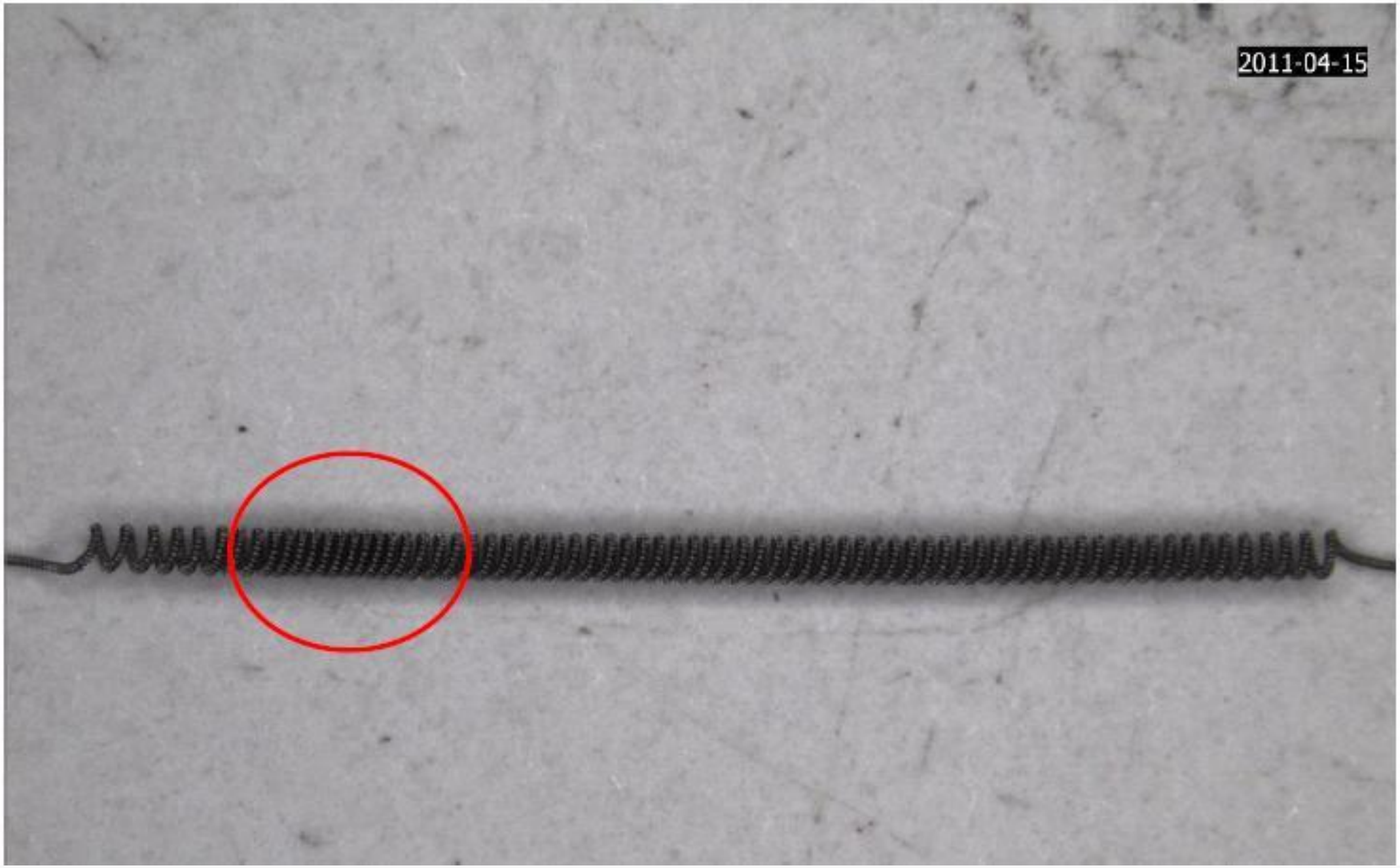
Technical diagnostics – problem of short circuits in coil body

- An element of a light-bulb



Technical diagnostics – problem of short circuits in coil body

- Problem– coil geometry failure – short circuits in coil body



Technical diagnostics – problem of short circuits in coil body

- Production process steps

Technical diagnostics – problem of short circuits in coil body

- Issues:
  - Wire diameter (W ~20μm; Mo ~175μm)
  - Batch throughput time (avg. 10 days)
  - Many factors suspected for failure, including interactions
    - Materials
    - Subprocesses
    - People
  - Such coil geometry failure occurred first time in history
  - Defects are hardly visible on machines

Technical diagnostics – problem of short circuits in coil body

- Data table – 550 lots described by 10 attributes

	B	G	I	J	L	N	P	V	W	X
1	Month & day of the month	ID of wolfram lot	ID of 1st coiling machine	No. of breaks	ID of 2nd coiling machine	ID of annealing furnace	ID of cutting machine	Day of the week	Number of days in productio	Failure
2	11_08	M296	26/3	42	75/3	5	6	Mo	15	NO
3	11_02	M297	7/2	37	6/1	5	3	Fr	34	YES
4	10_15	M288	7/4	13	7/1	5	6	Tu	21	NO
5	11_09	M296	48/6	0	6/1	5	1	Mo	32	NO
6	11_21	M286	26/6	14	7/4	5	1	We	33	NO
7	11_37	M293	26/4	12	75/3	5	1	Tu	21	NO
8	11_04	M292	7/1	10	75/3	5	1	Fr	21	YES
9	11_43	M296	7/2	16	75/4	5	1	Th	13	NO
10	11_20	M296	48/3	0	34/4	12	3	Mo	35	YES
11	11_10	M286	7/3	0	70/2	5	3	We	35	NO
12	11_17	M298	7/4	18	4/1	12	3	Tu	43	YES
13	11_07	M296	7/3	0	73/4	5	3	Fr	30	YES
14	11_34	M293	23/1	0	6/3	5	1	Th	20	NO
15	10_44	M288	48/5	0	70/1	5	1	We	29	NO
16	11_01	M288	7/3	28	3/4	1	3	Tu	0	NO
17	11_12	M296	7/2	14	73/1	12	1	Mo	67	YES
18	11_20	M300	48/6	0	33/3	9	6	Tu	33	NO
19	10_22	M287	48/4	0	71/4	6	6	We	26	YES
20	11_18	M299	7/4	17	36/4	2	6	Th	33	NO
21	11_44	M296	48/1	0	6/4	9	1	Fr	14	YES

Technical diagnostics – problem of short circuits in coil body

- Results of the DRSA analysis:
- Quality of approximation of the classification into Yes/No failure: 100%
- Reducts: 61 with 8 to 5 attributes
- Adopted reduct (5 attributes):
  - I – ID of the 1st coiling machine
  - L – ID of the 2nd coiling machine
  - N – ID of the annealing furnace
  - B – lot ID: day of the month
  - V – lot ready: day of the week

Technical diagnostics – problem of short circuits in coil body

- Rules:

If 1st coiling machine = 13, then YES failure support = 8%

If 1st coiling machine = 48 & Furnace = 12 & Cutting machine = 3,

then YES failure support = 12%

If Furnace = 12 & Cutting machine = 3 & Day = Friday, then YES failure

support = 9%

If Furnace = 5 & Cutting machine = 6, then NO failure

support = 14.67%

If Furnace = 5 & Month of the lot = 12, then NO failure

support = 20.22%

Other applications of DRSA

- Prediction of antimicrobial activity of quaternary chlorides by analysis of structure-activity-relationships (SAR)
- Complications after open-heart operations
- Colon cancer surgery
- Pediatric hip surgery, asthma treatment
- Prostate cancer teatment
- Brest cancer treatment
- HSV treatment of duodenal ulcer
- Extracorporeal shockwave lithotripsy (ESWL)
- Prediction of antifungal activity of gemini-imidazolium compounds
- Green chemistry classification of silver nanoparticles synthesis
- Comparative analysis of targeted metabolomics
- Triggerfish and cardiovascular data analysis of glaucoma patients
- …

DRSA for group decision

DRSA for group decision

- Example: students described by scores (1–20) in mathematics (M), physics (Ph) and literature (L) are classified by 3 professors (P1, P2, P3) to preference ordered classes: Bad, Medium, Good
- Decisions of P1, P2, P3 have to be aggregated, so as to select students which will be finally accepted for a graduate program
- The aggregate decision represents a consensus between professors
- Possible consenus:
  - 2 professors classify as „at least Medium” + 1 professor classifies as „Good”
    [Medium, Medium, Good], [Medium, Good, Medium], [Good, Medium, Medium]
- Resulting rules, e.g.:

if student x gained at least 15 in M, and at least 18 in L, then x is accepted

if student x gained at most 10 in M, and at most 13 in Ph, then x is not accepted

Interpretation of recommendation provided by an MCDA method in terms of decision rules

Illustrative example – ranking of students

Student	Mathematics	Physics	Literature
S1	medium	medium	good
S2	good	good	medium
S3	medium	good	medium
S4	medium	medium	medium
S5	good	good	bad
S6	medium	bad	good

Preference information given by the DM

- **Pairwise comparisons of some students**
  - S2 $\succ$ S1
  - S4 $\succ$ S5
  - S5 $\succ$ S6
- **Overall intensity of preference**
  - (S5,S6) $\succ^*$ (S2,S1)
- **Intensity of preference relative to a single criterion**
  - (good, medium) $\succ^*_{Mathematics}$ (medium, bad)

GRIP results

Dominance relation

Necessary Ranking Graph | Representative Ranking | Marginal Utilities
Dominance Relation | Possible Preference Relation | Necessary Preference Relation

	S1	S2	S3	S4	S5	S6
S1	False	False	False	True	False	True
S2	False	False	True	True	True	False
S3	False	False	False	True	False	False
S4	False	False	False	False	False	False
S5	False	False	False	False	False	False
S6	False	False	False	False	False	False

Necessary preference

Necessary Ranking Graph | Representative Ranking | Marginal Utilities
Dominance Relation | Possible Preference Relation | Necessary Preference Relation

	S1	S2	S3	S4	S5	S6
S1	True	False	False	True	True	True
S2	True	True	True	True	True	True
S3	False	False	True	True	True	True
S4	False	False	False	True	True	True
S5	False	False	False	False	True	True
S6	False	False	False	False	False	True

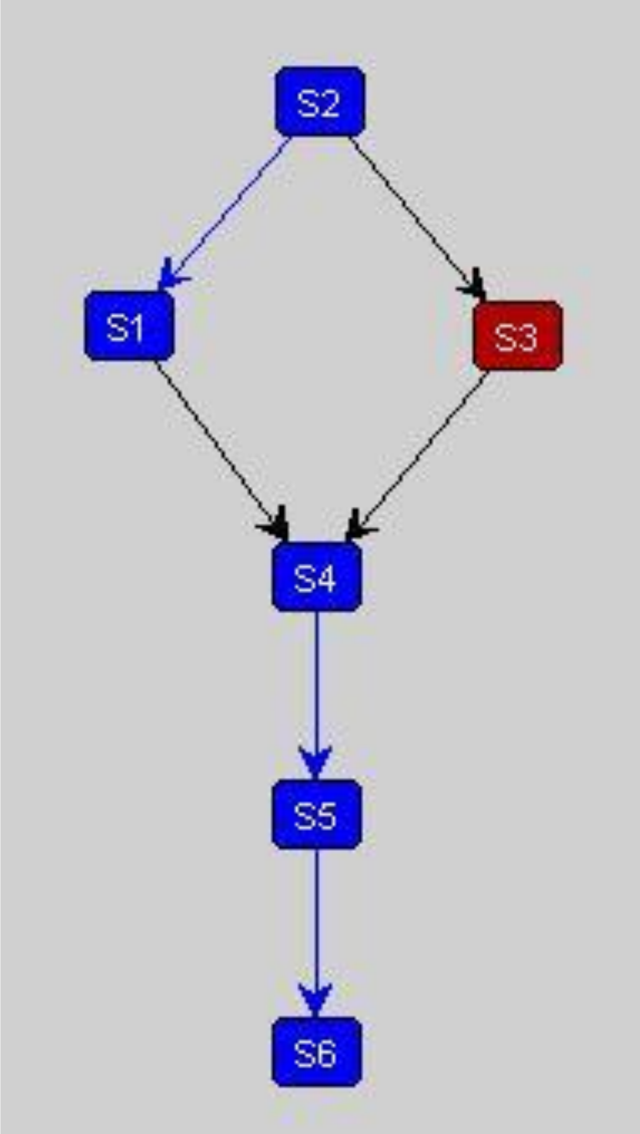
Possible preference

Necessary Ranking Graph | Representative Ranking | Marginal Utilities
Dominance Relation | Possible Preference Relation | Necessary Preference Relation

	S1	S2	S3	S4	S5	S6
S1	True	False	True	True	True	True
S2	True	True	True	True	True	True
S3	True	False	True	True	True	True
S4	False	False	False	True	True	True
S5	False	False	False	False	True	True
S6	False	False	False	False	False	True

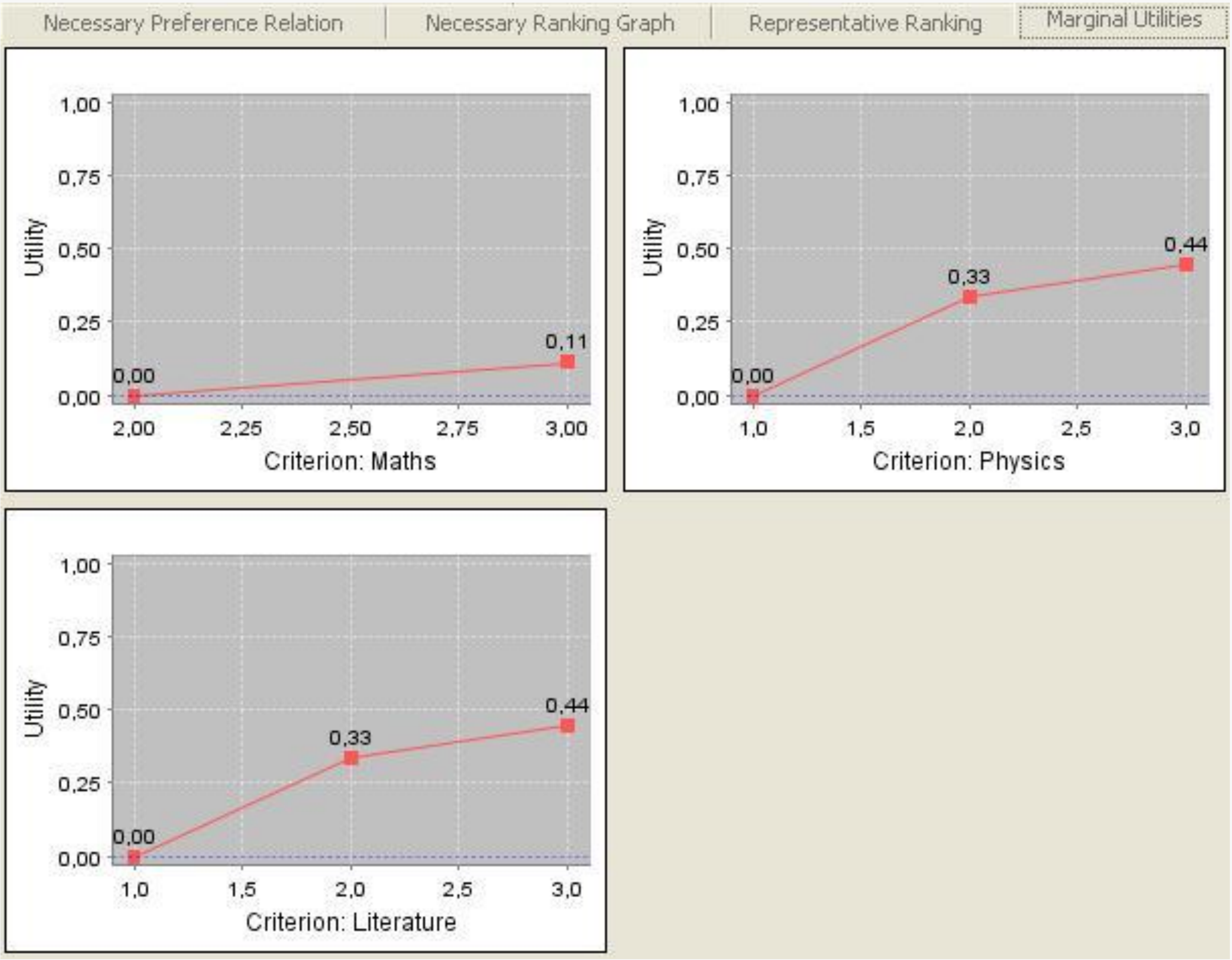
GRIP results

Necessary ranking



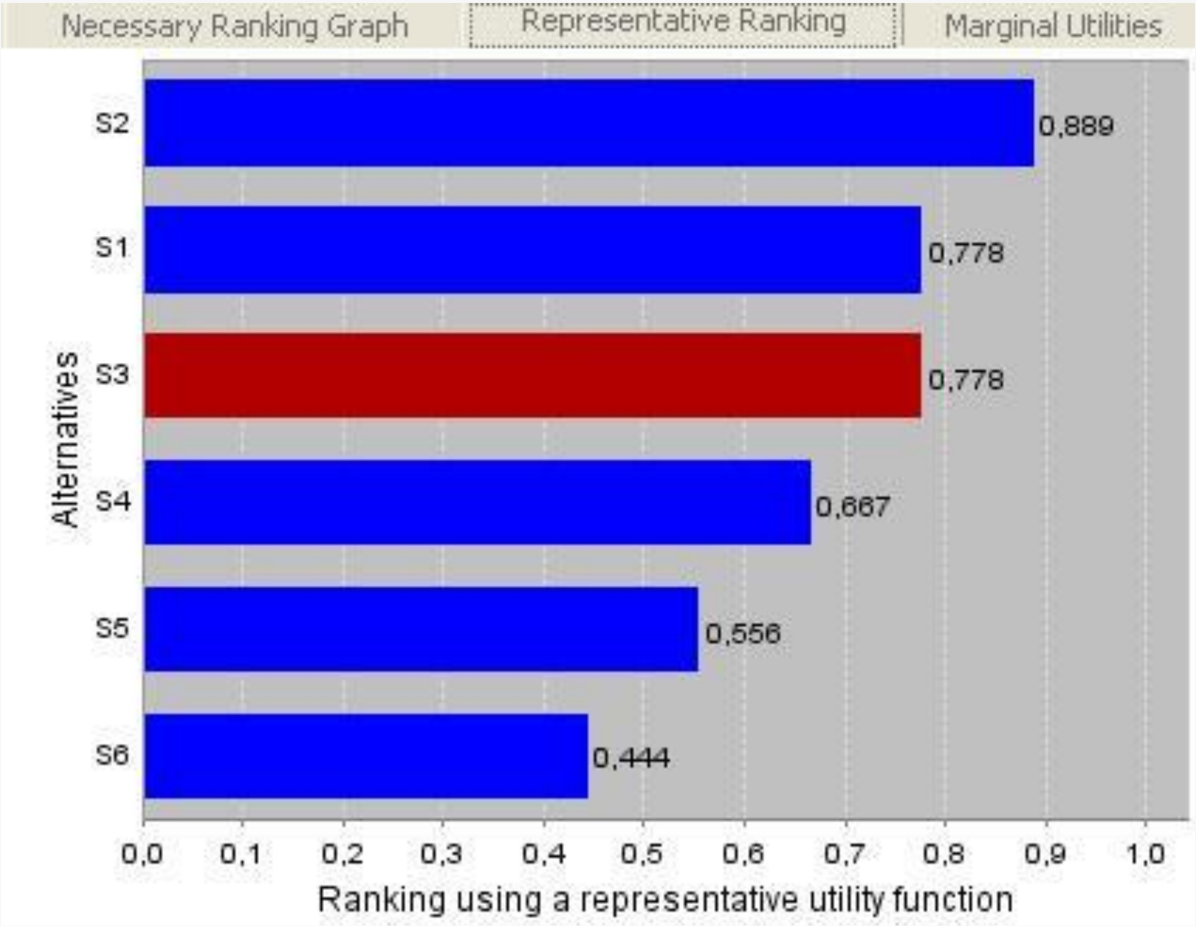
GRIP results

Representative value function



GRIP results

Representative ranking



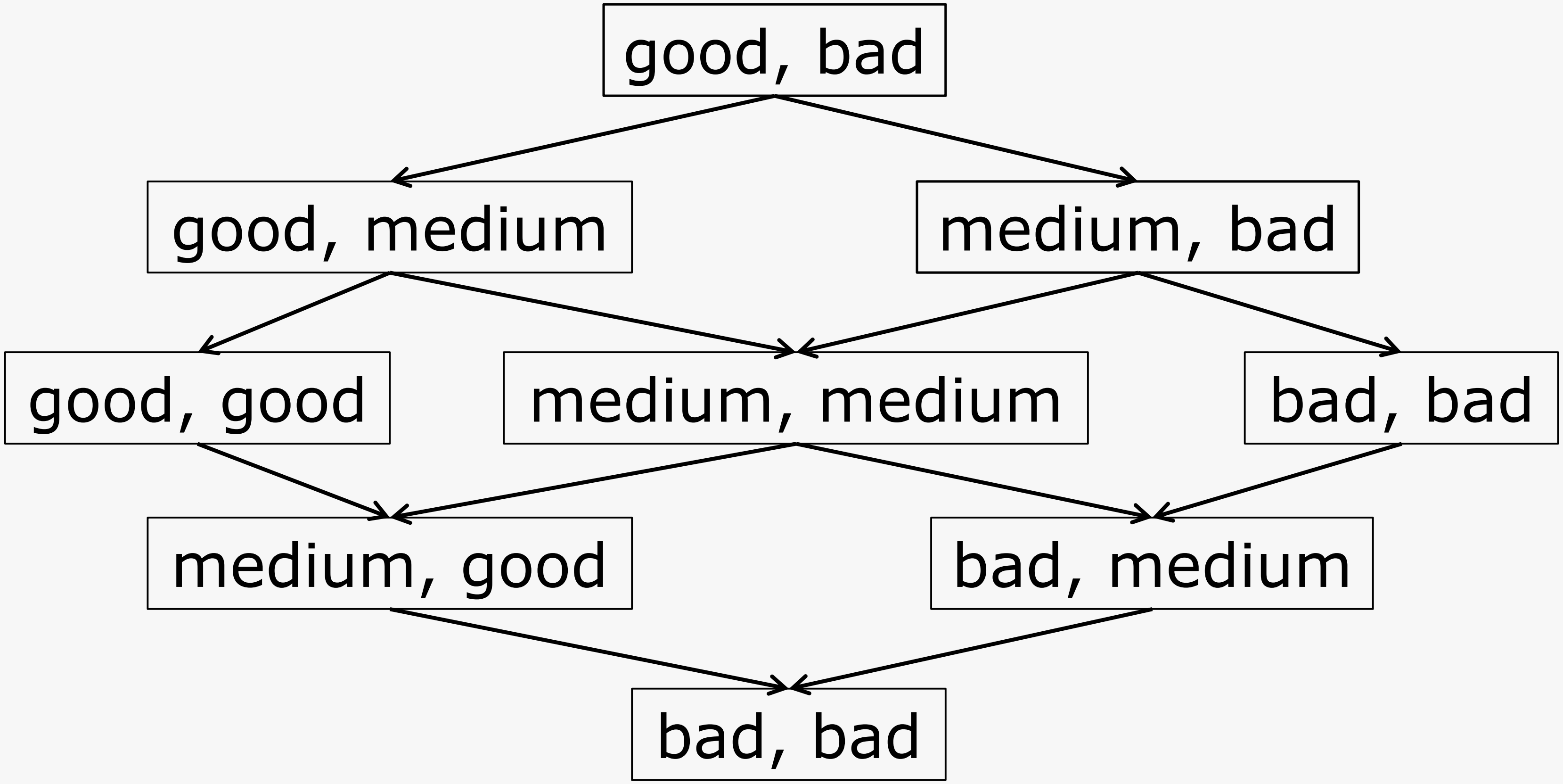
Illustrative example – ranking of students

Pairwise comparison table (PCT) and the necessary preference relation resulting from GRIP – input data for DRSA

Pair (S_i,S_j)	Maths S_i	Maths S_j	Physics S_i	Physics S_j	Literature S_i	Literature S_j	Nec. Pref. N
(S_1,S_1)	medium	medium	medium	medium	good	good	$\succsim^N$
(S_1,S_2)	medium	good	medium	good	good	medium	$\not\succsim^N$
(S_1,S_3)	medium	medium	medium	good	good	medium	$\not\succsim^N$
…	…		…	…	…	…	…
(S_6,S_4)	medium	medium	bad	medium	good	bad	$\not\succsim^N$
(S_6,S_5)	medium	good	bad	good	good	good	$\not\succsim^N$
(S_6,S_6)	medium	medium	bad	bad	good	good	$\succsim^N$

Preference on single criteria for pairs of evaluations

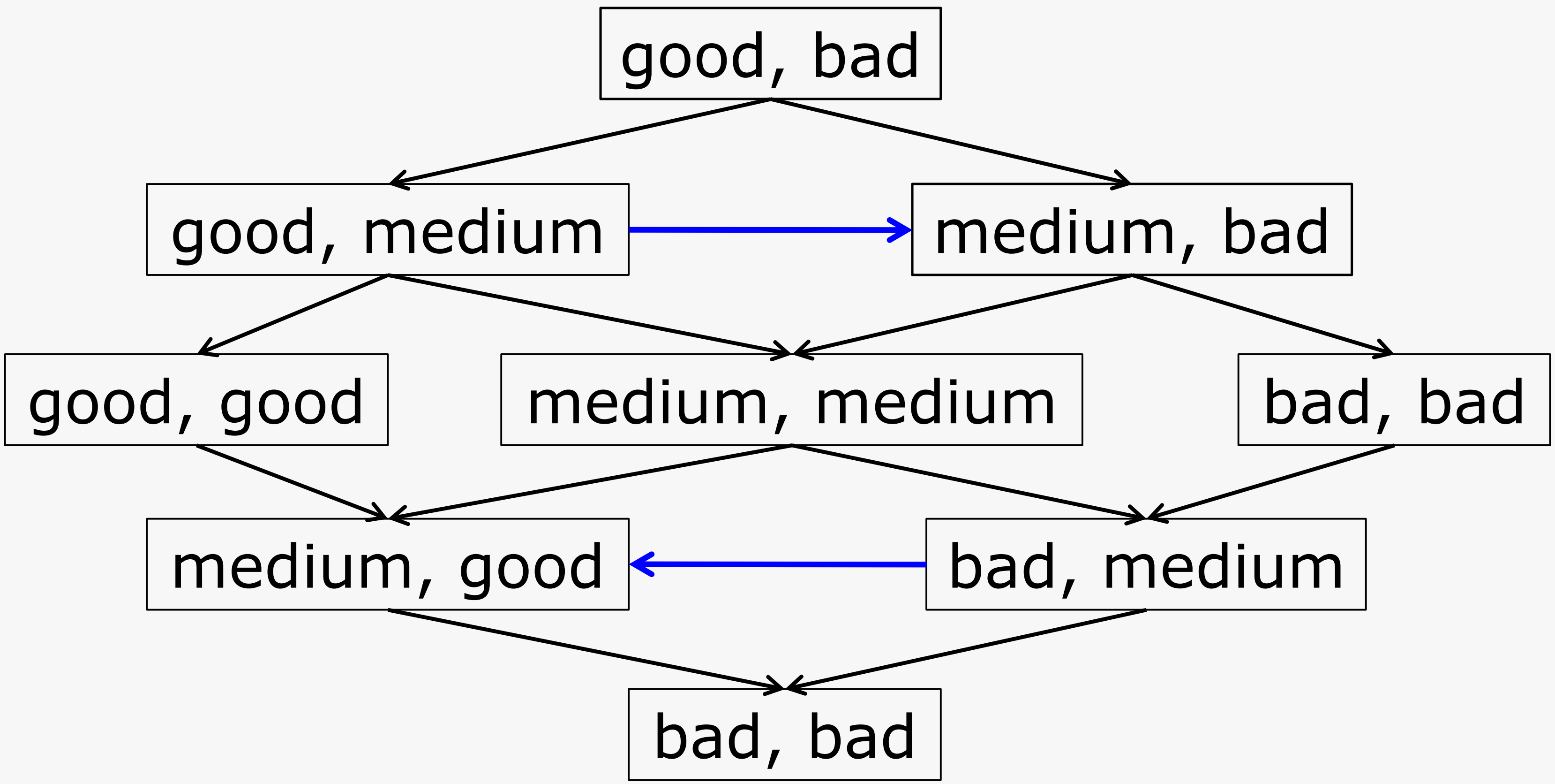
- In case of no information about the intensity of preference: Physics, Literature



Preference on single criteria for pairs of evaluations

- Use of preference information about intensity of preference on a single criterion – the case of Mathematics:

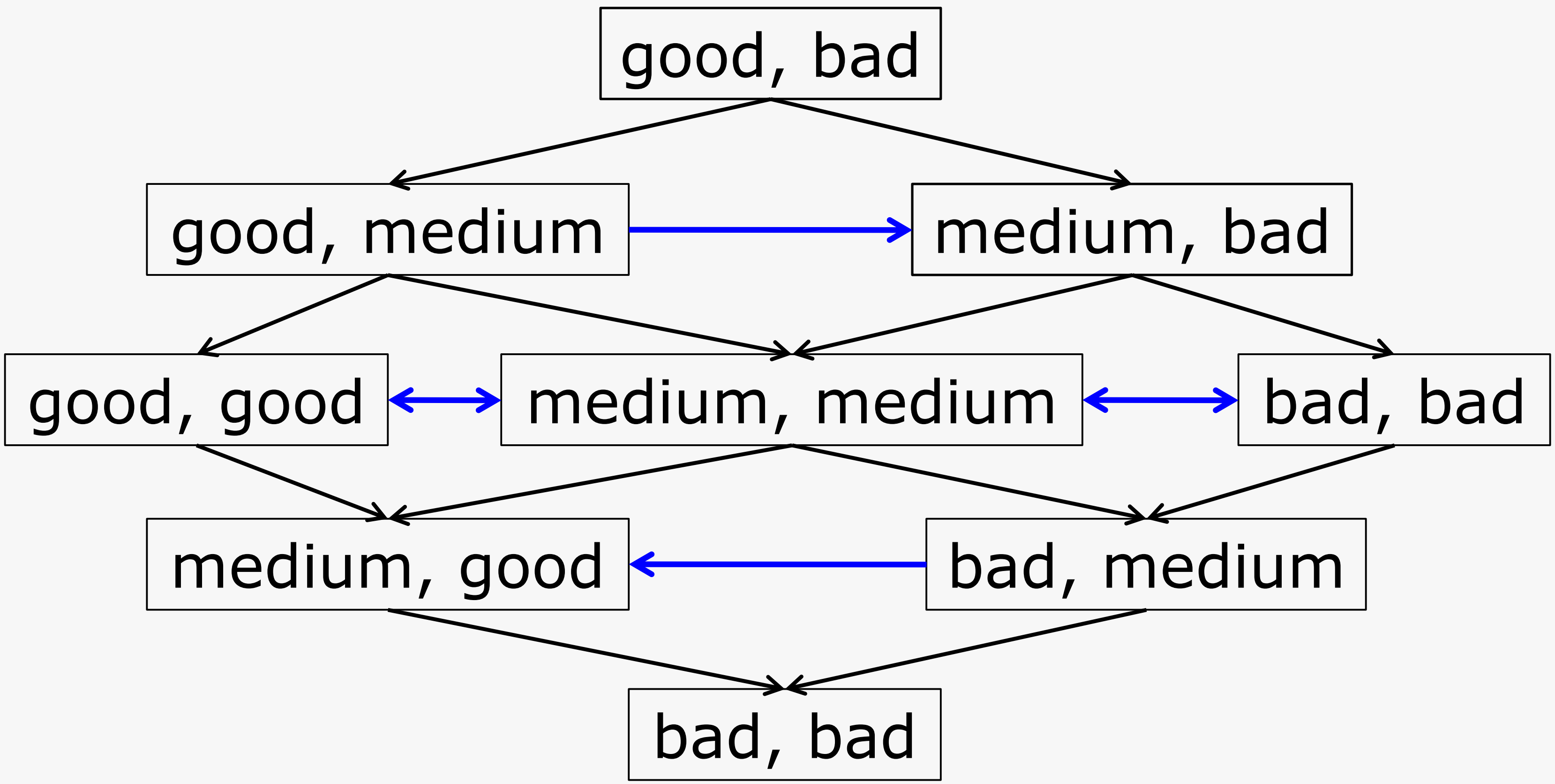
(good, medium) $\succ^{*}_{Mathematics}$ (medium, bad)



Preference on single criteria for pairs of evaluations

$$(\text{good, good}) \sim^*_{Mathematics} (\text{medium, medium}) \sim^*_{Mathematics} (\text{bad, bad})$$

$$(\text{good, medium}) \succ^*_{Mathematics} (\text{medium, bad})$$



Preference on single criteria for pairs of evaluations

- Intensity of preference wrt differences of evaluations on Mathematics

x_{Math}=good & y_{Math}=bad : x is **much better** than y

x_{Math}=good & y_{Math}=medium : x is **better** than y

x_{Math}=medium & y_{Math}=bad : x is **weakly better** than y

$x_{Math}= y_{Math}$: x and y are **indifferent**

All rules representing the necessary preference relation

#1: if $y_{Phys} \leq$ bad, then $x \succeq^N y$

(S1,S6),(S2,S6),(S3,S6),(S4,S6),(S5,S6),(S6,S6)

#2: if x_{Math} is better than y_{Math} & $x_{Lit} \geq$ medium, then $x \succeq^N$ y,

(S2,S1),(S2,S3),(S2,S4),(S2,S6)

#3: if $x_{Phys} \geq$ medium & $y_{Phys} \leq$ medium & $x_{Lit} \geq$ good, then $x \succeq^N y$

(S1,S1),(S1,S4),(S1,S6)

#4: if $x_{Phys} \geq$ medium & $y_{Lit} \leq$ bad, then $x \succeq^N y$,

(S1,S5),(S2,S5),(S3,S5),(S4,S5),(S5,S5)

#5: if x_{Math} is weakly better than y_{Math} & $x_{Phys} \geq$good &
$x_{Lit} \geq$ medium & $y_{Lit} \leq$ medium, then $x \succeq^N y$

(S2,S2),(S2,S3),(S2,S4),(S2,S5),(S3,S3),(S3,S4)

#6: if $x_{Phys} \geq$medium & $y_{Phys} \leq$medium & $x_{Lit} \geq$medium & $y_{Lit} \leq$medium,
then $x \succeq^N y$ (S1,S4),(S2,S4),(S3,S4),(S4,S4)

#7: if x_{Math} is weakly better than y_{Math} & $y_{Lit} \leq$ bad, then $x \succeq^N y$

(S2,S5),(S5,S5)

Minimal cover rules representing necessary preference relation

#1: if $y_{Phys} \leq$ bad, then $x \succeq^N y$

(S1,S6),(S2,S6),(S3,S6),(S4,S6),(S5,S6),(S6,S6)

#2: if x_{Math} is better than y_{Math} & $x_{Lit} \geq$ medium, then $x \succeq^N y$,

(S2,S1),(S2,S3),(S2,S4),(S2,S6)

#3: if $x_{Phys} \geq$ medium & $y_{Phys} \leq$ medium & $x_{Lit} \geq$ good, then $x \succeq^N y$

(S1,S1),(S1,S4),(S1,S6)

#4: if $x_{Phys} \geq$ medium & $y_{Lit} \leq$ bad, then $x \succeq^N y$,

(S1,S5),(S2,S5),(S3,S5),(S4,S5),(S5,S5)

#5: if x_{Math} is weakly better than y_{Math} & $x_{Phys} \geq$good &
$x_{Lit} \geq$ medium & $y_{Lit} \leq$ medium, then $x \succeq^N y$

(S2,S2),(S2,S3),(S2,S4),(S2,S5),(S3,S3),(S3,S4)

#6: if $x_{Phys} \geq$medium & $y_{Phys} \leq$medium & $x_{Lit} \geq$medium & $y_{Lit} \leq$medium,
then $x \succeq^N y$ (S1,S4),(S2,S4),(S3,S4),(S4,S4)

Observation of new rules after addition of preference information

- The DM adds new preference information:
  - S1 $\succ$ S3

- New rule appears to cover the new necessary preference relation:

#8: if x_{Math} is weakly better than y_{Math} & $x_{Phys} \geq$ medium & $x_{Lit} \geq$ good,
then $x \succeq^N y$

(S1,S1),(S1,S3),(S1,S4),(S1,S6)

Illustrative example - technical ranking of buses

Bus Id	MaxSpeed	ComprPressure	Blacking	Torque	SummerCons	WinterCons	OilCons	HorsePower
b01	90	2	49	477	21	25	1	138
b02	85	2	52	460	21	25	1	130
b03	72	2	73	425	23	27	2	112
b04	88	2	50	480	21	24	1	140
b05	60	1	95	400	23	24	4	96
b06	78	2	63	448	21	26	1	120
b07	90	2	26	482	22	24	0	148
b08	65	2	67	402	22	23	2	103
b09	90	2	51	468	22	26	1	138
b10	76	2	65	428	27	33	2	116
b11	85	2	50	454	21	26	1	129
b12	85	2	58	450	22	25	1	126
b13	88	2	48	458	22	25	1	130
b14	75	2	64	432	22	25	1	114
b15	68	2	70	400	22	26	2	100
b16	88	2	44	478	21	25	0	138
b17	85	2	55	445	23	26	1	120
b18	90	2	40	480	22	25	0	139
b19	72	2	64	428	21	25	2	111
b20	75	2	60	440	22	26	1	120
b21	85	2	61	458	21	25	1	126
b22	68	2	88	422	22	25	3	108
b23	82	2	65	430	23	25	2	115
b24	90	2	38	482	20	24	0	146
b25	90	2	45	479	21	25	1	145
b26	90	2	34	486	21	25	0	148
b27	86	2	60	444	22	25	1	122
b28	88	2	50	475	22	25	1	142
b29	85	2	63	440	21	26	2	120
b30	72	2	85	420	22	25	3	110
b31	65	2	94	400	24	27	4	98
b32	87	2	60	460	22	25	1	131

Preference information given by the DM – 1st iteration

- **Pairwise comparisons of some buses**
  - b04 $\succ$ b08
  - b05 $\succ$ b22
- **Overall intensity of preference**
  - (b01, b06) $\sim$ (b20, b30)
- **Intensity of preference on criterion *MaxSpeed***
  - (b01, b04) $\succ_{MaxSpeed}$ (b13, b15)

  i.e. (90, 88) $\succ_{MaxSpeed}$ (88, 68)

Dominance relation – no preference information

Dominance Relation | Possible Preference Relation | Necessary Preference Relation | Necessary Ranking Graph | Representative Ranking | Marginal Utilities

	b01	b02	b03	b04	b05	b06	b07	b08	b09	b10	b11	b12	b13	b14	b15	b16	b17	b18	b19	b20	b21	b22	b23	b24	b25	b26	b27	b28	b29	b30	b31	b32
b01	False	True	True	False	False	True	False	False	True	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b02	False	False	True	False	False	True	False	False	False	True	False	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	False	False	True	True	True	False
b03	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b04	False	True	True	False	True	True	False	False	False	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b05	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False
b06	False	False	True	False	False	False	False	False	False	True	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b07	False	False	True	False	True	False	False	False	True	True	False	True	True	True	True	False	True	True	False	True	False	True	True	False	False	False	True	True	False	True	True	True
b08	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b09	False	False	True	False	False	False	False	False	False	True	False	False	False	False	True	False	True	False	False	True	False	False	False	False	False	False	False	False	False	False	True	False
b10	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False
b11	False	False	True	False	False	True	False	False	False	True	False	False	False	False	True	False	True	False	False	True	False	False	False	False	False	False	False	False	True	False	True	False
b12	False	False	True	False	False	False	False	False	False	True	False	False	False	True	True	False	False	False	False	True	False	True	True	False	False	False	False	False	False	True	True	False
b13	False	False	True	False	False	False	False	False	False	True	False	True	False	True	True	False	True	False	False	True	False	True	True	False	False	False	True	False	False	True	True	False
b14	False	False	True	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	True	True	False
b15	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b16	False	True	True	False	False	True	False	False	False	True	True	True	True	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b17	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b18	False	False	True	False	False	False	False	False	True	True	False	True	True	True	True	False	True	False	False	True	False	True	True	False	False	False	True	False	False	True	True	True
b19	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	True	True	False
b20	False	False	True	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b21	False	False	True	False	False	True	False	False	False	True	False	False	False	True	True	False	False	False	True	False	False	True	True	False	False	False	False	False	True	True	True	False
b22	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b23	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b24	True	True	True	True	True	True	False	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b25	True	True	True	False	False	True	False	False	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b26	True	True	True	False	False	True	False	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b27	False	False	True	False	False	False	False	False	False	True	False	False	False	True	True	False	False	False	False	True	False	True	True	False	False	False	False	False	False	True	True	False
b28	False	False	True	False	False	False	False	False	False	True	False	True	False	True	True	False	True	False	False	True	False	True	True	False	False	False	True	False	False	True	True	True
b29	False	False	True	False	False	False	False	False	False	True	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b30	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b31	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False
b32	False	False	True	False	False	False	False	False	False	True	False	False	False	True	True	False	False	False	False	True	False	True	True	False	False	False	True	False	False	True	True	False

Necessary preference relation – 1st iteration

Dominance Relation | Possible Preference Relation | Necessary Preference Relation | Necessary Ranking Graph | Representative Ranking | Marginal Utilities

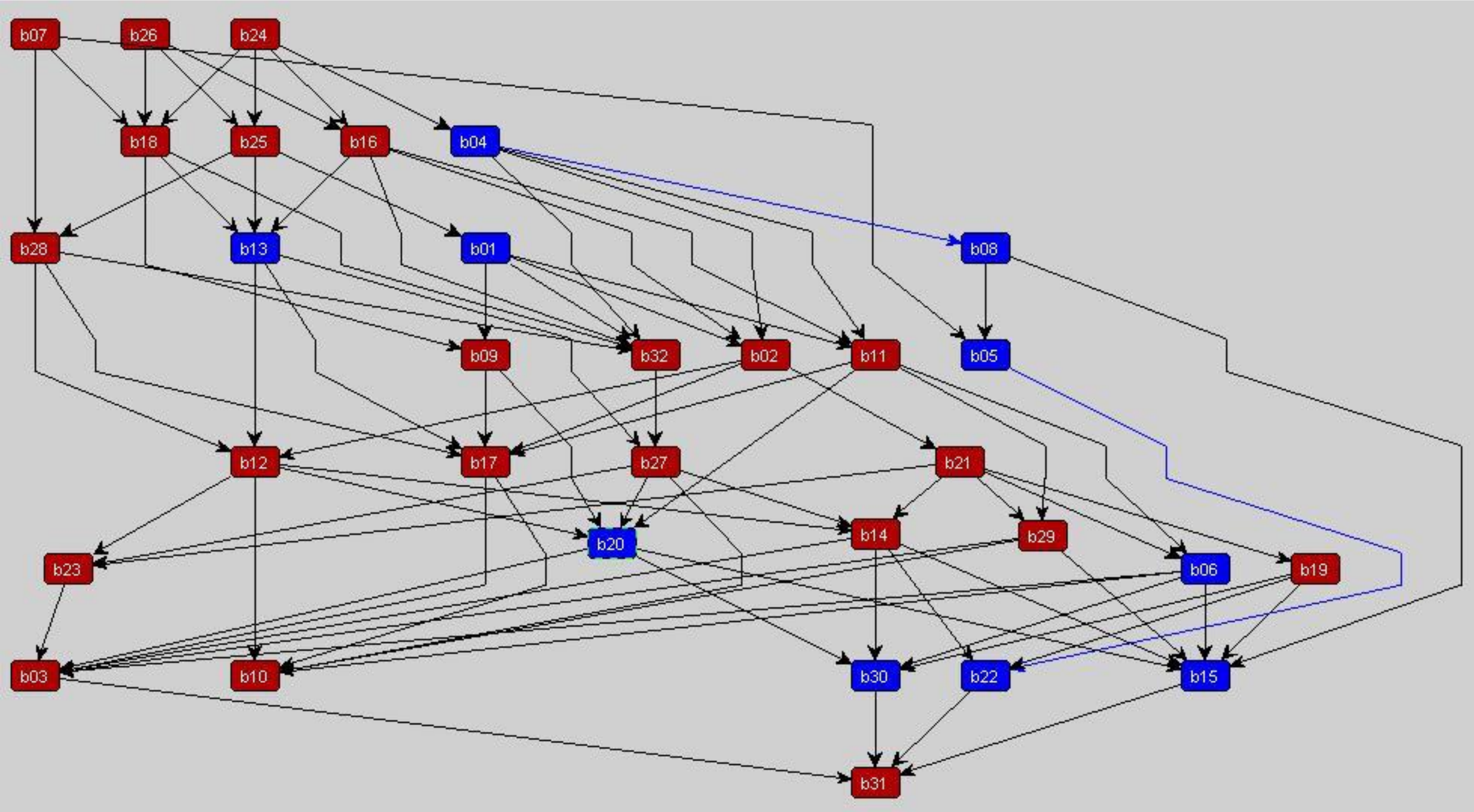
	b01	b02	b03	b04	b05	b06	b07	b08	b09	b10	b11	b12	b13	b14	b15	b16	b17	b18	b19	b20	b21	b22	b23	b24	b25	b26	b27	b28	b29	b30	b31	b32
b01	True	True	True	False	False	True	False	False	True	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b02	False	True	True	False	False	True	False	False	False	True	False	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	False	False	True	True	True	False
b03	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b04	False	True	True	True	True	True	False	True	False	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b05	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	True	False
b06	False	False	True	False	False	True	False	False	False	True	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	True	False
b07	False	False	True	False	True	False	True	False	True	True	False	True	True	True	True	False	True	True	False	True	False	True	True	False	False	False	True	True	False	True	True	True
b08	False	False	False	False	True	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	True	False
b09	False	False	True	False	False	False	False	False	True	True	False	False	False	False	True	False	True	False	False	True	False	False	False	False	False	False	False	False	False	True	True	False
b10	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False
b11	False	False	True	False	False	True	False	False	False	True	True	False	False	False	True	False	True	False	False	True	False	False	False	False	False	False	False	False	True	True	True	False
b12	False	False	True	False	False	False	False	False	False	True	False	True	False	True	True	False	False	False	False	True	False	True	True	False	False	False	False	False	False	True	True	False
b13	False	False	True	False	False	False	False	False	False	True	False	True	True	True	True	False	True	False	False	True	False	True	True	False	False	False	True	False	False	True	True	False
b14	False	False	True	False	False	False	False	False	False	False	False	False	False	True	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	True	True	False
b15	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b16	False	True	True	False	False	True	False	False	False	True	True	True	True	True	True	True	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b17	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b18	False	False	True	False	False	False	False	False	True	True	False	True	True	True	True	False	True	True	False	True	False	True	True	False	False	False	True	False	False	True	True	True
b19	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	True	False	False	True	False	False	False	False	False	False	False	True	True	False
b20	False	False	True	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	True	False	False	False	False	False	False	False	False	False	True	True	False
b21	False	False	True	False	False	True	False	False	False	True	False	False	False	True	True	False	False	False	True	False	True	True	True	False	False	False	False	False	True	True	True	False
b22	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	True	False
b23	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	True	False
b24	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True
b25	True	True	True	False	False	True	False	False	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b26	True	True	True	False	False	True	False	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True
b27	False	False	True	False	False	False	False	False	False	True	False	False	False	True	True	False	False	False	False	True	False	True	True	False	False	False	True	False	False	True	True	False
b28	False	False	True	False	False	False	False	False	False	True	False	True	False	True	True	False	True	False	False	True	False	True	True	False	False	False	True	True	False	True	True	True
b29	False	False	True	False	False	False	False	False	False	True	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	True	False
b30	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	True	False
b31	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b32	False	False	True	False	False	False	False	False	False	True	False	False	False	True	True	False	False	False	False	True	False	True	True	False	False	False	True	False	False	True	True	True

Possible preference relation – 1st iteration

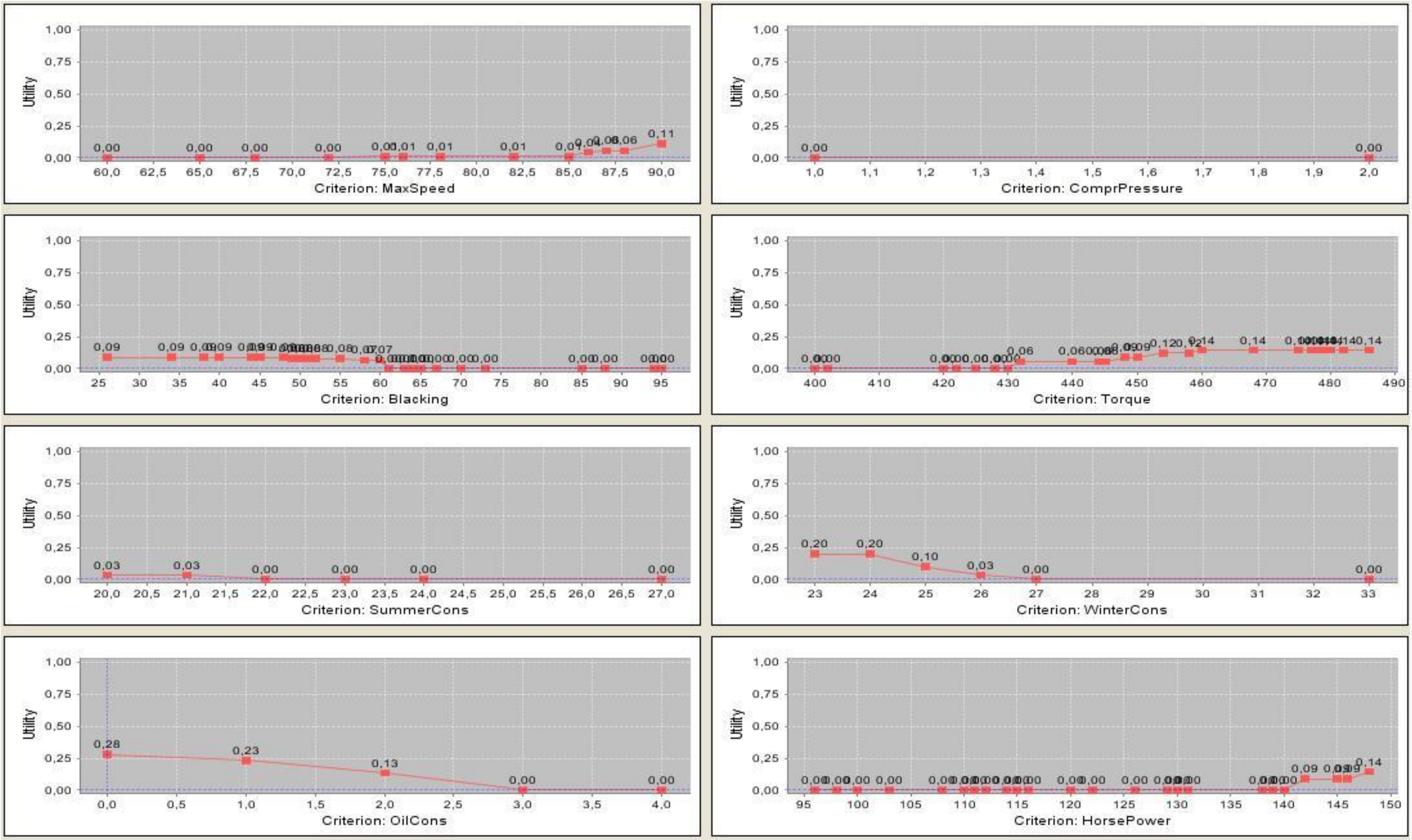
Dominance Relation | Possible Preference Relation | Necessary Preference Relation | Necessary Ranking Graph | Representative Ranking | Marginal Utilities

	b01	b02	b03	b04	b05	b06	b07	b08	b09	b10	b11	b12	b13	b14	b15	b16	b17	b18	b19	b20	b21	b22	b23	b24	b25	b26	b27	b28	b29	b30	b31	b32
b01	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b02	False	True	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b03	False	False	True	False	True	False	False	True	False	True	False	False	False	False	True	False	False	False	True	False	False	True	False	False	False	False	False	False	False	True	True	False
b04	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True
b05	True	True	True	False	True	True	False	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True
b06	False	False	True	False	True	True	True	True	True	True	False	True	True	True	True	False	True	True	True	True	False	True	True	False	False	False	True	True	True	True	True	True
b07	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True
b08	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True
b09	False	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b10	False	False	True	False	True	False	False	True	False	True	False	False	False	True	True	False	False	False	True	True	False	True	True	False	False	False	False	False	False	True	True	False
b11	False	True	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b12	False	False	True	False	True	True	False	True	True	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b13	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b14	False	False	True	False	True	True	False	True	True	True	True	False	False	True	True	False	True	False	True	True	False	True	True	False	False	False	False	False	True	True	True	False
b15	False	False	True	False	True	False	False	False	False	True	False	False	False	False	True	False	True	False	False	False	False	True	True	False	False	False	False	False	False	True	True	False
b16	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b17	False	False	True	False	True	True	False	True	False	True	False	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b18	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b19	False	False	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	False	True	True	False	False	False	True	True	True	True	True	True
b20	False	False	True	False	True	True	False	True	False	True	False	False	False	True	True	False	True	False	True	True	True	True	True	False	False	False	False	False	True	True	True	False
b21	False	False	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b22	False	False	True	False	False	True	False	False	True	True	True	False	False	False	True	False	True	False	False	True	False	True	True	False	False	False	False	False	True	True	True	False
b23	False	False	True	False	True	True	False	True	True	True	True	False	False	True	True	False	True	False	True	True	False	True	True	False	False	False	False	False	True	True	True	False
b24	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True
b25	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b26	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True
b27	False	True	True	False	True	True	False	True	True	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	False
b28	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b29	False	False	True	False	True	True	True	True	True	True	False	True	True	True	True	False	True	True	True	True	False	True	True	False	False	False	True	True	True	True	True	True
b30	False	False	True	False	True	False	False	True	False	True	False	False	False	False	True	False	True	False	False	False	False	True	True	False	False	False	False	False	True	True	True	False
b31	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b32	False	True	True	False	True	True	False	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True

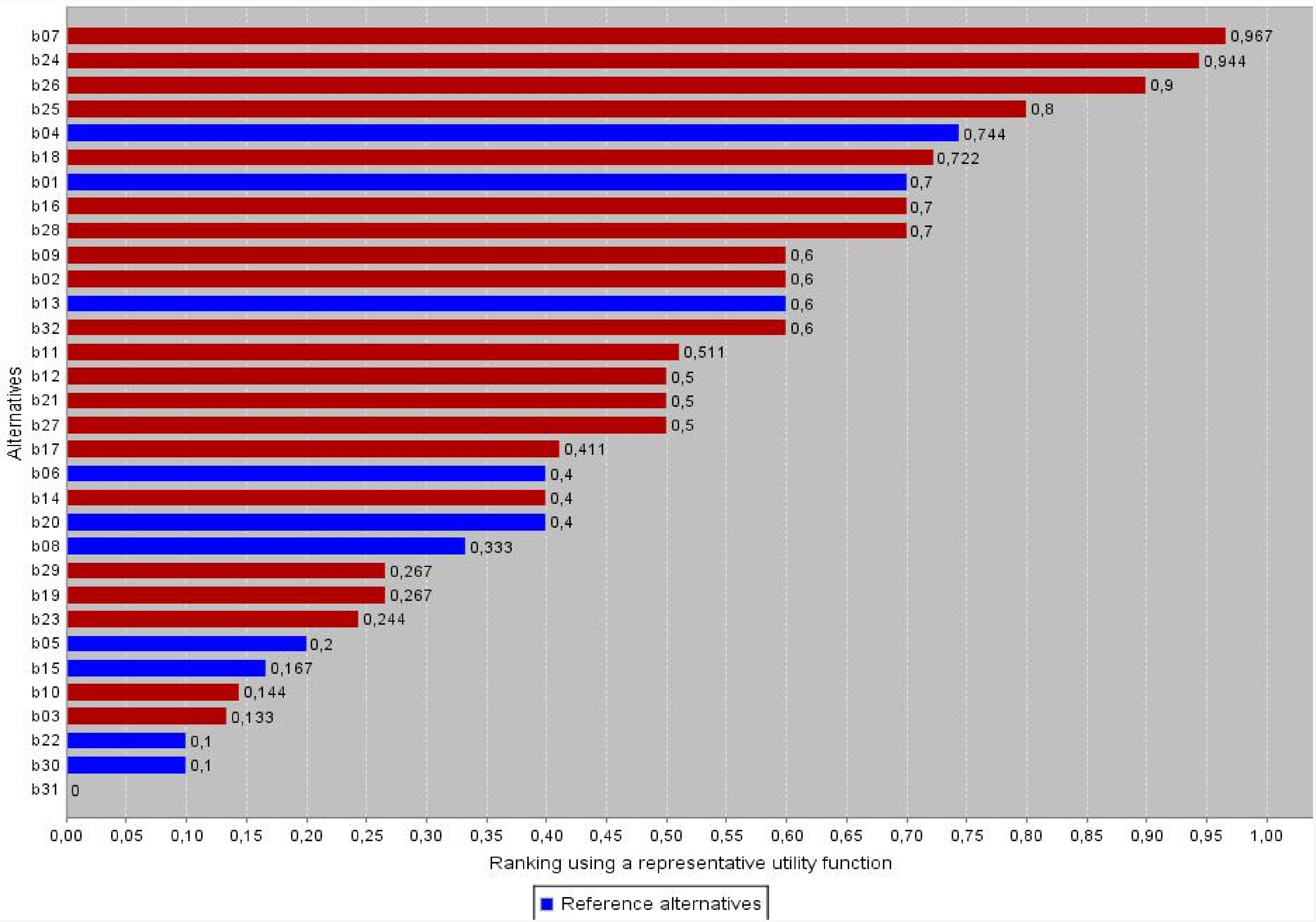
Necessary ranking graph – 1st iteration



Representative value function – 1st iteration



Representative ranking – 1st iteration



Rules induced from PCT representing necessary preference relation

All rules: 313 Minimal cover rules: 11

1	(SummerCons_x $\leq$ 20) & (HorsePower_y $\leq$ 146) => (Rel = NEC)
2	(Torque_x $\geq$ 486) & (WinterCons_y $\geq$ 25) => (Rel = NEC)
3	(SummerCons_y $\geq$ 24) & (HorsePower_y $\leq$ 115) & (WinterCons_x $\leq$ 27) => (Rel = NEC)
4	(WinterCons_y $\geq$ 27) & (Torque_x $\geq$ 444) => (Rel = NEC)
5	(Torque_x$\geq$468)&(MaxSpeed_y$\leq$87)&(SummerCons_x$\leq$21)&(WinterCons_y$\geq$25)=> (Rel=NEC)
6	(WinterCons_x $\leq$ 24) & (HorsePower_y $\leq$ 98) => (Rel = NEC)
7	(WinterCons_x $\leq$ 23) & (HorsePower_y $\leq$ 108) => (Rel = NEC)
8	(WinterCons_y $\geq$ 33) & (HorsePower_x $\geq$ 116) & (MaxSpeed_x $\geq$ 76) => (Rel = NEC)
9	(Blacking_x $\leq$ 26) & (SummerCons_y $\geq$ 22) & (WinterCons_y $\geq$ 24) => (Rel = NEC)
10	(MaxSpeed_x $\geq$ 90) & (WinterCons_y $\geq$ 26) & (SummerCons_y $\geq$ 22) => (Rel = NEC)
11	(SummerCons_y $\geq$ 23) & (Blacking_x $\leq$ 55) & (WinterCons_y $\geq$ 26) => (Rel = NEC)

Rules induced from PCT representing possible preference relation

All rules: 930 Minimal cover rules: 9

1	(HorsePower_x $\geq$ 146) => (Rel = POSSIBLE)
2	(WinterCons_x $\leq$ 24) & (WinterCons_y $\geq$ 25) => (Rel = POSSIBLE)
3	(HorsePower_y $\leq$ 110) & (HorsePower_x $\geq$ 110) => (Rel = POSSIBLE)
4	(WinterCons_y $\geq$ 33) => (Rel = POSSIBLE)
5	(SummerCons_x $\leq$ 21) & (SummerCons_y $\geq$ 22) => (Rel = POSSIBLE)
6	(HorsePower_y $\leq$ 100) & (WinterCons_x $\leq$ 24) => (Rel = POSSIBLE)
7	(WinterCons_x $\leq$ 23) & (SummerCons_y $\geq$ 22) => (Rel = POSSIBLE)
8	(MaxSpeed_y $\leq$ 76) & (HorsePower_x $\geq$ 114) => (Rel = POSSIBLE)
9	(Blacking_y $\geq$ 61) & (Blacking_x $\leq$ 61) => (Rel = POSSIBLE)

Preference information given by the DM – 2nd iteration

- The DM adds new preference information:
  - b05 $\sim$ b15

Necessary preference relation – 2nd iteration

Dominance Relation | Possible Preference Relation | Necessary Preference Relation | Necessary Ranking Graph | Representative Ranking | Marginal Utilities

	b01	b02	b03	b04	b05	b06	b07	b08	b09	b10	b11	b12	b13	b14	b15	b16	b17	b18	b19	b20	b21	b22	b23	b24	b25	b26	b27	b28	b29	b30	b31	b32
b01	True	True	True	False	True	True	False	False	True	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b02	False	True	True	False	True	True	False	False	False	True	False	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	False	False	True	True	True	False
b03	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b04	False	True	True	True	True	True	False	True	False	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b05	False	False	False	False	True	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	True	False
b06	False	False	True	False	True	True	False	False	False	True	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	True	True	False
b07	False	False	True	False	True	False	True	False	True	True	False	True	True	True	True	False	True	True	False	True	False	True	True	False	False	False	True	True	False	True	True	True
b08	False	False	False	False	True	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	True	False
b09	False	False	True	False	True	False	False	False	True	True	False	False	False	False	True	False	True	False	False	True	False	True	False	False	False	False	False	False	False	True	True	False
b10	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False
b11	False	False	True	False	True	True	False	False	False	True	True	False	False	False	True	False	True	False	False	True	False	True	False	False	False	False	False	False	True	True	True	False
b12	False	False	True	False	True	False	False	False	False	True	False	True	False	True	True	False	False	False	False	True	False	True	True	False	False	False	False	False	False	True	True	False
b13	False	False	True	False	True	False	False	False	False	True	False	True	True	True	True	False	True	False	False	True	False	True	True	False	False	False	True	False	False	True	True	False
b14	False	False	True	False	True	False	False	False	False	False	False	False	False	True	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	True	True	False
b15	False	False	False	False	True	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	True	False
b16	False	True	True	False	True	True	False	False	False	True	True	True	True	True	True	True	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b17	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b18	False	False	True	False	True	False	False	False	True	True	False	True	True	True	True	False	True	True	False	True	False	True	True	False	False	False	True	False	False	True	True	True
b19	False	False	False	False	True	False	False	False	False	False	False	False	False	False	True	False	False	False	True	False	False	True	False	False	False	False	False	False	False	True	True	False
b20	False	False	True	False	True	False	False	False	False	False	False	False	False	False	True	False	False	False	False	True	False	True	False	False	False	False	False	False	False	True	True	False
b21	False	False	True	False	True	True	False	False	False	True	False	False	False	True	True	False	False	False	True	False	True	True	True	False	False	False	False	False	True	True	True	False
b22	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	True	False
b23	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	True	False
b24	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True
b25	True	True	True	False	True	True	False	False	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b26	True	True	True	False	True	True	False	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True
b27	False	False	True	False	True	False	False	False	False	True	False	False	False	True	True	False	False	False	False	True	False	True	True	False	False	False	True	False	False	True	True	False
b28	False	False	True	False	True	False	False	False	False	True	False	True	False	True	True	False	True	False	False	True	False	True	True	False	False	False	True	True	False	True	True	True
b29	False	False	True	False	True	False	False	False	False	True	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	True	False
b30	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	True	False
b31	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b32	False	False	True	False	True	False	False	False	False	True	False	False	False	True	True	False	False	False	False	True	False	True	True	False	False	False	True	False	False	True	True	True

Necessary preference relation – 1st iteration

Dominance Relation | Possible Preference Relation | Necessary Preference Relation | Necessary Ranking Graph | Representative Ranking | Marginal Utilities

	b01	b02	b03	b04	b05	b06	b07	b08	b09	b10	b11	b12	b13	b14	b15	b16	b17	b18	b19	b20	b21	b22	b23	b24	b25	b26	b27	b28	b29	b30	b31	b32
b01	True	True	True	False	False	True	False	False	True	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b02	False	True	True	False	False	True	False	False	False	True	False	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	False	False	True	True	True	False
b03	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b04	False	True	True	True	True	True	False	True	False	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b05	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	True	False
b06	False	False	True	False	False	True	False	False	False	True	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	True	False
b07	False	False	True	False	True	False	True	False	True	True	False	True	True	True	True	False	True	True	False	True	False	True	True	False	False	False	True	True	False	True	True	True
b08	False	False	False	False	True	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	True	False
b09	False	False	True	False	False	False	False	False	True	True	False	False	False	False	True	False	True	False	False	True	False	False	False	False	False	False	False	False	False	True	True	False
b10	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False
b11	False	False	True	False	False	True	False	False	False	True	True	False	False	False	True	False	True	False	False	True	False	False	False	False	False	False	False	False	True	True	True	False
b12	False	False	True	False	False	False	False	False	False	True	False	True	False	True	True	False	False	False	False	True	False	True	True	False	False	False	False	False	False	True	True	False
b13	False	False	True	False	False	False	False	False	False	True	False	True	True	True	True	False	True	False	False	True	False	True	True	False	False	False	True	False	False	True	True	False
b14	False	False	True	False	False	False	False	False	False	False	False	False	False	True	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	True	True	False
b15	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b16	False	True	True	False	False	True	False	False	False	True	True	True	True	True	True	True	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b17	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b18	False	False	True	False	False	False	False	False	True	True	False	True	True	True	True	False	True	True	False	True	False	True	True	False	False	False	True	False	False	True	True	True
b19	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	True	False	False	True	False	False	False	False	False	False	False	True	True	False
b20	False	False	True	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	True	False	False	False	False	False	False	False	False	False	True	True	False
b21	False	False	True	False	False	True	False	False	False	True	False	False	False	True	True	False	False	False	True	False	True	True	True	False	False	False	False	False	True	True	True	False
b22	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	True	False
b23	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	True	False
b24	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True
b25	True	True	True	False	False	True	False	False	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b26	True	True	True	False	False	True	False	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True
b27	False	False	True	False	False	False	False	False	False	True	False	False	False	True	True	False	False	False	False	True	False	True	True	False	False	False	True	False	False	True	True	False
b28	False	False	True	False	False	False	False	False	False	True	False	True	False	True	True	False	True	False	False	True	False	True	True	False	False	False	True	True	False	True	True	True
b29	False	False	True	False	False	False	False	False	False	True	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False	True	False
b30	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	True	False
b31	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b32	False	False	True	False	False	False	False	False	False	True	False	False	False	True	True	False	False	False	False	True	False	True	True	False	False	False	True	False	False	True	True	True

Possible preference relation – 2nd iteration

Dominance Relation | Possible Preference Relation | Necessary Preference Relation | Necessary Ranking Graph | Representative Ranking | Marginal Utilities

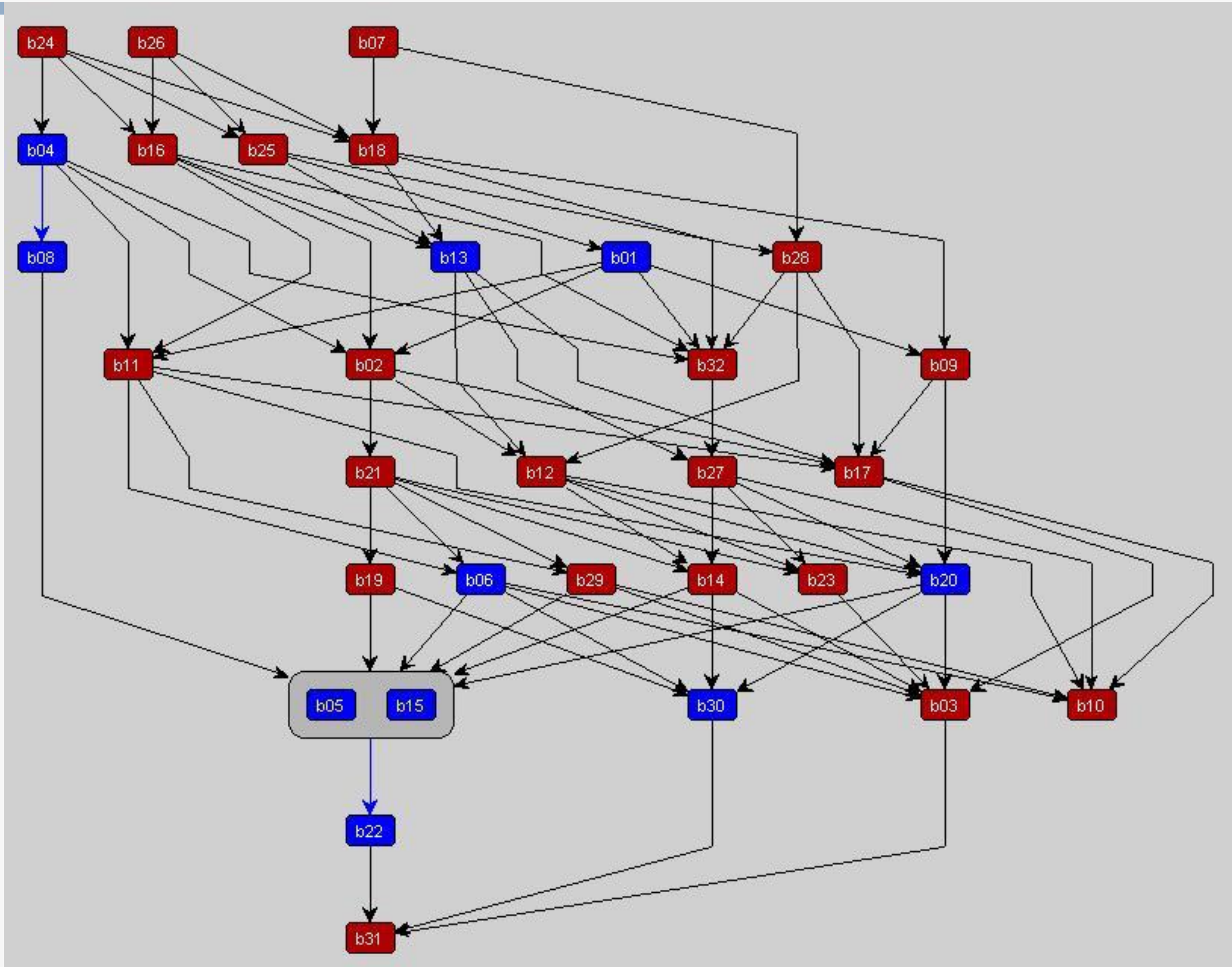
	b01	b02	b03	b04	b05	b06	b07	b08	b09	b10	b11	b12	b13	b14	b15	b16	b17	b18	b19	b20	b21	b22	b23	b24	b25	b26	b27	b28	b29	b30	b31	b32
b01	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b02	False	True	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b03	False	False	True	False	True	False	False	True	False	True	False	False	False	False	True	False	False	False	True	False	False	True	False	False	False	False	False	False	False	True	True	False
b04	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True
b05	False	False	True	False	True	False	False	False	False	True	False	False	False	False	True	False	True	False	False	False	False	True	True	False	False	False	False	False	False	True	True	False
b06	False	False	True	False	True	True	True	True	True	True	False	True	True	True	True	False	True	True	True	True	False	True	True	False	False	False	True	True	True	True	True	True
b07	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True
b08	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True
b09	False	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b10	False	False	True	False	True	False	False	True	False	True	False	False	False	True	True	False	False	False	True	True	False	True	True	False	False	False	False	False	False	True	True	False
b11	False	True	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b12	False	False	True	False	True	True	False	True	True	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b13	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b14	False	False	True	False	True	True	False	True	True	True	True	False	False	True	True	False	True	False	True	True	False	True	True	False	False	False	False	False	True	True	True	False
b15	False	False	True	False	True	False	False	False	False	True	False	False	False	False	True	False	True	False	False	False	False	True	True	False	False	False	False	False	False	True	True	False
b16	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b17	False	False	True	False	True	True	False	True	False	True	False	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b18	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b19	False	False	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	False	True	True	False	False	False	True	True	True	True	True	True
b20	False	False	True	False	True	True	False	True	False	True	False	False	False	True	True	False	True	False	True	True	True	True	True	False	False	False	False	False	True	True	True	False
b21	False	False	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b22	False	False	True	False	False	False	False	False	False	True	False	False	False	False	False	False	True	False	False	False	False	True	True	False	False	False	False	False	False	True	True	False
b23	False	False	True	False	True	True	False	True	True	True	True	False	False	True	True	False	True	False	True	True	False	True	True	False	False	False	False	False	True	True	True	False
b24	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True
b25	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b26	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True
b27	False	True	True	False	True	True	False	True	True	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	False
b28	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b29	False	False	True	False	True	True	True	True	True	True	False	True	True	True	True	False	True	True	True	True	False	True	True	False	False	False	True	True	True	True	True	True
b30	False	False	True	False	True	False	False	True	False	True	False	False	False	False	True	False	True	False	False	False	False	True	True	False	False	False	False	False	True	True	True	False
b31	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b32	False	True	True	False	True	True	False	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True

Possible preference relation – 1st iteration

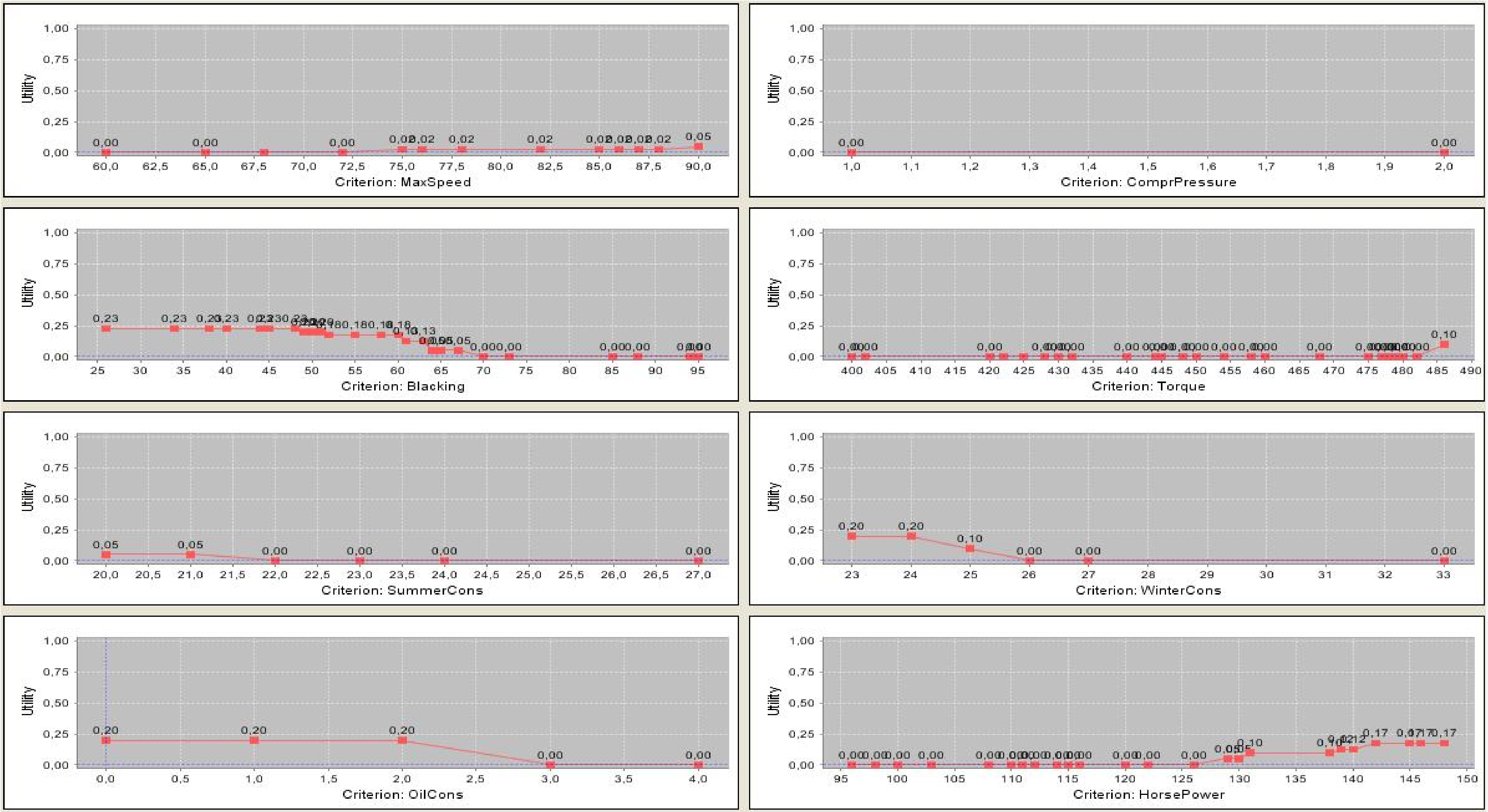
Dominance Relation | Possible Preference Relation | Necessary Preference Relation | Necessary Ranking Graph | Representative Ranking | Marginal Utilities

	b01	b02	b03	b04	b05	b06	b07	b08	b09	b10	b11	b12	b13	b14	b15	b16	b17	b18	b19	b20	b21	b22	b23	b24	b25	b26	b27	b28	b29	b30	b31	b32
b01	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b02	False	True	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b03	False	False	True	False	True	False	False	True	False	True	False	False	False	False	True	False	False	False	True	False	False	True	False	False	False	False	False	False	False	True	True	False
b04	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True
b05	True	True	True	False	True	True	False	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True
b06	False	False	True	False	True	True	True	True	True	True	False	True	True	True	True	False	True	True	True	True	False	True	True	False	False	False	True	True	True	True	True	True
b07	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True
b08	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True
b09	False	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b10	False	False	True	False	True	False	False	True	False	True	False	False	False	True	True	False	False	False	True	True	False	True	True	False	False	False	False	False	False	True	True	False
b11	False	True	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b12	False	False	True	False	True	True	False	True	True	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b13	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b14	False	False	True	False	True	True	False	True	True	True	True	False	False	True	True	False	True	False	True	True	False	True	True	False	False	False	False	False	True	True	True	False
b15	False	False	True	False	True	False	False	False	False	True	False	False	False	False	True	False	True	False	False	False	False	True	True	False	False	False	False	False	False	True	True	False
b16	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b17	False	False	True	False	True	True	False	True	False	True	False	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True
b18	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b19	False	False	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	False	True	True	False	False	False	True	True	True	True	True	True
b20	False	False	True	False	True	True	False	True	False	True	False	False	False	True	True	False	True	False	True	True	True	True	True	False	False	False	False	False	True	True	True	False
b21	False	False	True	False	True	True	True	True	True	True	True	True	True	True	True	False	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b22	False	False	True	False	False	True	False	False	True	True	True	False	False	False	True	False	True	False	False	True	False	True	True	False	False	False	False	False	True	True	True	False
b23	False	False	True	False	True	True	False	True	True	True	True	False	False	True	True	False	True	False	True	True	False	True	True	False	False	False	False	False	True	True	True	False
b24	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True
b25	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	True
b26	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True
b27	False	True	True	False	True	True	False	True	True	True	True	True	False	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	False
b28	True	True	True	True	True	True	False	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	True	False	False	False	True	True	True	True	True	True
b29	False	False	True	False	True	True	True	True	True	True	False	True	True	True	True	False	True	True	True	True	False	True	True	False	False	False	True	True	True	True	True	True
b30	False	False	True	False	True	False	False	True	False	True	False	False	False	False	True	False	True	False	False	False	False	True	True	False	False	False	False	False	True	True	True	False
b31	False	False	False	False	False	False	False	False	False	True	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	False	True	False
b32	False	True	True	False	True	True	False	True	True	True	True	True	True	True	True	False	True	False	True	True	True	True	True	False	False	False	True	False	True	True	True	True

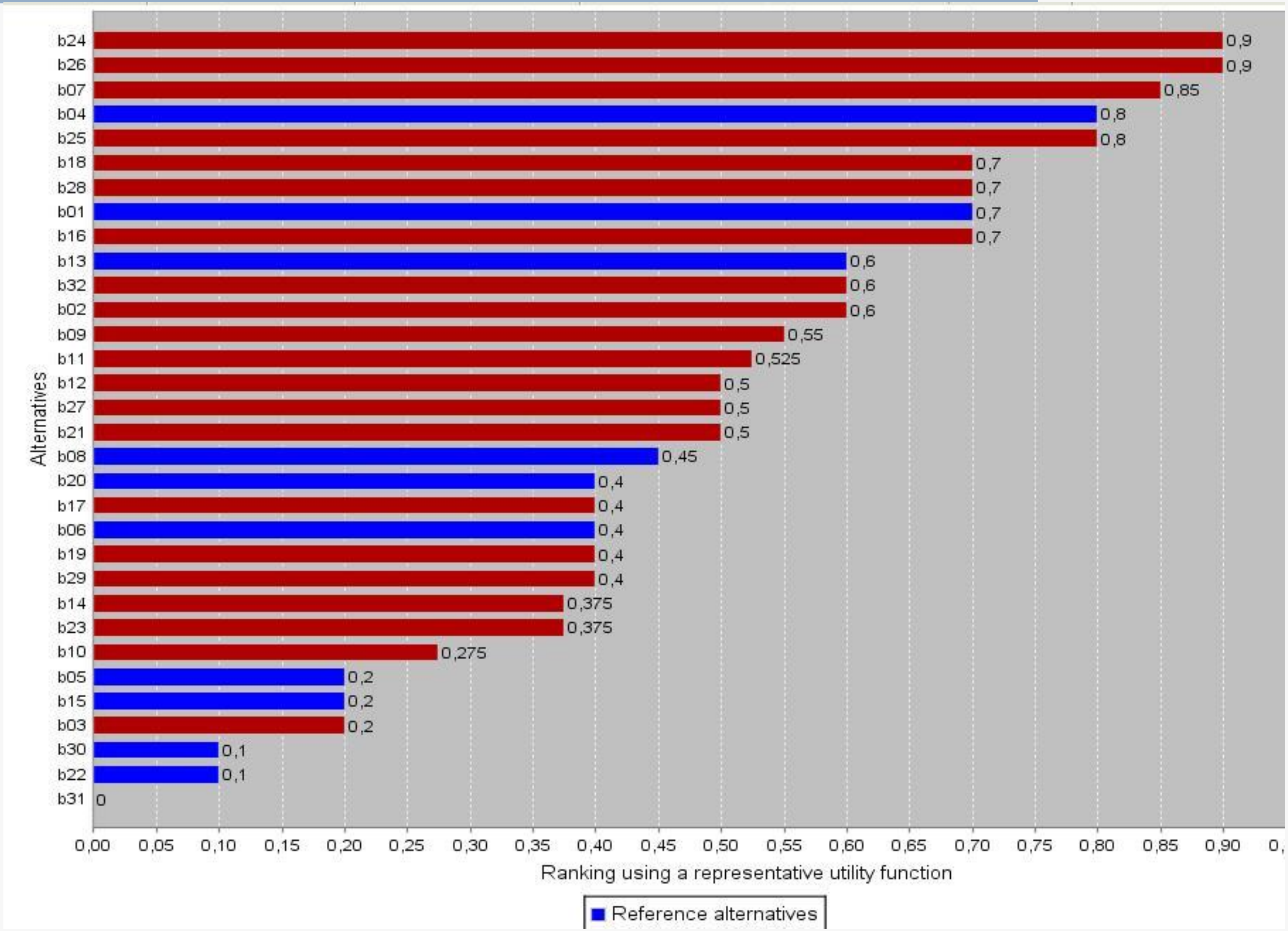
Necessary ranking graph – 2nd iteration



Representative value function – 2^{nd} iteration



Representative ranking – 2nd iteration



Rules induced from PCT representing necessary preference relation

All rules: 426 (313) Minimal cover rules: 11 (4,5,6,8,9,10 are new)

1	(SummerCons_x $\leq$ 20) & (HorsePower_y $\leq$ 146) => (Rel = NEC)
2	(Torque_x $\geq$ 486) & (WinterCons_y $\geq$ 25) => (Rel = NEC)
3	(HorsePower_y $\leq$ 108) & (WinterCons_x $\leq$ 23) => (Rel = NEC)
4	(WinterCons_y $\geq$ 27) & (Blacking_x $\leq$ 58) => (Rel = NEC)
5	(HorsePower_y $\leq$ 100) & (Blacking_x $\leq$ 73) & (SummerCons_x $\leq$ 22) => (Rel = NEC)
6	(Torque_x$\geq$468)&(SummerCons_x$\leq$21)&WinterCons_y$\geq$24)&(MaxSpeed_y$\leq$87)=> (Rel=NEC)
7	(WinterCons_y $\geq$ 33) & (HorsePower_x $\geq$ 116) & (MaxSpeed_x $\geq$ 76) => (Rel = NEC)
8	(WinterCons_x $\leq$ 24) & (HorsePower_y $\leq$ 100) => (Rel = NEC)
9	(OilCons_y $\geq$ 3) & (WinterCons_x $\leq$ 24) & (HorsePower_y $\leq$ 108) => (Rel = NEC)
10	(Blacking_x $\leq$ 26) & (SummerCons_y $\geq$ 22) & (WinterCons_y $\geq$ 24) => (Rel = NEC)
11	(SummerCons_y $\geq$ 22) & (WinterCons_y $\geq$ 26) & (MaxSpeed_x $\geq$ 90) => (Rel = NEC)

Rules induced from PCT representing possible preference relation

All rules: 904 (930)

Minimal cover rules: 9 (2,4,6,8 are new)

1	(HorsePower_x $\geq$ 146) => (Rel = POSSIBLE)
2	(SummerCons_y $\geq$ 24) => (Rel = POSSIBLE)
3	(HorsePower_x $\geq$ 110) & (HorsePower_y $\leq$ 110) => (Rel = POSSIBLE)
4	(HorsePower_y $\leq$ 100) & (OilCons_x $\leq$ 2) => (Rel = POSSIBLE)
5	(SummerCons_x $\leq$ 21) & (SummerCons_y $\geq$ 22) => (Rel = POSSIBLE)
6	(WinterCons_x $\leq$ 24) & (SummerCons_y $\geq$ 23) => (Rel = POSSIBLE)
7	(WinterCons_x $\leq$ 23) & (SummerCons_y $\geq$ 22) => (Rel = POSSIBLE)
8	(WinterCons_x $\leq$ 24)&(SummerCons_x $\leq$ 22)&(WinterCons_y $\geq$ 25)=> (Rel = POS)
9	(Blacking_x $\leq$ 61) & (Blacking_y $\geq$ 61) => (Rel = POSSIBLE)

Rules changing from possible to necessary in new iteration

- Observe the pair: (b11, b05)
- In the 1st iteration: b11 $\succeq^P$ b05 (covered by 459 possible rules)
- In the 2nd iteration: b11 $\succeq^N$ b05 (covered by 97 necessary rules

Strength	Minimal cover rules supported by (b11, b05)
266	(Blacking_x $\leq$ 61) & (Blacking_y $\geq$ 61) => (Rel = POSSIBLE)
264	(MaxSpeed_y $\leq$ 76) & (HorsePower_x $\geq$ 114) => (Rel = POSSIBLE)
240	(SummerCons_x $\leq$ 21) & (SummerCons_y $\geq$ 22) => (Rel = POSSIBLE)
162	(HorsePower_x $\geq$ 110) & (HorsePower_y $\leq$ 110) => (Rel = POSSIBLE)
72	(Blacking_x$\leq$73)&(SummerCons_x$\leq$22)&(HorsePower_y$\leq$100)=> (Rel = NEC)

Analysing the strongest rules covering the pair (b11, b05)

- 1st iteration

Strength	Possible rules
345	(Blacking_x <= 64) & (HorsePower_y <= 120) => (Rel = POSSIBLE)
345	(Torque_x >= 430) & (HorsePower_y <= 120) => (Rel = POSSIBLE)
323	(Torque_x >= 444) & (Torque_y <= 450) => (Rel = POSSIBLE)
323	(MaxSpeed_x >= 85) & (Torque_y <= 450) => (Rel = POSSIBLE)
315	(MaxSpeed_x >= 78) & (HorsePower_y <= 120) => (Rel = POSSIBLE)
304	(MaxSpeed_x >= 85) & (HorsePower_y <= 122) => (Rel = POSSIBLE)
304	(Torque_x >= 444) & (HorsePower_y <= 122) => (Rel = POSSIBLE)

- 2nd iteration

Strength	Necessary rules
120	(SummerCons_x <= 22) & (OilCons_x <= 1) & (Blacking_y >= 70) => (Rel = NECESSARY)
108	(Blacking_x <= 61) & (SummerCons_x <= 22) & (Blacking_y >= 70) => (Rel = NECESSARY)
108	(Torque_x >= 444) & (SummerCons_x <= 22) & (Blacking_y >= 70) => (Rel = NECESSARY)
102	(Torque_x >= 448) & (Blacking_y >= 70) => (Rel = NECESSARY)
102	(HorsePower_x >= 122) & (Blacking_y >= 70) => (Rel = NECESSARY)
102	(Torque_x >= 448) & (Blacking_y >= 67) & (WinterCons_y >= 24) => (Rel = NECESSARY)
102	(Torque_x >= 448) & (Torque_y <= 425) & (WinterCons_y >= 24) => (Rel = NECESSARY)
102	(HorsePower_x >= 122) & (Blacking_y >= 67) & (WinterCons_y >= 24) => (Rel = NECESSARY)
102	(HorsePower_x >= 122) & (Torque_y <= 425) & (WinterCons_y >= 24) => (Rel = NECESSARY)

Other methodological extensions of DRSA

- DRSA for Choice and Ranking with multi-graded preference relations
- DRSA as a Way of Handling Fuzzy-Rough Hybridization
- DRSA for Case-Based Reasoning
- DRSA for Decision Under Uncertainty and Time Preference
- DRSA for Ordinal Classification with Imprecise or Missing Evaluations and Assignments
- DRSA for Hierarchical Structure of Attributes and Criteria
- DRSA for Financial Portfolio Decision
- DRSA for Customer Satisfaction Analysis
- Robustness analysis for multiple criteria ranking and choice
- Robustness analysis for decision under uncertainty and group decision

Conclusions

Conclusions

- Monotonic "*if*..., *then*..." decision rules give account of most complex interactions among attributes, require weaker axioms than other preference models, and can represent inconsistent preferences
- Heterogeneous information (attributes, criteria) and attribute scales (ordinal, cardinal) can be handled by DRSA.
- DRSA exploits ordinal information only, and decision rules do not convert ordinal information into numeric one.
- DRSA supplies useful elements of knowledge about decision situation:
  - certain and doubtful knowledge distinguished by lower and upper appx.
  - relevance of particular attributes and information about their interaction,
  - reducts & core of attributes conveying important knowledge contained in data,
  - decision rules can be used for explanation of past decisions, for decision support and for strategic interventions.
- DRSA has sound theoretical foundations (bipolar algebra, bitopology, Bayesian confirmation theory)

Software available on the web

ROSE

ROugh Set data Explorer

http://idss.cs.put.poznan.pl/site/rose.html

jMAF & jRank

Decision Support Tools for Rule-based Analysis and Solving of Multi-Attribute and Multi-Criteria Decision Problems

http://www.cs.put.poznan.pl/jblaszczynski/Site/jRS.html

http://www.cs.put.poznan.pl/mszelag/Software/jRank/jRank.html

THANK YOU!

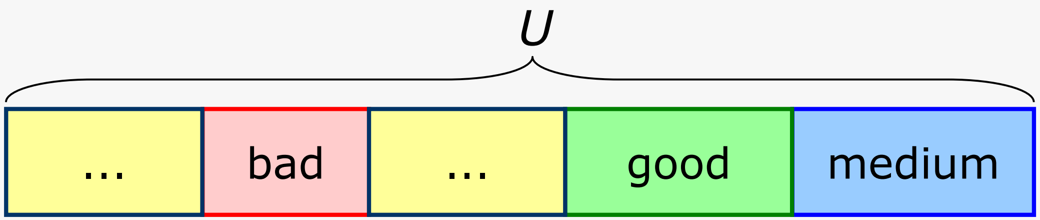
http://idss.cs.put.poznan.pl/site/software.html


Thank you

Algebra for DRSA

Algebraic models for DRSA: bipolarity

- In Classical Rough Set Theory, one approximates subsets of *U*, e.g.:
  - bad objects,
  - medium objects,
  - good objects.



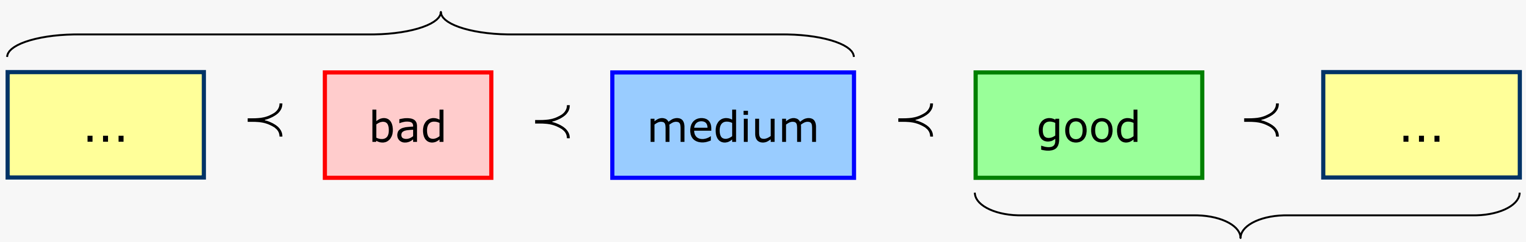
- In Dominance-based Rough Set Theory, one approximates unions of ordered subsets of *U* – downward unions, e.g.:
  - at most bad objects (i.e., bad or worse objects)
  - at most medium objects (i.e., medium or worse objects)

... $\prec$ bad $\prec$ medium $\prec$ good $\prec$ ...

- and upward unions, e.g.:
  - at least medium objects (i.e. medium or better objects)
  - at least good objects (i.e. good or better objects)

Algebraic models for DRSA: bipolarity

- Important remarks
  - Lower & upper approximations for downward unions are different operators from lower & upper approximations for upward unions
  - The complement of a downward union is an upward union, and vice versa, e.g.:



Bipolar disjoint representation using DRSA

- In general, given a finite set of objects (universe) U, we consider a partial preorder relation R on U (i.e., R is reflexive and transitive)
- R can be a dominance relation w.r.t. a subset of properties
- For any object $y \in U$, the elementary sets (granules) used for building rough approximations are:

  $$R^{+}(y)=\{x \in U\text{: } xRy\} \quad \text{and} \quad R^{-}(y)=\{x \in U\text{: } yRx\}$$

  (e.g., positive and negative dominance cones)

Bipolar disjoint representation using DRSA

- For every set $X \subseteq U$, we define its upward lower approximation and its upward upper approximation :

$$\underline{R}^{(>)}(X) = \left\{x \in U \colon R^{+}(x) \subseteq X\right\}$$

$$\overline{R}^{(>)}(X) = \left\{x \in U \colon R^{-}(x) \cap X \neq \varnothing\right\}$$

- Analogously, we define downward lower approximation and downward upper approximation of set $X \subseteq U$:

$$\underline{R}^{(<)}(X) = \left\{x \in U \colon R^{-}(x) \subseteq X\right\}$$

$$\overline{R}^{(<)}(X) = \left\{x \in U \colon R^{+}(x) \cap X \neq \varnothing\right\}$$

Bipolar disjoint representation using DRSA

- According to Rough Set philosophy, each concept X is represented by the pair (I, E), where
  - I (the interior) is the lower approximation of set $X \subseteq U$
  - E (the exterior) is the complement in U of the upper approx. of X

I is the set of objects that certainly belong to the concept

E is the set of objects that certainly <u>do not</u> belong to the concept

Bipolar disjoint representation using DRSA

- The algebraic structures for DRSA are based on representation of the approximations of X in terms of pairs (I^+, E^+) and (I^-, E^-), called bipolar disjoint representation (BDR):

positive interior and exterior of concept X:

$$I^+(X) = \underline{R}^{(>)}(X), \quad E^+(X) = U - \overline{R}^{(>)}(X)$$

negative interior and exterior of concept X:

$$I^-(X) = \underline{R}^{(<)}(X), \quad E^-(X) = U - \overline{R}^{(<)}(X)$$

Bipolar disjoint representation using DRSA

- In the context of BDR, union and intersection of sets is represented by the operation of **join** $\vee$ and **meet** $\wedge$

$$\big(I(X), E(X)\big) \vee \big(I(Y), E(Y)\big) = \big(I(X) \cup I(Y),\ E(X) \cap E(Y)\big)$$
$$\big(I(X), E(X)\big) \wedge \big(I(Y), E(Y)\big) = \big(I(X) \cap I(Y),\ E(X) \cup E(Y)\big)$$

- The same formula holds for pairs (I^+, E^+) and (I^-, E^-)

Bipolar disjoint representation using DRSA

- Moreover, one can use different bipolar negations, e.g.:
  - Kleene complementations $'^{+} : \Sigma^{+} \rightarrow \Sigma^{-}$ and $'^{-} : \Sigma^{-} \rightarrow \Sigma^{+}$

$$\left(I^{+}(X), E^{+}(X)\right)'^{+} = \left(E^{+}(X), I^{+}(X)\right)$$
$$\left(I^{-}(X), E^{+}(X)\right)'^{-} = \left(E^{-}(X), I^{-}(X)\right)$$

  - Brouwer complementations $\sim^{+} : \Sigma^{+} \rightarrow \Sigma^{-}$ and $\sim^{-} : \Sigma^{-} \rightarrow \Sigma^{+}$

$$\left(I^{+}(X), E^{+}(X)\right)^{\sim +} = \left(E^{+}(X), U - E^{+}(X)\right)$$
$$\left(I^{-}(X), E^{+}(X)\right)^{\sim -} = \left(E^{-}(X), U - E^{-}(X)\right)$$

Algebraic models for DRSA: bipolarity

- A typical algebra for Classical Rough Set Theory:
  - System $\langle \Sigma, \wedge, \vee, ', \sim, 0, 1 \rangle$

    is a Brouwer-Zadeh distributive lattice

    if the following properties (required by rough approximations) hold …

- A typical algebra for Dominance-based Rough Set Theory:
  - System $\langle \Sigma, \Sigma^{+}, \Sigma^{-}, \wedge, \vee, '^{+}, '^{-}, \sim^{+}, \sim^{-}, 0, 1 >$

    is a bipolar Brouwer-Zadeh distributive lattice

    if the following properties (required by DRSA approximations) hold…

Algebraic models for DRSA: bipolarity

- One can generalize all algebra models proposed for Classical Rough Sets to Dominance-based Rough Sets:
  - Nelson algebra $\rightarrow$ Bipolar Nelson algebra
  - Heyting algebra $\rightarrow$ Bipolar Heyting algebra
  - Wajsberg algebra $\rightarrow$ Bipolar Wajsberg algebra
  - Stone algebra $\rightarrow$ Bipolar Stone algebra
  - Łukasiewicz algebra $\rightarrow$ Bipolar Łukasiewicz algebra
  - Brouwer-Zadeh algebra $\rightarrow$ Bipolar Brouwer-Zadeh algebra
  - ...
- These algebra models give elegant representations of basic properties of Dominance–based Rough Sets

S.Greco, B.Matarazzo, R.Słowiński: Algebra and Topology for Dominance-based Rough Set Approach. [In]: Z.W.Raś, L.-S.Tsay (eds.), *Advances in Intelligent Information Systems*, Studies in Computational Intelligence, vol. 265, Springer, Berlin, 2010, pp. 43-78

Bipolar de Morgan Brouwer-Zadeh lattice as a model for DRSA

- A proper algebraic structure for ordinal classification with monotonicity constraints is a bipolar de Morgan Brouwer-Zadeh lattice

$$\boxed{<\Sigma,\Sigma^{+},\Sigma^{-},\wedge,\vee,{}'^{+},{}'^{-},\sim^{+},\sim^{-},0,1>}$$

where

$\Sigma=\{(I,E)\colon\ I,E\subseteq U\ \text{ and }\ I\cap E=\varnothing\}$ - set of concepts

$\Sigma^{+}=\{(I,E)\colon\ \exists X\in U\ \text{ such that }\ I=I^{+}(X),\ \ E=E^{+}(X)\}$ - positive concepts

$\Sigma^{-}=\{(I,E)\colon\ \exists Y\in U\ \text{ such that }\ I=I^{-}(Y),\ \ E=E^{-}(Y)\}$ - negative concepts

S.Greco, B.Matarazzo, R.Słowiński: The bipolar complemented de Morgan Brouwer-Zadeh distributive lattice as an algebraic structure for the Dominance-based Rough Set Approach. *Fundamenta Informaticae*, 115 (2012) 25–56

S.Greco, B.Matarazzo, R.Słowiński: On Topological Dominance-based Rough Set Approach. *Transactions on Rough Sets XII* (LNCS series, vol. 6190), Springer, Berlin, 2010, pp.21-45.

Bipolar quasi Brouwer-Zadeh Distributive Lattices (1)

- A system $<\Sigma,\Sigma^+,\Sigma^-,\wedge,\vee,'^+,'^-,\sim^+,\sim^-,0,1>$ is a bipolar quasi Brouwer-Zadeh distributive lattice if the following properties hold:
- $<\Sigma,\wedge,\vee,0,1>$ is a distributive lattice
- $<\Sigma^+,\wedge,\vee,0,1>$, $<\Sigma^-,\wedge,\vee,0,1>$ are distributive lattices with $\Sigma^+,\Sigma^-\subseteq\Sigma$
- $'^+: \Sigma^+\to\Sigma^-$ and $'^-: \Sigma^-\to\Sigma^+$ are bipolar Kleene complementations, that is for all $a,b\in\Sigma^+$ and $c,d\in\Sigma^-$
  - (K1b) $a'^{+'-} = a$, $c'^{-'+} = c$
  - (K2b) $(a \vee b)'^+ = a'^+ \wedge b'^+$, $(c \vee d)'^- = c'^- \wedge d'^-$
  - (K3b) $a \wedge a'^+ \leq b \vee b'^+$, $c \wedge c'^- \leq d \vee d'$

Bipolar Quasi Brouwer-Zadeh Distributive Lattices (2)

- $\sim^{+}:\Sigma^{+}\rightarrow\Sigma^{-}$ and $\sim^{-}:\Sigma^{-}\rightarrow\Sigma^{+}$ are bipolar Brouwer complementations, that is for all $a,b\in\Sigma^{+}$ and $c,d\in\Sigma^{-}$:
  - (B1b) $a \wedge a\sim^{+}\sim^{-} = a$, $c \wedge c\sim^{-}\sim^{+} = c$
  - (B2b) $(a \vee b)\sim^{+} = a\sim^{+} \wedge b\sim^{+}$, $(c \vee d)\sim^{+} = c\sim^{+} \wedge d\sim^{+}$
  - (B3b) $a \wedge a\sim^{+} = 0$, $c \wedge c\sim^{-} = 0$
- (win-b) for all $a\in\Sigma^{+}$ and $b\in\Sigma^{-}$, $a\sim^{+} \leq a'^{+}$ and $b\sim^{-} \leq b'^{-}$
- A bipolar quasi Brouwer-Zadeh lattice is a bipolar Brouwer-Zadeh lattice if stronger interconnection rule is satisfied:
- (in-b) for all $a\in\Sigma^{+}$ and $b\in\Sigma^{-}$, $a\sim^{+}\sim^{-} = a\sim^{+\prime-}$ and $b\sim^{-}\sim^{+} = b\sim^{-\prime+}$

Bipolar Brouwer-Zadeh Distributive De Morgan Lattices

- A bipolar Brouwer-Zadeh lattice is a bipolar de Morgan Brouwer-Zadeh lattice if it satisfies the $\wedge$-de Morgan property:
- (B2a-b) for all $a,b \in \Sigma^+$ and $c,d \in \Sigma^-$
- $(a \wedge b)\sim^+ = a\sim^+ \vee b\sim^+$, $(c \wedge d)\sim^- = c\sim^- \vee d\sim^-$

Topology for DRSA

Bitopological spaces

- A bitopological space is a triple (X,τ_1,τ_2) where X is a set and τ_1 and τ_2 are two topologies (Kelly1963).
- Using τ_1 and τ_2, one can define two interior operators I_1 and I_2
- Then, the bitopological space can be represented by the triple (X,I_1,I_2)
- From interior operators I_1 and I_2 one can be define closure operators C_1 and C_2 in the usual way: for all $A\subseteq X$

$$C_1(A)=X-I_2(X-A), \quad C_2(A)=X-I_1(X-A)$$

- A bitopological space satisfies the biclopen sets property if for all $A\subseteq X$

$$C_1(I_1(A))=I_1(A), \quad C_2(I_2(A))=I_2(A), \quad I_1(C_1(A))=C_1(A), \quad I_2(C_2(A))=C_2(A)$$

Bitopological spaces and DRSA

- **Theorem** (Bezhanishvili et al. 2010) If (X, I_1, I_2) is a bitopological space having biclopen sets property, then there exists a partial preorder $\succeq$ in X such that there exist two bases for τ_1 and τ_2, resp.,

$$\{\{y \in X: y \succeq x\}: x \in X\} \cup \{\emptyset\}, \quad \{\{y \in X: x \succeq y\}: x \in X\} \cup \{\emptyset\}.$$

- DRSA: (U, R^+, R^-) is a bitopological space and the two bases for R^+, R^-:

$$\{\{R^+(x): x \in U\} \cup \{\emptyset\}\}, \quad \{\{R^-(x): x \in U\} \cup \{\emptyset\}\}.$$

- Lower & upper appx of DRSA are interior & closure operators of (U, R^+, R^-)
- (U, τ, R) with R the partial preorder (dominance) relation of DRSA and

$$\tau = \{R^+(x): x \in U\} \cup \{R^-(x): x \in U\} \cup \{\emptyset\}$$

can also be seen as a Priestley topological space (Priestley 1971)

DRSA as a way of handling Fuzzy-Rough Hybridization

DRSA as a proper way of handling graduality in Rough Set Theory

- Rough set concept refers to some ideas of Gottlob Frege (vague concepts), Gottfried Leibniz (indiscernibility), George Boole (reasoning methods), Jan Łukasiewicz (multi-valued logic), and Thomas Bayes (inductive reasoning)

- Gottfried Leibniz (Leibniz's law)
  „identity of indiscernibles" is a principle of analytic ontology :

  *if **x** and **y** are indiscernible, then **x** has the same properties as **y*** (*i.e.* ***x***=***y***)

  the converse principle is called „indiscernibility of identicals":

  *if **x** has the same properties as **y*** (*i.e.* ***x***=***y***)*, then **x** and **y** are indiscernible*

- Rough set theory by Zdzisław Pawlak uses a weaker Leibniz's law to classify objects falling under the same concept – weakened „identity of indiscernibles":

  *if **x** and **y** are indiscernible, then **x** and **y** **belong** to the same class*

  „Indiscernibility of identicals" cannot be reformulated analogously, because it is not true that *if **x** and **y** belong to the same class, then **x** and **y** are indiscernible*

DRSA as a proper way of handling graduality in Rough Set Theory

- From the viewpoint of granular computing, „class” is a synonym of „granule” :

*„if **x** and **y** are indiscernible,*
*then **x** and **y** **belong** to the same classification granule”*

- The relaxation in the consequence of the „identity of indiscernibles” implicitly implies a relaxation in the antecedent :

*„if **x** and **y** are indiscernible **taking into account a given set of properties**,*
*then **x** and **y** **belong** to the same classification granule”*

- This weakening in the antecedent means also that the objects indiscernible with respect to a given set of properties can be seen as a granule:

*„if **x** and **y** **belong to the same granule wrt a given set of properties**,*
*then **x** and **y** **belong** to the same classification granule”*

- Rough set theory needs a still weaker form of „identity of indiscernibles”

DRSA as a proper way of handling graduality in Rough Set Theory

- According to Gottlob Frege:
  „A concept must have a sharp boundary.
  To the (vague) concept without a sharp boundary there would correspond an area that had not a sharp boundary-line all around"

- Following this intuition, one can further reformulate the „identity of indiscernibles":

  *„if **x** and **y** are indiscernible, then **x** and **y** **should belong** to the same class"*

  This formulation implies that there is an inconsistency if ***x*** and ***y*** are indiscernible and they belong to different classes

- In terms of granular computing:

  *„if **x** and **y** **belong to the same granule wrt a given set of properties**, then **x** and **y** **should belong** to the same classification granule"*

- This corresponds exactly to the rough set concept proposed by Pawlak

DRSA as a proper way of handling graduality in Rough Set Theory

- The Pawlak's rough set should be completed, however, by referring to another idea, given by George Boole, and concerning presence (truth) or absence (falsity) of a property for an object.

- Jan Łukasiewicz has enriched the 0-1 truth values by considering gradual truth in many-valued logic – thus, the property can be true to some degree

- The Łukasiewicz's idea of graduality has been reconsidered and fully exploited by Lotfi Zadeh within fuzzy set theory, where graduality concerns membership to a set

DRSA as a proper way of handling graduality in Rough Set Theory

- Any proposal of **putting rough sets and fuzzy sets together** can be seen as a reconstruction of the rough set concept, where the Boole's binary logic is substituted by Łukasiewicz's multi-valued logic, such that the Leibniz's identity of indiscernibles and the Frege's intuition about vagueness are combined through the idea that a property is true to some degree:

> „*if the **degree of each property** for **x** is **at least as high as** the degree for **y**, then **x should belong** to the considered class **in degree at least as high as y**"*

- This formulation is perfectly concordant with the **Dominance-based Rough Set Approach** – it handles the monotonic relationship in exacly the same way

DRSA as a proper way of handling graduality in Rough Set Theory

- In terms of granular computing, the hybridized concept of rough-fuzzy set, which is concordant with DRSA, can be summarized as:

*„if **x** belongs to the granule defined by considered properties not less than **y**, then **x** should belong to the classification granule in degree at least as high as **y**”*

Remarks on fuzzy set extensions of rough sets (before DRSA)

- Nakamura & Gao 1991; Dubois & Prade 1992; Lin 1992; Słowiński 1995; Pal 1996; Słowiński & Stefanowski 1996; Yao 1997; Cattaneo 1998; Morsi & Yakout 1998; Greco, Matarazzo & Słowiński 1999, 2000; Thiele 2000; Inuiguchi & Tanino 2002; Polkowski 2002, Greco, Inuiguchi & Słowiński 2002, Radzikowska & Kerre 2003; Wu, Mi & Zhang 2003; ...

- The fuzzy extensions of Pawlak's definition of lower and upper approximations use fuzzy connectives (t-norm, t-conorm, fuzzy implication)

- In general, fuzzy connectives depend on cardinal properties of membership degrees, i.e. the result is sensitive to order preserving transformation of membership degrees

An example of a fuzzy logic operator: the t-conorm

- Within fuzzy logic t-conorm corresponds to „or" operator in classical logic.
- A t-conorm is a function $T^*:[0,1]\times[0,1]\rightarrow[0,1]$ such that if
  - credibility of proposition p is $\alpha\in [0,1]$, and
  - credibility of proposition q is $\beta\in [0,1]$

  then
  - credibility of proposition $p\vee q$ is $T^*(\alpha, \beta)$.
- E.g., using the t-conorm of Łukasiewicz credibility of proposition $p\vee q$ is

$$T^*(\alpha,\beta) = \min\{\alpha+\beta, 1\}.$$

- Formally a t-conorm is a function $T^*:[0,1]\times[0,1]\rightarrow[0,1]$ being non decreasing in its two arguments, associative, commutative and such that for all $\alpha\in[0,1]$, $T^*(\alpha,1)=\alpha$.

Remarks on fuzzy extensions of rough sets

- Consider the t-conorm of Łukasiewicz: $T^*(\alpha,\beta) = \min\{\alpha+\beta, 1\}$, the following values of arguments:

  α=0.5, β=0.3, γ=0.2, δ=0.1

  and their order preserving transformation:

  α'=0.4, β'=0.3, γ'=0.2, δ'=0.05.

  The values of the t-conorm are:

  $T^*(\alpha,\delta) = 0.6 \quad > \quad T^*(\beta,\gamma) = 0.5$

  $T^*(\alpha',\delta') = 0.45 \quad < \quad T^*(\beta',\gamma') = 0.5$

- The order of the results has changed after the order preserving transformation of the arguments.
- This means that the Łukasiewicz t-conorm takes into account not only the ordinal properties of the membership degrees, but also their cardinal properties.

Which t-conorm to choose? Is there some „right" one?

- Max: $T^*(\alpha,\beta) = \max\{\alpha, \beta\}$?
- t-conorm of Łukasiewicz: $T^*(\alpha,\beta) = \min\{\alpha+\beta, 1\}$?
- Probabilistic sum: $T^*(\alpha,\beta) = \alpha+\beta-\alpha\beta$?
- Drastic t-conorm: $T^*(\alpha,\beta) = \begin{cases} 0 & \text{if } a = 0 \text{ or } \beta = 0 \\ 1 & \text{otherwise} \end{cases}$?
- Nilpotent maximum: $T^*(\alpha,\beta) = \begin{cases} \max(a, \beta) & \text{if } a + \beta < 1 \\ 1 & \text{otherwise} \end{cases}$?
- Frank t-conorm: $T^*(\alpha,\beta) = 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-x} - 1)(\lambda^{1-y} - 1)}{\lambda - 1}\right)$?
- …

Remarks on fuzzy extensions of rough sets

- **A natural question arises**: is it reasonable to expect from membership degree a cardinal meaning instead of ordinal only?
- In other words, is it realistic to think that a human is able to express in a meaningful way not only that

  „object *x* belongs to fuzzy set *X* more likely than object *y*”

  but even something like

  „object *x* belongs to fuzzy set *X* <u>two times</u> more likely than object *y*”?

S.Greco, M.Inuiguchi, R.Słowiński: Fuzzy rough sets and multiple-premise gradual decision rules. *International Journal of Approximate Reasoning*, 41 (2005) 179-211

Dominance-based (monotonic) Rough Approximation of a Fuzzy Set

- The dominance-based rough approximation of a fuzzy set avoids arbitrary choice of fuzzy connectives and not meaningful operations on membership degrees
- Approximation of knowledge about Y using knowledge about X is based on positive or negative relationships between premises and conclusions, called *gradual rules*, i.e.:

  i) „the more x is X, the more it is Y" (positive relationship)

  ii) „the more x is X, the less it is Y" (negative relationship)
- Example:

  „the larger the market share of a company, the larger its profit"

  „the larger the debt of a company, the smaller its profit"

Dominance-based (monotonic) Rough Approximation of a Fuzzy Set

- These monotonic relationships have the form of *gradual decision rules*:

  „if a car is speedy with credibility at least 0.8
  and it has high fuel consumption with credibility at most 0.7,
  then it is a good car with a credibility at least 0.9"

  „if a car is speedy with credibility at most 0.5
  and it has high fuel consumption with credibility at least 0.8,
  then it is a good car with a credibility at most 0.6"

- The syntax of gradual decision rules is based on monotonic relationships between degrees of credibility, as in monotonic decision rules induced from preference-ordered data.

- This explains why one can build a fuzzy-rough approximation using DRSA

DRSA as an approach to computing with words

- Classical fuzzy set approach to **computing with words**:

  i) qualitative inputs, such as „very bad”, „bad”, „medium”, „good”, „very good”

  ii) numerical codification of the inputs (fuzzification): e.g.

  „very bad”=0, „bad”=0.25, „medium”=0.5, „good”=0.75, „very good”=1

  iii) algebraic operations on numerical codes : e.g.
  „comprehensive evalaution of a student good in mathematics and medium in physics”=(0.75+0.5)/2=0.625

  iv) recodification in qualitiative terms of the calculation result (defuzzification): e.g., 0.625=between medium and good

- Dominance-based Rough Set Approach does not need fuzzification and defuzzification: e.g.

  „*if* the student is at least medium in Mathematics *and* at least medium in Literature, *then* the student is at least medium”